Starlikeness Properties of a New Integral Operator for Meromorphic Functions

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We define here an integral operator \( H_{\gamma_1, \ldots, \gamma_n} \) for meromorphic functions in the punctured open unit disk. Several starlikeness conditions for the integral operator \( H_{\gamma_1, \ldots, \gamma_n} \) are derived.

1. Introduction

Let \( \Sigma \) denotes the class of functions of the form

\[
 f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,
\]

which are analytic in the punctured open unit disk

\[
 \mathbb{U}^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{U} \setminus \{0\},
\]

where \( \mathbb{U} \) is the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \).

We say that a function \( f \in \Sigma \) is meromorphic starlike of order \( \alpha \) (\( 0 \leq \alpha < 1 \)), and belongs to the class \( \Sigma^*(\alpha) \), if it satisfies the inequality

\[
 -\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha. \tag{1.3}
\]
A function \( f \in \Sigma \) is a meromorphic convex function of order \( \alpha \) \((0 \leq \alpha < 1)\), if \( f \) satisfies the following inequality

\[
-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \tag{1.4}
\]

and we denote this class by \( \Sigma_\alpha(\alpha) \).

Analogous to the integral operator defined by Breaz et al. [1] on the normalized analytic functions, we now define the following integral operator on the space meromorphic functions in the class \( \Sigma \).

**Definition 1.1.** Let \( n \in \mathbb{N}, \gamma_i > 0, i \in \{1, 2, 3, \ldots, n\} \). We define the integral operator \( \mathcal{H}_{n,\ldots,\gamma_n}(f_1, f_2, \ldots, f_n) : \Sigma^n \rightarrow \Sigma \) by

\[
\mathcal{H}_{n,\ldots,\gamma_n}(f_1, \ldots, f_n)(z) = \frac{1}{z^2} \int_0^z (-u^2 f_1'(u))^{\gamma_1} \cdots (-u^2 f_n'(u))^{\gamma_n} du. \tag{1.5}
\]

For the sake of simplicity, from now on we will write \( \mathcal{H}_{n,\ldots,\gamma_n}(z) \) instead of \( \mathcal{H}_{n,\ldots,\gamma_n}(f_1, \ldots, f_n)(z) \).

By \( \Sigma_\beta(\beta) \) \((-1 \leq \beta < 1)\), we denote the class of functions \( f \in \Sigma \) such that

\[
\left|\frac{zf''(z)}{f'(z)} + 2\right| < -\Re\left(\frac{zf''(z)}{f'(z)} + \beta\right) - 1. \tag{1.6}
\]

In order to derive our main results, we have to recall here the following preliminary results.

**Lemma 1.2** (see [2]). Suppose that the function \( \Psi : \mathbb{C}^2 \rightarrow \mathbb{C} \) satisfies the following condition:

\[
\Re\{\Psi(is, t)\} \leq 0, \quad s, t \in \mathbb{R}; \quad t \leq -\left(1 + \frac{s^2}{2}\right). \tag{1.7}
\]

If the function \( p(z) = 1 + p_1z + \cdots \) is analytic in \( \mathbb{U} \) and

\[
\Re\{\Psi(p(z), zp'(z))\} > 0, \quad (z \in \mathbb{U}), \tag{1.8}
\]

then,

\[
\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}). \tag{1.9}
\]

**Proposition 1.3** (see [3]). If \( f \in \Sigma \) satisfying

\[
-\Re\left\{\frac{zf''(z) + 3f'(z)}{zf'(z) + 2f(z)}\right\} > \alpha, \quad 0 \leq \alpha < 1,
\]

\[
\left|\frac{zf'(z)}{f(z)} + 1\right| < 1, \tag{1.10}
\]

then...
In this section, we investigate sufficiency of the operator $H$.

Then, $H$ is defined in Definition 1.1, to be in the class $\Sigma^*(\alpha)$, $0 \leq \alpha < 1$.

**Theorem 2.1.** Let $f_i \in \sum$, $\gamma_i > 0$ for all $i \in \{1, \ldots, n\}$. If

$$-\Re\left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{-1}{n\gamma_i} + 2,$$

then $H_{\gamma_1, \ldots, \gamma_n}(z)$ belongs to $\Sigma^*(0)$.

**Proof.** On successive differentiation of $H_{\gamma_1, \ldots, \gamma_n}(z)$, which is defined in (1.5), we get

$$2zH''_{\gamma_1, \ldots, \gamma_n}(z) + z^2H'_{\gamma_1, \ldots, \gamma_n}(z) = (-z^2f'_1(z))^n \cdots (-z^2f'_n(z))^n,$$

$$z^2H''_{\gamma_1, \ldots, \gamma_n}(z) + 4zH'_{\gamma_1, \ldots, \gamma_n}(z) + 2H_{\gamma_1, \ldots, \gamma_n}(z)$$

$$= \sum_{i=1}^n \gamma_i (-z^2f'_i(z))^{n-1} (-z^2f''_i(z) - 2zf'_i(z)) \prod_{j=1, j \neq i}^n (-z^2f'_j(z))^{\gamma_j}.$$

Then from (2.2), we obtain

$$\frac{z^2H''_{\gamma_1, \ldots, \gamma_n}(z) + 4zH'_{\gamma_1, \ldots, \gamma_n}(z) + 2H_{\gamma_1, \ldots, \gamma_n}(z)}{z^2H'_{\gamma_1, \ldots, \gamma_n}(z) + 2zH_{\gamma_1, \ldots, \gamma_n}(z)} = \sum_{i=1}^n \gamma_i \left( \frac{z f''_i(z)}{f'_i(z)} + \frac{2}{z} \right).$$

By multiplying (2.3) with $z$ yield,

$$\frac{z^2H''_{\gamma_1, \ldots, \gamma_n}(z) + 4zH'_{\gamma_1, \ldots, \gamma_n}(z) + 2H_{\gamma_1, \ldots, \gamma_n}(z)}{zeH'_{\gamma_1, \ldots, \gamma_n}(z) + 2zeH_{\gamma_1, \ldots, \gamma_n}(z)} = \sum_{i=1}^n \gamma_i \left( \frac{zf''_i(z)}{f'_i(z)} + 2 \right).$$

That is equivalent to

$$\left\{ \frac{z \left( zH''_{\gamma_1, \ldots, \gamma_n}(z) + 3H'_{\gamma_1, \ldots, \gamma_n}(z) \right)}{zeH''_{\gamma_1, \ldots, \gamma_n}(z) + 2zeH_{\gamma_1, \ldots, \gamma_n}(z)} \right\} + 1 = \sum_{i=1}^n \gamma_i \left( \frac{zf''_i(z)}{f'_i(z)} + 2 \right).$$
Or

\[
- \left\{ \frac{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\mu (z) + 3 \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z)}{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z) + 2 \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\rho (z)} \right\} = \sum_{i=1}^{n} y_i \left( \frac{-zf_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.6}
\]

We can write the left-hand side of (2.6), as the following:

\[
- \left( \frac{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\mu (z) / \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z)}{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z) / \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\rho (z)} + 3 \right)
\]

\[
= \sum_{i=1}^{n} y_i \left( \frac{-zf_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.7}
\]

We define the regular function \( p \) in \( \mathbb{U} \) by

\[
p(z) = - \frac{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\mu (z)}{\partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z)}, \tag{2.8}
\]

and \( p(0) = 1 \). Differentiating \( p(z) \) logarithmically, we obtain

\[
-p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z \partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\mu (z)}{\partial \mathcal{E}_{\gamma_1,\ldots,\gamma_n}^\nu (z)} \tag{2.9}
\]

From (2.7), (2.8), and (2.9), we obtain

\[
p(z) + \frac{zp'(z)}{-p(z) + 2} = \sum_{i=1}^{n} y_i \left( \frac{-zf_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.10}
\]

Let us put

\[
\Psi(u, v) = u + \frac{v}{-u + 2}. \tag{2.11}
\]

From (2.1), (2.10), and (2.11), we obtain

\[
\Re \{ \Psi(p(z), zp'(z)) \} = y_1 \left( -\Re \frac{zf_1''(z)}{f_1'(z)} \right) + \cdots + \left( -\Re \frac{zf_n''(z)}{f_n'(z)} \right) - 2(y_1 + \cdots + y_n) + 1
\]

\[
> y_1 \left( \frac{-1}{ny_1} + 2 \right) + \cdots + y_n \left( \frac{-1}{ny_n} + 2 \right) - 2(y_1 + \cdots + y_n) + 1 = 0. \tag{2.12}
\]
Now, we proceed to show that
\[ \Re \{ \Psi(is,t) \} \leq 0, \quad (s,t \in \mathbb{R}; t \leq \frac{-(1+s^2)}{2}) \]  \hspace{1cm} (2.13)

Indeed, from (2.11), we have
\[ \Re \{ \Psi(is,t) \} = \Re \left\{ is + \frac{t}{-is+2} \right\} = \frac{2t}{4+s^2} \leq \frac{1+s^2}{4+s^2} < 0. \]  \hspace{1cm} (2.14)

Thus, from (2.12), (2.14), and by using Lemma 1.2, we conclude that \( \Re \{ p(z) \} > 0 \), and so
\[ -\Re \left\{ \frac{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)}{\mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)} \right\} > 0 \]  \hspace{1cm} (2.15)

that is, \( \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) \) is starlike of order 0. \( \square \)

**Theorem 2.2.** For \( i \in \{1, \ldots, n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Sigma_{k}(\alpha_i) \) (\( 0 \leq \alpha_i < 1 \)). If \( 0 < \sum_{i=1}^{n} \gamma_i(1-\alpha_i) \leq 1 \), \( \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) \) be the integral operator given by (1.5) and
\[ \left| \frac{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)}{\mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)} + 1 \right| < 1. \]  \hspace{1cm} (2.16)

Then \( \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) \) belong to \( \Sigma^{*}(\mu) \), where \( \mu = 1 - \sum_{i=1}^{n} \gamma_i(1-\alpha_i) \).

**Proof.** Following the same steps as in Theorem 2.1, we obtain
\[ -\left\{ \frac{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) + 3 \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)}{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) + 2 \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)} \right\} = \sum_{i=1}^{n} \gamma_i \left\{ -\Re \left( \frac{z f_i''(z)}{f_i'(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \]  \hspace{1cm} (2.17)

Taking the real part of both terms of the last expression, we have
\[ -\Re \left\{ \frac{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) + 3 \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)}{z \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z) + 2 \mathcal{K}_{\gamma_1, \ldots, \gamma_n}(z)} \right\} = \sum_{i=1}^{n} \gamma_i \left\{ -\Re \left( \frac{z f_i''(z)}{f_i'(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \]  \hspace{1cm} (2.18)
Since $f_i \in \Sigma_k(\alpha_i)$, for $i \in \{1, \ldots, n\}$, we receive

\[
-\Re\left\{\frac{z(z\mathcal{H}''_{y_1,\ldots,y_n}(z) + 3z\mathcal{H}'_{y_1,\ldots,y_n}(z))}{z\mathcal{H}'_{y_1,\ldots,y_n}(z) + 2z\mathcal{H}_{y_1,\ldots,y_n}(z)}\right\} > \sum_{i=1}^{n} y_i \alpha_i + 1 - \sum_{i=1}^{n} y_i. \tag{2.19}
\]

Therefore,

\[
-\Re\left\{\frac{z(z\mathcal{H}''_{y_1,\ldots,y_n}(z) + 3z\mathcal{H}'_{y_1,\ldots,y_n}(z))}{z\mathcal{H}'_{y_1,\ldots,y_n}(z) + 2z\mathcal{H}_{y_1,\ldots,y_n}(z)}\right\} > 1 - \sum_{i=1}^{n} y_i(1 - \alpha_i). \tag{2.20}
\]

Using (2.16), (2.20), and applying Proposition 1.3, we get $\mathcal{H}_{y_1,\ldots,y_n}(z) \in \Sigma^*(\mu)$, where $\mu = 1 - \sum_{i=1}^{n} y_i(1 - \alpha_i)$. \(\square\)

Letting $\alpha_i = \alpha$, $i \in \{1, \ldots, n\}$ in Theorem 2.2, we get the following.

**Corollary 2.3.** For $i \in \{1, \ldots, n\}$, let $y_i > 0$ and $f_i \in \Sigma_k(\alpha)$ ($0 \leq \alpha < 1$). If

\[
0 < \sum_{i=1}^{n} y_i \leq \frac{1}{1 - \alpha'}, \tag{2.21}
\]

$\mathcal{H}_{y_1,\ldots,y_n}$ be the integral operator given by (1.5) and

\[
\left|\frac{z\mathcal{H}'_{y_1,\ldots,y_n}(z)}{\mathcal{H}_{y_1,\ldots,y_n}(z)} + 1\right| < 1. \tag{2.22}
\]

Then $\mathcal{H}_{y_1,\ldots,y_n}(z)$ is starlike of order $1 - (1 - \alpha')\sum_{i=1}^{n} y_i$.

**Theorem 2.4.** For $i \in \{1, \ldots, n\}$, let $y_i > 0$ and $f_i \in \Sigma_k(\beta_i)$ ($-1 \leq \beta_i < 1$). If

\[
0 < \left(\sum_{i=1}^{n} y_i(1 - \beta_i) \leq 1, \tag{2.23}
\]

$\mathcal{H}_{y_1,\ldots,y_n}(z)$ be the integral operator given by (1.5) and

\[
\left|\frac{z\mathcal{H}'_{y_1,\ldots,y_n}(z)}{\mathcal{H}_{y_1,\ldots,y_n}(z)} + 1\right| < 1. \tag{2.24}
\]

Then $\mathcal{H}_{y_1,\ldots,y_n}(z)$ is starlike of order $1 - \sum_{i=1}^{n} y_i(1 - \beta_i)$. 

Proof. Following the same steps as in Theorem 2.1, we obtain

\[
- \left\{ \frac{z(ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 3e^{u_{\gamma_1,\ldots,\gamma_n}}(z))}{ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 2e^{u_{\gamma_1,\ldots,\gamma_n}}(z)} \right\} = - \sum_{i=1}^{n} y_i \left( \frac{zf_i''(z)}{f_i'(z)} + 2 \right) + 1
\]

\[
= \sum_{i=1}^{n} y_i \left( \frac{-zf_i''(z)}{f_i'(z)} + \beta_i \right) - 1 + 1 - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} y_i \beta_i
\]

\[
= \sum_{i=1}^{n} y_i \left( \frac{-zf_i''(z)}{f_i'(z)} + \beta_i \right) - 1 + 1 - \sum_{i=1}^{n} y_i (1 - \beta_i).
\]

(2.25)

We calculate the real part from both terms of the above equality and obtain

\[
- \Re \left\{ \frac{z(ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 3e^{u_{\gamma_1,\ldots,\gamma_n}}(z))}{ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 2e^{u_{\gamma_1,\ldots,\gamma_n}}(z)} \right\}
\]

\[
= \sum_{i=1}^{n} y_i \left( -\Re \left( \frac{zf_i''(z)}{f_i'(z)} + \beta_i \right) - 1 \right) + 1 - \sum_{i=1}^{n} y_i (1 - \beta_i).
\]

(2.26)

Since \( f_i \in \Sigma_{\kappa_i}(\beta_i) \) for all \( i \in \{1, \ldots, n\} \), the above relation then yields

\[
- \Re \left\{ \frac{z(ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 3e^{u_{\gamma_1,\ldots,\gamma_n}}(z))}{ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 2e^{u_{\gamma_1,\ldots,\gamma_n}}(z)} \right\}
\]

\[
> \sum_{i=1}^{n} y_i \left| \frac{zf_i''(z)}{f_i'(z)} + 2 \right| + 1 - \sum_{i=1}^{n} y_i (1 - \beta_i).
\]

(2.27)

Because \( \sum_{i=1}^{n} y_i |zf_i''(z)/f_i'(z) + 2| \geq 0 \), we obtain that

\[
- \Re \left\{ \frac{z(ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 3e^{u_{\gamma_1,\ldots,\gamma_n}}(z))}{ze^{u_{\gamma_1,\ldots,\gamma_n}}(z) + 2e^{u_{\gamma_1,\ldots,\gamma_n}}(z)} \right\}
\]

\[
> 1 - \sum_{i=1}^{n} y_i (1 - \beta_i).
\]

(2.28)

Using (2.24), (2.28) and applying Proposition 1.3, we get \( \mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \) is a starlike function of order \( 1 - \sum_{i=1}^{n} y_i (1 - \beta_i) \).

Letting \( \beta_i = \beta, i \in \{1, \ldots, n\} \) in Theorem 2.4, we get the following.

Corollary 2.5. For \( i \in \{1, \ldots, n\} \), let \( y_i > 0 \) and \( f_i \in \Sigma_{\kappa_i}(\beta) \) \((-1 \leq \beta < 1\)). If

\[
0 < \sum_{i=1}^{n} y_i \leq \frac{1}{1 - \beta}
\]

(2.29)
Let \( H_{\gamma_1,\ldots,\gamma_n}(z) \) be the integral operator given by (1.5) and
\[
\left| \frac{zH'(\gamma_1,\ldots,\gamma_n)(z)}{H_{\gamma_1,\ldots,\gamma_n}(z)} + 1 \right| < 1. \tag{2.30}
\]
Then \( H_{\gamma_1,\ldots,\gamma_n}(z) \) is starlike of order \( 1 - (1 - \beta) \sum_{i=1}^{n} \gamma_i \).

Letting \( n = 1, \gamma_1 = \gamma \) and \( f_1 = f \) in Corollary 2.5, we get the following.

**Corollary 2.6.** Let \( \gamma > 0 \), and \( f \in \Sigma_{k_p}(\beta) \) \((-1 \leq \beta < 1)\). If
\[
0 < \gamma \leq \frac{1}{1 - \beta},
\]
\( H_{\gamma}(z) \) be the integral operator,
\[
H_{\gamma}(z) = \frac{1}{z^2} \int_{0}^{z} \left( -u^2 f'(u) \right)^{\gamma} du,
\]
\[
\left| \frac{zH'(\gamma)(z)}{H_{\gamma}(z)} + 1 \right| < 1. \tag{2.32}
\]
Then \( H_{\gamma}(z) \) is starlike of order \( 1 - (1 - \beta)\gamma \).

Other work related to integral operator for different studies can also be found in [4–6].

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**References**
