Research Article

Strong Convergence Theorems for a Countable Family of Total Quasi- ϕ -Asymptotically Nonexpansive Nonsself Mappings

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The purpose of this paper is to introduce a class of total quasi- ϕ -asymptotically nonexpansive-nonsself mappings and to study the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results announced by some authors recently.

1. Introduction

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed and convex subset of E , E^* is the dual space of E , and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f^* \in E^*, \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in E. \quad (1.1)$$

Recall that a Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex*, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be *smooth*, if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (**) \quad$$

exists for all $x, y \in U$. And E is said to be *uniformly smooth*, if the above limit exists uniformly for $x, y \in U$.

In the sequel, we shall denote the fixed point set of a mapping T by $F(T)$. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x .

A mapping $T : C \rightarrow C$ is said to be *nonexpansive*, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.2)$$

A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C. \quad (1.3)$$

Recall that a subset C of E is said to be *retract* of E , if there exists a continuous mapping $P : E \rightarrow C$ such that $Px = x$, for all $x \in C$.

It is well known that every nonempty closed and convex subset of a uniformly convex Banach space is a retract of E . A mapping $P : E \rightarrow C$ is said to be a *retraction*, if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P . A mapping $P : E \rightarrow C$ is said to be a *nonexpansive retraction*, if it is nonexpansive and it is a retraction from E to C .

In the sequel, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E . Throughout this paper we assume that $\phi : E \times E \rightarrow \mathcal{R}^+$ is the Lyapunov function which is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (1.4)$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E, \quad (1.5)$$

$$\phi\left(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)\right) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \quad \forall x, y \in E. \quad (1.6)$$

Following Alber [1], the generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \quad (1.7)$$

Lemma 1.1 (see [1]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E . Then the following conclusions hold:*

- (1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;
- (2) If $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0$, for all $y \in C$;
- (3) For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Remark 1.2. If E is a real Hilbert space H , then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

A mapping $T : C \rightarrow C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Definition 1.3. Let $P : E \rightarrow C$ be the nonexpansive retraction.

- (1) $T : C \rightarrow E$ is said to be *quasi- ϕ -nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and

$$\phi(u, Tx) \leq \phi(u, x), \quad \forall x \in C, u \in F(T). \quad (1.8)$$

- (2) $T : C \rightarrow E$ is said to be *quasi- ϕ -asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that

$$\phi(u, T(PT)^{n-1}x) \leq k_n \phi(u, x), \quad \forall x \in C, u \in F(T), n \geq 1. \quad (1.9)$$

- (3) $T : C \rightarrow E$ is said to be *total quasi- ϕ -asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ with $\rho(0) = 0$ such that for all $x \in C, u \in F(T)$

$$\phi(u, T(PT)^{n-1}x) \leq \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \geq 1. \quad (1.10)$$

- (4) A countable family of nonself mappings $\{T_i\} : C \rightarrow E$ is said to be *uniformly total quasi- ϕ -asymptotically nonexpansive*, if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ with $\rho(0) = 0$ such that for each $i \geq 1$ and all $x \in C, u \in \bigcap_{i=1}^{\infty} F(T_i)$

$$\phi(u, T_i(PT_i)^{n-1}x) \leq \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \geq 1. \quad (1.11)$$

Remark 1.4. From the definitions, it is easy to know that

- (1) If T is a quasi- ϕ -nonexpansive nonself mapping, then it must be a quasi- ϕ -asymptotically nonexpansive nonself mapping with $\{k_n = 1\}$.
 (2) Taking $\rho(t) = t, t > 0, \nu_n = (k_n - 1)$ and $\mu_n = 0$, then (1.9) can be rewritten as

$$\phi(u, T(PT)^{n-1}x) \leq \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \geq 1, x \in C, u \in F(T). \quad (1.12)$$

This implies that each quasi- ϕ -asymptotically nonexpansive nonself mapping must be a total quasi- ϕ -asymptotically nonexpansive nonself mapping, but the converse is not true.

A nonself mapping $T : C \rightarrow E$ is said to be *uniformly L -Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \forall x, y \in C, \quad n \geq 1. \quad (1.13)$$

Lemma 1.5 (see [2]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ (as $n \rightarrow \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$ (as $n \rightarrow \infty$).*

Lemma 1.6. *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed and convex subset E . Let $T : C \rightarrow E$ be a closed and total quasi- ϕ -asymptotically nonexpansive nonself mapping with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ such that $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ and $\rho(0) = 0$. Then the fixed point set $F(T)$ is a closed and convex subset of C .*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow u$ (as $n \rightarrow \infty$). Since $Tx_n = x_n \rightarrow u$, by the closeness of T , we have $u = Tu$, that is, $u \in F(T)$. This shows that $F(T)$ is a closed set in C .

Next, we prove that $F(T)$ is convex. For any $x, y \in F(T)$, $t \in (0, 1)$, putting $q = tx + (1 - t)y$, we prove that $q \in F(T)$. Indeed, let $\{u_n\}$ be a sequence generated by

$$\begin{aligned} u_1 &= Tq, \quad u_2 = TPTq = TPu_1, \quad u_3 = T(PT)^2q = TPu_2, \dots, \\ u_n &= T(PT)^{n-1}q = TPu_{n-1}, \dots, \end{aligned} \quad (1.14)$$

we have

$$\begin{aligned} \phi(q, u_n) &= \|q\|^2 - 2\langle q, Ju_n \rangle + \|u_n\|^2 \\ &= \|q\|^2 - 2t\langle x, Ju_n \rangle - 2(1-t)\langle y, Ju_n \rangle + \|u_n\|^2 \\ &= \|q\|^2 + t\phi(x, u_n) + (1-t)\phi(y, u_n) - t\|x\|^2 - (1-t)\|y\|^2. \end{aligned} \quad (1.15)$$

Since

$$\begin{aligned} &t\phi(x, u_n) + (1-t)\phi(y, u_n) \\ &\leq t(\phi(x, q) + \nu_n\rho(\phi(x, q)) + \mu_n) + (1-t)(\phi(y, q) + \nu_n\rho(\phi(y, q)) + \mu_n) \\ &= t(\|x\|^2 - 2\langle x, Jq \rangle + \|q\|^2 + \nu_n\rho(\phi(x, q)) + \mu_n) \\ &\quad + (1-t)(\|y\|^2 - 2\langle y, Jq \rangle + \|q\|^2 + \nu_n\rho(\phi(y, q)) + \mu_n) \\ &= t\|x\|^2 + (1-t)\|y\|^2 - \|q\|^2 + t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n. \end{aligned} \quad (1.16)$$

Substituting (1.16) into (1.15), and simplifying we have

$$\phi(q, u_n) \leq t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n \rightarrow 0 \quad (n \rightarrow \infty). \quad (1.17)$$

By Lemma 1.5, we have $u_n \rightarrow q$ ($n \rightarrow \infty$). This implies that $u_{n+1} \rightarrow q$ ($n \rightarrow \infty$).

Since $u_{n+1} = T(PT)^n q = TPT(PT)^{n-1} q = TPu_n$ and T is closed, we have $q = TPq$. Since $q \in C$, $Pq = q$, thus $q = Tq$. this implies that $F(T)$ is a convex set in C . $\square$

Concerning the strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi- ϕ -nonexpansive and quasi- ϕ -asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see e.g., [2–19]).

The purpose of this paper is to modify the Halpern and Mann-type iteration algorithm for a family of of total quasi- ϕ -asymptotically nonexpansive nonself mappings and to have the strong convergence under removing $F(T)$ is a convex set of condition and a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results of Chang et al. [4–7], W. P. Guo and W. Guo [8], Hao et al. [9], Kamimura and Takahashi [10], Kiziltunc and Temir [11], Nilsrakoo and Saejung [2], Pathak et al. [12], Qin et al. [13], Su et al. [14], Thianwan [15], Wang et al. [16], Yıldırım and Özdemir [17], Yang and Xie [18], Zegeye et al. [19], Kanjanasamranwong et al. [20], Saewan and Kumam [21–24] and Wattanawitoon and Kumam [25].

2. Main Results

Theorem 2.1. *Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed convex subset E . Let $T_i : C \rightarrow E, i = 1, 2, \dots$ be a family of closed and uniformly total quasi- ϕ -asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ such that $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ and $\rho(0) = 0$, and for each $i \geq 1, T_i$ be uniformly L_i -Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\beta_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:*

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &\in E \text{ chosen arbitrarily; } C_1 = C, \\ y_{n,i} &= J^{-1} \left[\alpha_n Jx_1 + (1 - \alpha_n) \left(\beta_n Jx_n + (1 - \beta_n) JT_i(PT_i)^{n-1} x_n \right) \right], \quad i \geq 1, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) + \theta_n \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{aligned} \tag{2.1}$$

where $\theta_n = \nu_n \sup_{u \in \mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$, for all $n \geq 1, \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty-bounded subset in C , then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. We divide the proof of Theorem 2.1 into five steps.

- (I) $\mathcal{F}$ and $C_n, n \geq 1$ are closed and convex subset in C .

In fact, it follows from Lemma 1.6 that $F(T_i)$, $i \geq 1$ is closed and convex subset of C . Therefore $\mathcal{F}$ is a closed and convex subset in C .

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have that

$$\begin{aligned}
 C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) + \theta_n \right\} \\
 &= \bigcap_{i \geq 1} \{ z \in C_n : \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) + \theta_n \} \cap C_n \\
 &= \bigcap_{i \geq 1} \left\{ z \in C_n : 2\alpha_n \langle z, Jx_1 \rangle + 2(1 - \alpha_n) \langle z, Jx_n \rangle - 2 \langle z, Jy_{n,i} \rangle \right. \\
 &\quad \left. \leq \alpha_n \|x_1\|^2 + (1 - \alpha_n) \|x_n\|^2 - \|y_{n,i}\|^2 + \theta_n \right\} \cap C_n.
 \end{aligned} \tag{2.2}$$

This implies that C_{n+1} is closed and convex. The conclusion is proved.

(II) Now we prove that $\mathcal{F} \subset C_n$, $n \geq 1$.

In fact, it is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 2$. Letting

$$w_{n,i} = J^{-1} \left(\beta_n Jx_n + (1 - \beta_n) JT_i (PT_i)^{n-1} x_n \right), \tag{2.3}$$

it follows from (1.6) that for any $u \in \mathcal{F} \subset C_n$ we have

$$\begin{aligned}
 \phi(u, y_{n,i}) &= \phi \left(u, J^{-1} (\alpha_n Jx_1 + (1 - \alpha_n) Jw_{n,i}) \right) \\
 &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, w_{n,i}),
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 \phi(u, w_{n,i}) &= \phi \left(u, J^{-1} (\beta_n Jx_n + (1 - \beta_n) JT_i (PT_i)^{n-1} x_n) \right) \\
 &\leq \beta_n \phi(u, x_n) + (1 - \beta_n) \phi(u, T_i (PT_i)^{n-1} x_n) \\
 &\leq \beta_n \phi(u, x_n) + (1 - \beta_n) \{ \phi(u, x_n) + v_n \rho(\phi(u, x_n)) + \mu_n \} \\
 &= \phi(u, x_n) + (1 - \beta_n) (v_n \rho(\phi(u, x_n)) + \mu_n) \\
 &\leq \phi(u, x_n) + v_n \rho(\phi(u, x_n)) + \mu_n,
 \end{aligned} \tag{2.5}$$

therefore we have

$$\begin{aligned}
 \sup_{i \geq 1} \phi(u, y_{n,i}) &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \{ \phi(u, x_n) + v_n \rho(\phi(u, x_n)) + \mu_n \} \\
 &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + v_n \sup_{u \in \mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \\
 &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + \theta_n,
 \end{aligned} \tag{2.6}$$

where $\theta_n = \nu_n \sup_{u \in \mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$. This shows that $u \in C_{n+1}$, and so $\mathcal{F} \subset C_{n+1}$. The conclusion is proved.

(III) Next we prove that $\{x_n\}$ is a Cauchy sequence in C .

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 1.1(2) we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in C_n. \quad (2.7)$$

Again since $\mathcal{F} \subset C_n$, for all $n \geq 1$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in \mathcal{F}. \quad (2.8)$$

It follows from Lemma 1.1(1) that for each $u \in \mathcal{F}$ and for each $n \geq 1$

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \quad (2.9)$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (1.5), $\{x_n\}$ is also bounded.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, for all $n \geq 1$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence the limit $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists. By the construction of C_n , for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$. This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \rightarrow 0, \quad \text{as } n, m \rightarrow \infty. \quad (2.10)$$

It follows from Lemma 1.5 that $\lim_{m, n \rightarrow \infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C . Since C is a nonempty closed subset of Banach space E , it is complete, without loss of generality, we can assume that $x_n \rightarrow x^* (n \rightarrow \infty)$.

By the assumption, it is easy to see that

$$\lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \left(\nu_n \sup_{u \in \mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \right) = 0. \quad (2.11)$$

(IV) Now we prove that $x^* \in \mathcal{F}$.

In fact, since $x_{n+1} \in C_{n+1}$ and $\alpha_n \rightarrow 0$, it follows from (2.1) and (2.11) that

$$\sup_{i \geq 1} \phi(x_{n+1}, y_{n,i}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \theta_n \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (2.12)$$

Since $x_n \rightarrow x^*$, by virtue of Lemma 1.5 for each $i \geq 1$, we have

$$\lim_{n \rightarrow \infty} y_{n,i} = x^*. \quad (2.13)$$

Since $\{x_n\}$ is bounded, $\{T_i\}_{i=1}^\infty$ is uniformly total quasi- ϕ -asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous

function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ such that $\nu_n \rightarrow 0$, $\mu_n \rightarrow 0$, and $\rho(0) = 0$, for any given $u \in \mathcal{F}$, we have

$$\phi\left(u, T_i(PT_i)^{n-1}x_n\right) \leq \phi(u, x_n) + \nu_n\rho(\phi(u, x_n)) + \mu_n. \quad (2.14)$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded. Since

$$\begin{aligned} \|w_{n,i}\| &= \left\| J^{-1}\left(\beta_n Jx_n + (1 - \beta_n)JT_i(PT_i)^{n-1}x_n\right) \right\| \\ &\leq \beta_n\|x_n\| + (1 - \beta_n)\left\| T_i\|PT_i\|^{n-1}x_n \right\| \\ &\leq \|x_n\| + \left\| T_i(PT_i)^{n-1}x_n \right\|. \end{aligned} \quad (2.15)$$

This implies that $\{w_{n,i}\}$ is also uniformly bounded.

Since $\alpha_n \rightarrow 0$, from (2.1), for each $i \geq 1$ we have

$$\lim_{n \rightarrow \infty} \|Jy_{n,i} - Jw_{n,i}\| = \lim_{n \rightarrow \infty} \alpha_n \|Jx_1 - Jw_{n,i}\| = 0. \quad (2.16)$$

Since J^{-1} is uniformly continuous on each bounded subset of E^* , it follows from (2.13) and (2.16) that

$$\lim_{n \rightarrow \infty} w_{n,i} = x^* \quad \text{for each } i \geq 1. \quad (2.17)$$

Since J is uniformly continuous on each bounded subset of E , we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|Jw_{n,i} - Jx^*\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta_n Jx_n + (1 - \beta_n)JT_i(PT_i)^{n-1}x_n - Jx^* \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \beta_n (Jx_n - Jx^*) + (1 - \beta_n) \left(JT_i(PT_i)^{n-1}x_n - Jx^* \right) \right\| \\ &= \lim_{n \rightarrow \infty} (1 - \beta_n) \left\| JT_i(PT_i)^{n-1}x_n - Jx^* \right\|. \end{aligned} \quad (2.18)$$

By condition (b), we have that

$$\lim_{n \rightarrow \infty} \left\| JT_i(PT_i)^{n-1}x_n - Jx^* \right\| = 0. \quad (2.19)$$

Since J is uniformly continuous, this shows that $\lim_{n \rightarrow \infty} T_i(PT_i)^{n-1}x_n = x^*$ uniformly in $i \geq 1$.

Again by the assumptions that for each $i \geq 1$, T_i is uniformly L_i -Lipschitz continuous, thus we have

$$\begin{aligned}
& \left\| T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n \right\| \\
& \leq \left\| T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1} \right\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| \\
& \quad + \left\| x_{n+1} - x_n \right\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\| \\
& \leq (L_i + 1) \left\| x_n - x_{n+1} \right\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\|.
\end{aligned} \tag{2.20}$$

Since $\lim_{n \rightarrow \infty} T_i(PT_i)^{n-1} x_n = x^*$ and $x_n \rightarrow x^*$, these together with (2.20) imply that $\lim_{n \rightarrow \infty} \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n\| = 0$ and $\lim_{n \rightarrow \infty} T_i(PT_i)^n x_n = x^*$, that is,

$$\lim_{n \rightarrow \infty} T_i P(PT_i)^{n-1} x_n = x^*. \tag{2.21}$$

In view continuity of $T_i P$, it yields that $T_i P x^* = x^*$. Since $x^* \in C$, $P x^* = x^*$. This shows that $T x^* = x^*$. By the arbitrariness of $i \geq 1$, we have $x^* \in \mathcal{F}$.

(V) Finally we prove that $x_n \rightarrow x^* = \Pi_{\mathcal{F}} x_1$.

Let $w = \Pi_{\mathcal{F}} x_1$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, for all $n \geq 1$. This implies that

$$\phi(x^*, x_1) = \lim_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{2.22}$$

In view of the definition of $\Pi_{\mathcal{F}} x_1$, from (2.22) we have $x^* = w$. Therefore $x_n \rightarrow x^* = \Pi_{\mathcal{F}} x_1$. This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. Let $E, C, \{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.1. Let $T_i : C \rightarrow E, i = 1, 2, \dots$ be a family of closed and uniformly ϕ -asymptotically nonexpansive nonself mappings with sequence $\{k_n\} \subset [1, \infty), k_n \rightarrow 1$, and for each $i \geq 1$, T_i be uniformly L_i -Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned}
& x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
& y_{n,i} = J^{-1} \left[\alpha_n J x_1 + (1 - \alpha_n) \left(\beta_n J x_n + (1 - \beta_n) J T_i(PT_i)^{n-1} x_n \right) \right], \quad i \geq 1, \\
& C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) + \theta_n \right\}, \\
& x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned} \tag{2.23}$$

where $\theta_n = (k_n - 1) \sup_{u \in \mathcal{F}} \phi(u, x_n)$, $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty bounded subset in C , then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. By Remark 1.4 $T_i : C \rightarrow E, i = 1, 2, \dots$ be a family of closed and uniformly ϕ -asymptotically nonexpansive nonself mappings that it is a family of closed and uniformly

total quasi- ϕ -asymptotically nonexpansive nonself mappings with taking $\rho(t) = t$, $t > 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$. Therefore all conditions in Theorem 2.1 are satisfied. By the similar methods as given in the proof of Theorem 2.1, we can prove that the sequence $\{x_n\}$ defined by (2.23) converges strongly to $\Pi_{\mathcal{F}}x_1$. $\square$

Theorem 2.3. *Let $E, C, \{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.2. Let $T_i : C \rightarrow E$, $i = 1, 2, \dots$ be a family of quasi- ϕ -nonexpansive nonself mappings such that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and for each $i \geq 1$, T_i be uniformly L_i -Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &\in E \text{ chosen arbitrarily; } C_1 = C, \\ y_{n,i} &= J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)JT_i x_n)], \quad i \geq 1, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{aligned} \tag{2.24}$$

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}x_1$.

Proof. By Remark 1.4 $T_i : C \rightarrow E$, $i = 1, 2, \dots$ be a family of quasi- ϕ -nonexpansive nonself mappings that it is a family of uniformly quasi- ϕ -asymptotically nonexpansive nonself mappings with sequence $\{k_n\} = \{1\}$. Hence $\theta_n = (k_n - 1) \sup_{u \in \mathcal{F}} \phi(u, x_n) = 0$. Therefore all conditions in Theorem 2.2 are satisfied. By the similar methods, we can prove that the sequence $\{x_n\}$ defined by (2.24) converges strongly to $\Pi_{\mathcal{F}}x_1$. $\square$

3. Application and Example

In this section we utilize the results presented in Section 2 to prove a strong convergence theorem concerning maximal monotone operators in Hilbert spaces.

Let E be a real Hilbert space and let A be a maximal monotone operator from E to E . For each $r > 0$, we can define a single valued mapping $J_r^A : E \rightarrow E$ by $J_r^A = (I + rA)^{-1}$ and such a mapping J_r^A is called the *resolvent* of A . It is easy to prove that J_r^A is a nonexpansive mapping and $A^{-1}(0) = F(J_r^A)$ for all $r > 0$. Therefore it is a uniformly 1-Lipschitz continuous and quasi- ϕ -nonexpansive mapping. Hence for each $p \in F(J_r^A)$ and $w \in E$, we have

$$\phi(p, J_r^A w) \leq \phi(p, w), \tag{3.1}$$

and $F(J_r^A) = A^{-1}(0)$. These show that all conditions in Theorem 2.3 are satisfied. Hence from Theorem 2.3 we have the following.

Theorem 3.1. Let E be a real Hilbert space. Let A_1, A_2 be two maximal monotone operators from E to E such that $\mathcal{F} = A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. Let $J_r^{A_1}$ and $J_r^{A_2}$ be the resolvent of A_1 and A_2 , respectively, where $r > 0$. Let $\{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.3 and $\{x_n\}$ be the sequence defined by

$$\begin{aligned} x_1 &\in E \text{ chosen arbitrarily; } C_1 = E, \\ y_{n,i} &= J^{-1} \left[\alpha_n J x_1 + (1 - \alpha_n) \left(\beta_n J x_n + (1 - \beta_n) J J_r^{A_i} x_n \right) \right], \quad i = 1, 2, \\ C_{n+1} &= \left\{ z \in C_n : \max_{i=1,2} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n) \right\}, \\ x_{n+1} &= P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{aligned} \tag{3.2}$$

where P_C is the metric projection from H onto the subset $C \subset H$. Then the sequence $\{x_n\}$ defined by (3.2) converges strongly to $P_{\mathcal{F}} x_1$.

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