Letter to the Editor

Variational Iteration Method for q -Difference Equations of Second Order

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Recently, Liu extended He's variational iteration method to strongly nonlinear q -difference equations Liu (2010). In this study, the iteration formula and the Lagrange multiplier are given in a more accurate way. The q -oscillation equation of second order is approximately solved to show the new Lagrange multiplier's validity.

1. Introduction

Generally, applying the variational iteration method (VIM) [1, 2] in differential equations follows the three steps:

- (a) establishing the correction functional;
- (b) identifying the Lagrange multipliers;
- (c) determining the initial iteration.

Obviously, the step (b) is crucial and critical in the method.

For the strongly nonlinear q -difference equation,

$$\frac{d_q^2}{d_q t^2} x + (2 + \varepsilon x) \frac{d_q}{d_q t} x + \Omega^2 x + x^2 = 0, \quad (1.1)$$

where $d_q/d_q t$ is the q -derivative [3], Liu [4] used the Lagrange multiplier

$$\lambda(t, s) = s - t, \quad (1.2)$$

which results in the iteration formula (see [4, (4.10) and (4.11)]):

$$x_{n+1} = x_n + \int_0^t (s-t) \left(\frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \quad (1.3)$$

In this paper, it is pointed out that the iteration formula (1.3) can be given in a more accurate way and a new Lagrange multiplier is explicitly identified.

2. Properties of q -Calculus

2.1. q -Calculus

Let $f(x)$ be a real continuous function. The q -derivative is defined as

$$\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad 0 < q < 1, \quad (2.1)$$

and $(d_q/d_q x)f(x)|_{x=0} = \lim_{n \rightarrow \infty} ((f(q^n) - f(0))/q^n)$.

The partial q -derivative with respect to x is

$$\frac{\partial_q}{\partial_q x} f(x; y; \dots) = \frac{f(qx; y; \dots) - f(x; y; \dots)}{(q-1)x}. \quad (2.2)$$

The corresponding q -integral [5] is

$$\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x). \quad (2.3)$$

2.2. q -Leibniz Product Law

One has

$$\frac{d_q}{d_q x} [g(x)f(x)] = g(qx) \frac{d_q}{d_q x} [f(x)] + f(x) \frac{d_q}{d_q x} [g(x)]. \quad (2.4)$$

2.3. q -Integration by Parts

One has

$$\int_a^b g(qt) \frac{d_q}{d_q t} f(t) d_q t = f(t)g(t) \Big|_a^b - \int_a^b f(t) \frac{d_q}{d_q t} g(t) d_q t. \quad (2.5)$$

The properties above are needed in the construction of the correction functional for q -difference equations. For more results and properties in q -calculus, readers are referred to the recent monographs [5–8].

3. A q -Analogue of Lagrange Multiplier

In order to identify the Lagrange multipliers of the q -difference equations, we first establish the correctional functional for (1.1) as

$$x_{n+1} = x_n + \int_0^t \lambda(t, q^2 s) \left(\frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \quad (3.1)$$

The correction functional here is different from the one in ordinary calculus since the parameter q “disappears” after the integration by parts (2.5) each time. As a result, we use $\lambda(t, q^2 s)$ in the above functional.

We only need to consider the leading term $(d_q^2/d_q t^2)x$ when other terms are restricted variations in (1.1)

$$x_{n+1} = x_n + \int_0^t \lambda(t, q^2 s) \left(\frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \quad (3.2)$$

Through the integration by parts (2.5), we can have

$$\delta x_{n+1} = \left(1 - q \frac{\partial_q}{\partial_q s} \lambda(t, s) \Big|_{s=t} \right) \delta x_n + \lambda(t, qs) \Big|_{s=t} \delta x'_n - q \int_0^t \frac{\partial_q^2}{\partial_q s^2} \lambda(t, s) \delta x_n d_q s, \quad (3.3)$$

where δ is the variation operator and “ ∂_q ” denotes the q -derivative with respect to t . As a result, the system of the Lagrange multiplier can be obtained:

the coefficient of δx_n : $1 - q(\partial_q/\partial_q s)\lambda(t, s)|_{s=t} = 0$,

the coefficient of $\delta x'_n$: $\lambda(t, qs)|_{s=t} = 0$,

the coefficient of δx_n in the q -integral : $q(\partial_q^2/\partial_q s^2)\lambda(t, s) = 0$,

from which we can get

$$\lambda(t, s) = q^{-1}(s - tq), \quad (3.4)$$

instead of $\lambda(t, s) = s - t$ in [4]. More introductions to the identification of various Lagrange multipliers of the VIM can be found in [9, 10].

We also can show the above q -analogue of Lagrange multiplier's validness. For $0 < q < 1$, let T_q be the time scale: $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of positive integers. For the real continuous function $u(t) : T_q \rightarrow \mathbb{R}$, a q -oscillator equation of second order is

$$\frac{d_q^2}{d_q t^2} u - u = 0, \quad u(0) = 1, \quad \frac{d_q}{d_q t} u \Big|_{t=0} = 1. \quad (3.5)$$

From (3.4), the iteration formula can be given as

$$u_{n+1} = u_n + \int_0^t q^{-1}(q^2 s - tq) \left[\frac{d_q^2}{d_q s^2} u_n(s) - u_n(s) \right] d_q s. \quad (3.6)$$

Starting from the initial iteration $u_0 = 1 + t/[1]_q!$, the successive approximate solutions can be obtained as

$$\begin{aligned} u_0 &= 1 + \frac{t}{[1]_q!}, \\ u_1 &= 1 + \frac{t}{[1]_q!} + \frac{t^2}{[2]_q!} + \frac{t^3}{[3]_q!}, \\ &\vdots \\ u_n &= \sum_{k=0}^{2n+1} \frac{t^k}{[k]_q!}. \end{aligned} \quad (3.7)$$

The limit $u = \lim_{n \rightarrow \infty} u_n = e_q(t)$ is an exact solution of (3.5). Here $e_q(t)$ is one of the q -exponential functions.

4. Conclusions

In the past ten years, the VIM has been one of the often used nonlinear methods. The q -derivative is a deformation of the classical derivative and it has played a crucial role in quantum mechanics and quantum calculus. In this study, the method is successfully extended to q difference equations of second order. A q -analogue of Lagrange multiplier is presented. Readers who feel interested in the initial value problems of the q difference equations are referred to [11–17].

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