Research Article

New Sharp Bounds for the Bernoulli Numbers and Refinement of Becker-Stark Inequalities

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We obtain new sharp bounds for the Bernoulli numbers: $2(2n)!/(\pi^{2n}(2^{2n}-1)) < |B_{2n}| \leq (2(2^{2k}-1)/2^{2k})\zeta(2k)(2n)!/(\pi^{2n}(2^{2n}-1))$, $n = k, k+1, \dots$, $k \in \mathbb{N}^+$, and establish sharpening of Papenfuss's inequalities, the refinements of Becker-Stark, and Steckin's inequalities. Finally, we show a new simple proof of Ruehr-Shafer inequality.

1. Introduction

The classical Bernoulli numbers B_n ($n = 1, 2, \dots$) can be defined by (see [1])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \quad (1.1)$$

Reference [2] shows a upper bound for $|B_{2n}| = (-1)^{n+1} B_{2n}$

$$|B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{1-2n}}, \quad n = 1, 2, \dots \quad (1.2)$$

On the other hand, [3] presents a lower bound for $|B_{2n}|$ as follows:

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}}, \quad n = 1, 2, \dots \quad (1.3)$$

On the basis of (1.2) and (1.3), Alzer [4] obtains the further results.

Theorem A. For all integers $n \geq 1$ one has

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\alpha-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\beta-2n}}, \quad (1.4)$$

with the best possible constants $\alpha = 0$ and $\beta = 2 + \ln(1 - 6/\pi^2)/\ln 2 \approx 0.6491 \dots$.

In this paper, we obtain new bounds for the Bernoulli numbers as follows.

Theorem 1.1. Let $k \in \mathbb{N}^+$, $n = k, k+1, \dots$, then

$$\frac{2(2n)!}{\pi^{2n}(2^{2n}-1)} < |B_{2n}| \leq \frac{2(2^{2k}-1)}{2^{2k}} \zeta(2k) \frac{(2n)!}{\pi^{2n}(2^{2n}-1)}. \quad (1.5)$$

The equality holds in (1.5) if and only if $n = k$. Furthermore, 2 and $(2(2^{2k}-1)/2^{2k})\zeta(2k)$ are the best constants in (1.5).

In the following, we study on some trigonometric inequalities.

Mitrinovic [5] gives us a result which belongs to StecKin.

Theorem B. If $0 < x < \pi/2$, then

$$\frac{4}{\pi} \frac{x}{\pi - 2x} < \tan x. \quad (1.6)$$

Now, we show a upper bound for $\tan x$ and obtain the following sharp StecKin's inequalities.

Theorem 1.2. If $0 < x < \pi/2$, then

$$\frac{4}{\pi} \frac{x}{\pi - 2x} < \tan x < \pi \frac{x}{\pi - 2x} \quad (1.7)$$

or

$$\frac{4}{\pi} \frac{1}{\pi - 2x} < \frac{\tan x}{x} < \pi \frac{1}{\pi - 2x}. \quad (1.8)$$

Furthermore, $4/\pi$ and π are the best constants in (1.7) and (1.8).

Kuang [6] gives us the further results described as Becker-Stark inequalities

Theorem C. Let $0 < t < 1$, then

$$\frac{4}{\pi} \frac{t}{1-t^2} < \tan \frac{\pi}{2} t < \frac{\pi}{2} \frac{t}{1-t^2}. \quad (1.9)$$

Furthermore, $4/\pi$ and $\pi/2$ are the best constants in (1.9).

Let $x = (\pi/2)t$ in (1.9), then Theorem C is equivalent to.

Theorem D. Let $0 < x < \pi/2$, then

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \quad (1.10)$$

Furthermore, 8 and π^2 are the best constants in (1.10).

Clearly, Becker-Stark inequalities (1.10) are the generalization of the strengthened Steckin's inequalities (1.8).

On the other hand, Papenfuss [7] proposes an open problem described as the following statement.

Theorem E. Let $0 \leq x < \pi/2$, then

$$x \sec^2 x - \tan x \leq \frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2}. \quad (1.11)$$

Bach [8] prove Theorem E and obtain a further result.

Theorem F. Let $0 \leq x < \pi/2$, then

$$x \sec^2 x - \tan x \leq \frac{2\pi^4}{3} \frac{x^3}{(\pi^2 - 4x^2)^2}. \quad (1.12)$$

In this section, we first obtain sharp Papenfuss-Bach inequalities described as Theorem 1.3.

Theorem 1.3. Let $0 < x < \pi/2$, then

$$\frac{64x^3}{(\pi^2 - 4x^2)^2} < x \sec^2 x - \tan x < \frac{2\pi^4}{3} \frac{x^3}{(\pi^2 - 4x^2)^2}. \quad (1.13)$$

Furthermore, 64 and $2\pi^4/3$ are the best constants in (1.13).

The inequalities (1.13) are equivalent to

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \frac{x \sec^2 x - \tan x}{x^2} < \frac{2\pi^4}{3} \frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \quad (1.14)$$

That is,

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \left(\frac{\tan x}{x}\right)' < \frac{2\pi^4}{3} \frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \quad (1.15)$$

Then, integrating the three functions in (1.15) from 0 to x , where $x \in (0, \pi/2)$, we obtain the following refinement of Becker-Stark inequalities.

Theorem 1.4 (Refinement of Becker-Stark Inequalities). *Let $0 < x < \pi/2$, then*

$$\frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^4}{12} \frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right). \quad (1.16)$$

An application of Theorem 1.4 leads to Theorem 1.5 (the refinement of Steckin's inequalities).

Theorem 1.5 (Refinement of Steckin's Inequalities). *If $0 < x < \pi/2$, then*

$$\frac{4}{\pi} \frac{1}{\pi - 2x} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^3}{12} \frac{1}{\pi - 2x} + \left(1 - \frac{\pi^2}{12}\right). \quad (1.17)$$

Finally, we will show a new proof of Ruehr-Shafer inequality.

Theorem G (Ruehr-Shafer Inequality, see [8]). *Let $0 \leq x < \pi/2$, then*

$$x \sec^2 x - \tan x \leq 2\pi^2 \frac{\tan x - x}{\pi^2 - 4x^2}. \quad (1.18)$$

2. Two Lemmas

Lemma 2.1 (see [9, Lemma 2.1]). *The function $(1 - (1/2^n))\zeta(n)$ ($n = 1, 2, \dots$) is decreasing, where $\zeta(n)$ is the Riemann's zeta function.*

Lemma 2.2 (see [10]). *Let l_n and m_n ($n = 1, 2, \dots$) be real numbers, and let the power series $L(x) = \sum_{n=1}^{\infty} l_n x^n$ and $M(x) = \sum_{n=1}^{\infty} m_n x^n$ be convergent for $|x| < R$. If $m_n > 0$ for $n = 1, 2, \dots$ and if l_n/m_n is strictly decreasing for $n = 1, 2, \dots$, then the function $L(x)/M(x)$ is strictly decreasing on $(0, R)$.*

3. Proof of Theorem 1.1

Using the representation

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \quad n = 1, 2, \dots, \quad (3.1)$$

(cf. [11, page 266]), we have

$$G(n) = \frac{|B_{2n}| \pi^{2n} (2^{2n} - 1)}{(2n)!} = 2 \left(1 - \frac{1}{2^{2n}}\right) \zeta(2n), \quad n = k, k+1, \dots; \quad k \in \mathbf{N}^+. \quad (3.2)$$

From Lemma 2.1, we know that $G(n)$ is decreasing and $G(k) = (2(2^{2k} - 1)/2^{2k})\zeta(2k)$, $G(+\infty) = 2\lim_{n \rightarrow \infty} \zeta(2n) = 2$. Then, the proof of Theorem 1.1 is complete.

4. Proofs of Theorem 1.3 and G

4.1. Proof of Theorem 1.3.

The following power series expansion can be found in [12]:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (-1)^{n-1} B_{2n} x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}. \quad (4.1)$$

Then

$$\begin{aligned} \sec^2 x &= (\tan x)' = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-1) |B_{2n}| x^{2n-2}, \quad |x| < \frac{\pi}{2}, \\ x \sec^2 x - \tan x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-2) |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}. \end{aligned} \quad (4.2)$$

Let

$$A(x) = \frac{x \sec^2 x - \tan x}{x^3 / (\pi^2 - 4x^2)^2} = \frac{L(x)}{M(x)}, \quad (4.3)$$

where

$$\begin{aligned} L(x) &= x \sec^2 x - \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-2) |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}, \\ M(x) &= \frac{x^3}{(\pi^2 - 4x^2)^2} = \sum_{n=2}^{\infty} \frac{2n-2}{32} \left(\frac{2}{\pi} \right)^{2n} x^{2n-1}, \quad |x| < \frac{\pi}{2}. \end{aligned} \quad (4.4)$$

Then, $l_n = ((2^{2n}(2^{2n}-1)/(2n)!) (2n-2) |B_{2n}|$, $m_n = ((2n-2)/32) (2/\pi)^{2n} > 0$ ($n \geq 2$) and $l_n/m_n = 64(1 - 1/2^{2n})\zeta(2n)$. So, l_n/m_n is decreasing by Lemma 2.1. Therefore, $A(x) = L(x)/M(x)$ is decreasing on $(0, \pi/2)$ by Lemma 2.2. At the same time, $\lim_{x \rightarrow 0^+} A(x) = 2\pi^4/3$ and $\lim_{x \rightarrow (\pi/2)^-} A(x) = 64$, so 64 and $2\pi^4/3$ are the best constants in (1.13).

4.2. Proof of Theorem G.

By (4.1) and (4.2), we have

$$\begin{aligned} (x \sec^2 x - \tan x)(\pi^2 - 4x^2) &= 2\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (n-1) |B_{2n}| x^{2n-1} \\ &\quad - 4 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-2) |B_{2n}| x^{2n+1}, \\ 2\pi^2(\tan x - x) &= 2\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n-1}, \end{aligned} \quad (4.5)$$

so, (1.18) is equivalent to

$$2\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (n-2)|B_{2n}|x^{2n-1} \leq 4 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-2)|B_{2n}|x^{2n+1}, \quad (4.6)$$

that is,

$$2\pi^2 \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (n-2)|B_{2n}|x^{2n-1} \leq 4 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (2n-2)|B_{2n}|x^{2n+1}, \quad (4.7)$$

or

$$\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!} (n-1)|B_{2n+2}|x^{2n+1} \leq 4 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (n-1)|B_{2n}|x^{2n+1}. \quad (4.8)$$

From Lemma 2.1, we have that $(1 - 1/2^{2n})\zeta(2n)$ is decreasing or

$$\frac{2^{2n+2}-1}{4}\zeta(2n+2) < (2^{2n}-1)\zeta(2n) \quad (4.9)$$

holds. By (3.1), we get

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}|B_{2n}|, \quad (4.10)$$

so, (4.8) holds.

5. Remark

In 2010, Zhu and Hua [9] proved for $x \in (0, \pi/2)$ that

$$\frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + (\pi^2/3 - 4)x^2}{\pi^2 - 4x^2}. \quad (5.1)$$

Now, we can compare the results of (5.1) with (1.16). In fact, we can easily check that

$$\begin{aligned} \frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) &= \frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2 - 4x^2}, \\ \frac{\pi^4}{12} \frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right) &= \frac{\pi^2 + (\pi^2/3 - 4)x^2}{\pi^2 - 4x^2}. \end{aligned} \quad (5.2)$$

So, (5.1) is equivalent to (1.16).

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