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Research Article

Monotone Iterative Technique for Fractional Evolution Equations in Banach Spaces

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We investigate the initial value problem for a class of fractional evolution equations in a Banach space. Under some monotone conditions and noncompactness measure conditions of the nonlinearity, the well-known monotone iterative technique is then extended for fractional evolution equations which provides computable monotone sequences that converge to the extremal solutions in a sector generated by upper and lower solutions. An example to illustrate the applications of the main results is given.

1. Introduction

In this paper, we use the monotone iterative technique to investigate the existence and uniqueness of mild solutions of the fractional evolution equation in an ordered Banach space X,

$$\begin{gathered} D^{\alpha}u(t) + Au(t) = f(t, u(t)), \quad t \in I, \\ u(0) = x_0 \in X, \end{gathered} \tag{1.1}$$

where D^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$, $I = [0, T]$, $A : D(A) \subset X \to X$ is a linear closed densely defined operator, $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $T(t)$ $(t \geq 0)$, and $f : I \times X \to X$ is continuous.

The origin of fractional calculus (i.e., calculus of integrals and derivatives of any arbitrary real or complex order) goes back to Newton and Leibnitz in the seventieth century. We observe that the fractional order can be complex in viewpoint of pure mathematics, and there is much interest in developing the theoretical analysis and numerical methods to fractional equations, because they have recently proved to be valuable in various fields such as physics, chemistry, aerodynamics, viscoelasticity, porous media, electrodynamics of complex medium, electrochemistry, control, and electromagnetic. For instance, fractional calculus concepts have been used in the modeling of transmission lines [1], neurons [2], viscoelastic materials [3], and electrical capacitors [4–6]. References [5, 6] used modified Riemann-Liouville fractional derivatives (Jumarie's fractional derivatives) and proposed the method of fractional complex transform to find exact solutions which are much needed in engineering applications. Other examples from fractional-order dynamics can be found in [7, 8] and the references therein.

Fractional evolution equations are evolution equations where the integer derivative with respect to time is replaced by a derivative of any order. In recent years, fractional evolution equations have attracted increasing attention, see [9–23]. A strong motivation for investigating the Cauchy problem (1.1) comes from physics. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. The main physical purpose for investigating these type of equations is to describe phenomena of anomalous diffusion appearing in transport processes and disordered systems. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$, namely,

$$\partial_t^\alpha u(y,t) = Au(y,t), \quad t \geq 0, \; y \in R, \tag{1.2}$$

where A may be linear fractional partial differential operator. For fractional diffusion equations, we can see [24–26] and the references therein.

It is well known that the method of monotone iterative technique has been proved to be an effective and a flexible mechanism. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Early on, Du and Lakshmikantham [27] established a monotone iterative method for an initial value problem for ordinary differential equation. Later on, many papers used the monotone iterative technique to establish existence and comparison results for nonlinear problems. For evolution equations of integer order ($\alpha = 1$), Li [28–32] and Yang [33] used this method, in which positive C_0-semigroup plays an important role. Recently, there have been some papers which deal with the existence of the solutions of initial value problems or boundary value problems for fractional ordinary differential equations by using this method, see [34–43].

However, when many partial differential equations involving time-variable t turn to evolution equations in Banach spaces, they always generate an unbounded closed operator term A, such as (1.2). A is corresponding to linear partial differential operator with certain boundary conditions. In this case, the results in [34–43] are not suitable. To the best of the authors' knowledge, no results yet exist for the fractional evolution equations involving a closed operator term by using the monotone iterative technique. The approach via fractional differential inequalities is clearly better suited as in the case of classical results of differential equations, and therefore this paper choose to proceed in that setup.

If $-A$ is the infinitesimal generator of an analytic semigroup in a Banach space, then $-(A + qI)$ generates a uniformly bounded analytic semigroup for $q > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is uniformly bounded. Hence, for convenience, throughout this paper, we suppose that $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $T(t)$ $(t \geq 0)$.

Our contribution in this work is to establish the monotone iterative technique for the fractional evolution (1.1). Under some monotone conditions and noncompactness measure conditions of nonlinearity f, which are analogous to those in Li and liu [44], Li [28–32], Chen and li [45], Chen [46], and Yang [33, 47], we obtain results on the existence and uniqueness of mild solutions of the problem (1.1). In this paper, positive semigroup also plays an important role. At last, to illustrate our main results, we examine sufficient conditions for the main results to a fractional partial differential diffusion equation.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1 (see [7]). The Riemann-Liouville fractional integral of order $\alpha > 0$ with the lower limit zero, of function $f \in L_1(\mathbb{R}^+)$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 (see [7]). The Caputo fractional derivative of order $\alpha > 0$ with the lower limit zero, $n-1 < \alpha < n$, is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$. If $0 < \alpha < 1$, then

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds. \tag{2.3}$$

If f is an abstract function with values in X, then the integrals and derivatives which appear in (2.1) and (2.2) are taken in Bochner's sense.

Proposition 2.3. *For $\alpha, \beta > 0$ and f as a suitable function (for instance, [7]), one has*

(i) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$

(ii) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$

(iii) $I^\alpha (f(t) + g(t)) = I^\alpha f(t) + I^\alpha g(t)$

(iv) $I^\alpha D^\alpha f(t) = f(t) - f(0), \quad 0 < \alpha < 1$

(v) $D^\alpha I^\alpha f(t) = f(t)$

(vi) $D^\alpha D^\beta f(t) \neq D^{\alpha+\beta} f(t)$

(vii) $D^\alpha D^\beta f(t) \neq D^\beta D^\alpha f(t)$,

(viii) $D^\alpha C = 0$, *C is a constant.*

We observe from the above that the Caputo fractional differential operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For basic facts about fractional integrals and fractional derivatives, one can refer to the books [7, 48–50].

Let X be an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{y \in X \mid y \geq \theta\}$ (θ is the zero element of X) is normal with normal constant N. Let $C(I, X)$ be the Banach space of all continuous X-value functions on interval I with norm $\|u\|_C = \max_{t\in I}\|u(t)\|$. For $u, v \in C(I, X)$, $u \leq v \Leftrightarrow u(t) \leq v(t)$, for all $t \in I$. For $v, w \in C(I, X)$, denote the ordered interval $[v, w] = \{u \in C(I, X) \mid v \leq u \leq w\}$ and $[v(t), w(t)] = \{y \in X \mid v(t) \leq y \leq w(t)\}$, $t \in I$. Set $C^{\alpha,0}(I, X) = \{u \in C(I, X) \mid D^\alpha u$ exists and $D^\alpha u \in C(I, X)\}$. By X_1, we denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. We note that $-A$ is the infinitesimal generator of a uniformly bounded analytic semigroup $T(t)$ $(t \geq 0)$. This means that there exists $M \geq 1$ such that

$$\|T(t)\| \leq M, \quad t \geq 0. \tag{2.4}$$

Definition 2.4. If $v_0 \in C^{\alpha,0}(I, X) \cap C(I, X_1)$ and satisfies

$$D^\alpha v_0(t) + Av_0(t) \leq f(t, v_0(t)), \quad t \in I, \ v_0(0) \leq x_0, \tag{2.5}$$

then v_0 is called a lower solution of the problem (1.1); if all inequalities of (2.5) are inverse, we call it an upper solution of problem (1.1).

Lemma 2.5 (see [12, 19, 20]). *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0, 1]$, then the unique solution of the Cauchy problem*

$$\begin{gathered} D^\alpha u(t) + Au(t) = h(t), \quad t \in I, \\ u(0) = x_0 \in X \end{gathered} \tag{2.6}$$

is given by

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s) h(s) ds, \tag{2.7}$$

where

$$U(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha\theta) d\theta, \qquad V(t) = \alpha \int_0^\infty \theta\zeta_\alpha(\theta) T(t^\alpha\theta) d\theta, \tag{2.8}$$

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-(1/\alpha)}\rho_\alpha\left(\theta^{-1/\alpha}\right), \tag{2.9}$$

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0, \infty), \tag{2.10}$$

$\zeta_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$.

Remark 2.6 (see [19, 20, 22]). $\zeta_\alpha(\theta) \geq 0$, $\theta \in (0, \infty)$, $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1/\Gamma(1+\alpha)$.

Definition 2.7. By the mild solution of the Cauchy problem (2.6), we mean the function $u \in C(I, X)$ satisfying the integral equation

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1}V(t-s)h(s)ds, \tag{2.11}$$

where $U(t)$ and $V(t)$ are given by (2.8) and (2.9), respectively.

Definition 2.8. An operator family $S(t) : X \to X$ $(t \geq 0)$ in X is called to be positive if for any $u \in P$ and $t \geq 0$ such that $S(t)u \geq \theta$.

From Definition 2.8, if $T(t)$ $(t \geq 0)$ is a positive semigroup generated by $-A$, $h \geq \theta$, $x_0 \geq \theta$, then the mild solution $u \in C(I, X)$ of (2.6) satisfies $u \geq \theta$. For positive semigroups, one can refer to [28–32].

Now, we recall some properties of the measure of noncompactness will be used later. Let $\mu(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [51]. For any $B \subset C(I, X)$ and $t \in I$, set $B(t) = \{u(t) \mid u \in B\}$. If B is bounded in $C(I, X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \leq \mu(B)$.

Lemma 2.9 (see [52]). *Let $B = \{u_n\} \subset C(I, X)$ $(n = 1, 2, \ldots)$ be a bounded and countable set, then $\mu(B(t))$ is Lebesgue integral on I,*

$$\mu\left(\left\{\int_I u_n(t)dt \mid n = 1, 2, \ldots\right\}\right) \leq 2\int_I \mu(B(t))dt. \tag{2.12}$$

In order to prove our results, one also needs a generalized Gronwall inequality for fractional differential equation.

Lemma 2.10 (see [53]). *Suppose that $b \geq 0$, $\beta > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds \tag{2.13}$$

on this interval, then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds, \quad 0 \leq t < T. \tag{2.14}$$

3. Main Results

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ $(t \geq 0)$ is positive, the Cauchy problem (1.1) has a lower solution*

$v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \le w_0$, and the following conditions are satisfied.

(H_1) *There exists a constant $C \ge 0$ such that*

$$f(t,x_2) - f(t,x_1) \ge -C(x_2 - x_1), \tag{3.1}$$

for any $t \in I$, and $v_0(t) \le x_1 \le x_2 \le w_0(t)$, that is, $f(t,x) + Cx$ is increasing in x for $x \in [v_0(t), w_0(t)]$.

(H_2) *There exists a constant $L \ge 0$ such that*

$$\mu(\{f(t,x_n)\}) \le L\mu(\{x_n\}), \tag{3.2}$$

for any $t \in I$, and increasing or decreasing monotonic sequences $\{x_n\} \subset [v_0(t), w_0(t)]$,

then the Cauchy problem (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. It is easy to see that $-(A + CI)$ generates an analytic semigroup $S(t) = e^{-Ct}T(t)$, and $S(t)$ $(t \ge 0)$ is positive. Let $\Phi(t) = \int_0^\infty \zeta_\alpha(\theta)S(t^\alpha\theta)d\theta, \Psi(t) = \alpha\int_0^\infty \theta\zeta_\alpha(\theta)S(t^\alpha\theta)d\theta$. By Remark 2.6, $\Phi(t)$ $(t \ge 0)$ and $\Psi(t)$ $(t \ge 0)$ are positive. By (2.4) and Remark 2.6, we have that

$$\|\Phi(t)\| \le M, \quad \|\Psi(t)\| \le \frac{\alpha}{\Gamma(\alpha+1)}M \triangleq M_1, \quad t \ge 0. \tag{3.3}$$

Let $D = [v_0, w_0]$, we define a mapping $Q : D \to C(I,X)$ by

$$Qu(t) = \Phi(t)x_0 + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,u(s)) + Cu(s)\right]ds, \quad t \in I. \tag{3.4}$$

By Lemma 2.5 and Definition 2.7, $u \in D$ is a mild solution of the problem (1.1) if and only if

$$u = Qu. \tag{3.5}$$

For $u_1, u_2 \in D$ and $u_1 \le u_2$, from the positivity of operators $\Phi(t)$ and $\Psi(t)$, and (H_1), we have that

$$Qu_1 \le Qu_2. \tag{3.6}$$

Now, we show that $v_0 \leq Qv_0$, $Qw_0 \leq w_0$. Let $D^\alpha v_0(t) + Av_0(t) + Cv_0(t) \triangleq \sigma(t)$, by Definition 2.4, Lemma 2.5, and the positivity of operators $\Phi(t)$ and $\Psi(t)$, we have that

$$\begin{aligned} v_0(t) &= \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\sigma(s)ds \\ &\leq \Phi(t)x_0 + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,v_0(s)) + Cv_0(s)\right]ds \\ &= Qv_0(t), \quad t \in I, \end{aligned} \tag{3.7}$$

namely, $v_0 \leq Qv_0$. Similarly, we can show that $Qw_0 \leq w_0$. For $u \in D$, in view of (3.6), then $v_0 \leq Qv_0 \leq Qu \leq Qw_0 \leq w_0$. Thus, $Q : D \to D$ is an increasing monotonic operator. We can now define the sequences

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1,2,\ldots, \tag{3.8}$$

and it follows from (3.6) that

$$v_0 \leq v_1 \leq \cdots v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0. \tag{3.9}$$

Let $B = \{v_n\}$ $(n = 1,2,\ldots)$ and $B_0 = \{v_{n-1}\}$ $(n = 1,2,\ldots)$. It follows from $B_0 = B \cup \{v_0\}$ that $\mu(B(t)) = \mu(B_0(t))$ for $t \in I$. Let

$$\varphi(t) = \mu(B(t)) = \mu(B_0(t)), \quad t \in I. \tag{3.10}$$

For $t \in I$, from (H_2), (3.3), (3.4), (3.8), (3.10), Lemma 2.9, and the positivity of operator $\Psi(t)$, we have that

$$\begin{aligned} \varphi(t) &= \mu(B(t)) = \mu(QB_0(t)) \\ &= \mu\left(\left\{\int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,v_{n-1}(s)) + Cv_{n-1}(s)\right]ds \mid n = 1,2,\ldots\right\}\right) \\ &\leq 2\int_0^t \mu\left(\left\{(t-s)^{\alpha-1}\Psi(t-s)\left[f(s,v_{n-1}(s) + Cv_{n-1}(s)\right] \mid n = 1,2,\ldots\right\}\right)ds \\ &\leq 2M_1\int_0^t (t-s)^{\alpha-1}(L+C)\mu(B_0(s))ds \\ &= 2M_1(L+C)\int_0^t (t-s)^{\alpha-1}\varphi(s)ds. \end{aligned} \tag{3.11}$$

By (3.11) and Lemma 2.10, we obtain that $\varphi(t) \equiv 0$ on I. This means that $v_n(t)$ $(n = 1,2,\ldots)$ is precompact in X for every $t \in I$. So, $v_n(t)$ has a convergent subsequence in X. In view of (3.9), we can easily prove that $v_n(t)$ itself is convergent in X. That is, there exists $\underline{u}(t) \in X$ such that $v_n(t) \to \underline{u}(t)$ as $n \to \infty$ for every $t \in I$. By (3.4) and (3.8), for any $t \in I$, we have that

$$v_n(t) = \Phi(t)x_0 + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s, v_{n-1}(s)) + Cv_{n-1}(s)\right]ds. \tag{3.12}$$

Let $n \to \infty$, then by Lebesgue-dominated convergence theorem, for any $t \in I$, we have that

$$\underline{u}(t) = \Phi(t)x_0 + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s, \underline{u}(s)) + C\underline{u}(s)\right]ds, \tag{3.13}$$

and $\underline{u} \in C(I, X)$, then $\underline{u} = Q\underline{u}$. Similarly, we can prove that there exists $\overline{u} \in C(I, X)$ such that $\overline{u} = Q\overline{u}$. By (3.6), if $u \in D$, and u is a fixed point of Q, then $v_1 = Qv_0 \le Qu = u \le Qw_0 = w_1$. By induction, $v_n \le u \le w_n$. By (3.9) and taking the limit as $n \to \infty$, we conclude that $v_0 \le \underline{u} \le u \le \overline{u} \le w_0$. That means that $\underline{u}, \overline{u}$ are the minimal and maximal fixed points of Q on $[v_0, w_0]$, respectively. By (3.5), they are the minimal and maximal mild solutions of the Cauchy problem (1.1) on $[v_0, w_0]$, respectively. □

Remark 3.2. Theorem 3.1 extends [37, Theorem 2.1]. Even if $A = 0$ and $X = \mathbb{R}$, our results are also new.

Corollary 3.3. *Let X be an ordered Banach space, whose positive cone P is regular. Assume that $T(t)$ $(t \ge 0)$ is positive, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I, X)$ and an upper solution $w_0 \in C(I, X)$ with $v_0 \le w_0$, and (H_1) holds, then the Cauchy problem* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since (H_1) is satisfied, then (3.9) holds. In regular positive cone P, any monotonic and ordered-bounded sequence is convergent, then there exist $\underline{u} \in C(I, E)$, $\overline{u} \in C(I, E)$, and $\lim_{n\to\infty} v_n = \underline{u}$, $\lim_{n\to\infty} w_n = \overline{u}$. Then by the proof of Theorem 3.1, the proof is then complete. □

Corollary 3.4. *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ $(t \ge 0)$ is positive, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I, X)$ and an upper solution $w_0 \in C(I, X)$ with $v_0 \le w_0$, and (H_1) holds, then the Cauchy problem* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since X is an ordered and weakly sequentially complete Banach space, then the assumption (H_2) holds. In fact, by [54, Theorem 2.2], any monotonic and ordered bounded sequence is precompact. Let x_n be an increasing or decreasing sequence. By (H_1), $\{f(t, x_n) + Cx_n\}$ is a monotonic and ordered bounded sequence. Then, by the properties of the measure of noncompactness, we have

$$\mu(\{f(t, x_n)\}) \le \mu(\{f(t, x_n) + Cx_n\}) + \mu(\{Cx_n\}) = 0. \tag{3.14}$$

So, (H_2) holds. By Theorem 3.1, the proof is then complete. □

Theorem 3.5. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. Assume that $T(t)$ $(t \geq 0)$ is positive, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, (H_1) holds, and the following condition is satisfied:*

(H_3) *there is constant $S \geq 0$ such that*

$$f(t,x_2) - f(t,x_1) \leq S(x_2 - x_1), \tag{3.15}$$

for any $t \in I$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.

Then the Cauchy problem (1.1) *has the unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof. We can find that (H_1) and (H_3) imply (H_2). In fact, for $t \in I$, let $\{x_n\} \subset [v_0(t), w_0(t)]$ be an increasing sequence. For $m,n = 1,2,\ldots$ with $m > n$, by (H_1) and (H_3), we have that

$$\theta \leq f(t,x_m) - f(t,x_n) + C(x_m - x_n) \leq (S+C)(x_m - x_n). \tag{3.16}$$

By (3.16) and the normality of positive cone P, we have

$$\|f(t,x_m) - f(t,x_n)\| \leq (NS + NC + C)\|x_m - x_n\|. \tag{3.17}$$

From (3.17) and the definition of the measure of noncompactness, we have that

$$\mu(\{f(t,x_n)\}) \leq L\mu(\{x_n\}), \tag{3.18}$$

where $L = NS + NC + C$. Hence, (H_2) holds.

Therefore, by Theorem 3.1, the Cauchy problem (1.1) has the minimal mild solution $\underline{u}$ and the maximal mild solution $\overline{u}$ on $D = [v_0, w_0]$. In view of the proof of Theorem 3.1, we show that $\underline{u} = \overline{u}$. For $t \in I$, by (3.3), (3.4), (3.5), (H_3), and the positivity of operator $\Psi(t)$, we have that

$$\begin{aligned}
\theta \leq \overline{u}(t) - \underline{u}(t) &= Q\overline{u}(t) - Q\underline{u}(t) \\
&= \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,\overline{u}(s)) - f(s,\underline{u}(s)) + C(\overline{u}(s) - \underline{u}(s))\right]ds \\
&\leq \int_0^t (t-s)^{\alpha-1}\Psi(t-s)(S+C)(\overline{u}(s) - \underline{u}(s))ds \\
&\leq M_1(S+C)\int_0^t (t-s)^{\alpha-1}\left[\overline{u}(s) - \underline{u}(s)\right]ds.
\end{aligned} \tag{3.19}$$

By (3.19) and the normality of the positive cone P, for $t \in I$, we obtain that

$$\|\overline{u}(s) - \underline{u}(s)\| \leq NM_1(S+C)\int_0^t (t-s)^{\alpha-1}\|\overline{u}(s) - \underline{u}(s)\|ds. \tag{3.20}$$

By Lemma 2.10, then $\underline{u}(t) \equiv \overline{u}(t)$ on I. Hence, $\underline{u} = \overline{u}$ is the the unique mild solution of the Cauchy problem (1.1) on $[v_0, w_0]$. By the proof of Theorem 3.1, we know it can be obtained by a monotone iterative procedure starting from v_0 or w_0. □

By Corollary 3.3, Corollary 3.4, Theorem 3.5, we have the following results.

Corollary 3.6. *Assume that $T(t)$ $(t \geq 0)$ is positive, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I, X)$ and an upper solution $w_0 \in C(I, X)$ with $v_0 \leq w_0$, (H_1) and (H_3) hold, and one of the following conditions is satisfied:*

(i) *X is an ordered Banach space, whose positive cone P is regular,*

(ii) *X is an ordered and weakly sequentially complete Banach space, whose positive cone P is normal with normal constant N,*

then the Cauchy problem (1.1) *has the unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

4. Examples

Example 4.1. In order to illustrate our main results, we consider the fractional partial differential diffusion equation in X,

$$\begin{gathered} \partial_t^\alpha u - \Delta u = g(y, t, u), \quad (y, t) \in \Omega \times I, \\ u|_{\partial\Omega} = 0, \\ u(y, 0) = \varphi(y), \quad y \in \Omega, \end{gathered} \tag{4.1}$$

where ∂_t^α is the Caputo fractional partial derivative with order $0 < \alpha < 1$, Δ is the Laplace operator, $I = [0, T]$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and $g : \overline{\Omega} \times I \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let $X = L^2(\Omega)$, $P = \{v \mid v \in L^2(\Omega), v(y) \geq 0 \ a.e.\ y \in \Omega\}$, then X is a Banach space, and P is a normal cone in X. Define the operator A as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = -\Delta u. \tag{4.2}$$

Then $-A$ generates an analytic semigroup of uniformly bounded analytic semigroup $T(t)$ $(t \geq 0)$ in X (see [18]). $T(t)$ $(t \geq 0)$ is positive (see [31, 32, 55, 56]). Let $u(t) = u(\cdot, t)$, $f(t, u) = g(\cdot, t, u(\cdot, t))$, then the problem (4.1) can be transformed into the following problem:

$$\begin{gathered} D^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in I, \\ u(0) = \varphi. \end{gathered} \tag{4.3}$$

Let λ_1 be the first eigenvalue of A, and φ_1 is the corresponding eigenfunction, then $\lambda_1 \geq 0$, $\varphi_1(y) \geq 0$. In order to solve the problem (4.1), we also need the following assumptions:

(O_1) $\varphi(y) \in H^2(\Omega) \cap H_0^1(\Omega)$, $0 \leq \varphi(y) \leq \varphi_1(y)$, $g(y, t, 0) \geq 0$, $g(y, t, \varphi_1(y)) \leq \lambda_1 \varphi_1(y)$,

(O_2) the partial derivative $g'_u(y, t, u)$ is continuous on any bounded domain.

Theorem 4.2. *If* (O_1) *and* (O_2) *are satisfied, then the problem* (4.1) *has the unique mild solution.*

Proof. From Definition 2.4 and (O_1), we obtain that 0 is a lower solution of (4.3), and $\psi_1(y)$ is an upper solution of (4.3). Form (O_2), it is easy to verify that (H_1) and (H_3) are satisfied. Therefore, by Theorem 3.5, the problem (4.1) has the unique mild solution. □

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