Research Article

Projection Algorithms for Variational Inclusions

Youli Yu,¹ Pei-Xia Yang,² and Khalida Inayat Noor³

¹ School of Mathematics and Information Engineering, Taizhou University, Linhai 317000, China

² Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China

³ Mathematics Department, COMSATS Institute of Information Technology, Islamabad 44000, Pakistan

Correspondence should be addressed to Youli Yu, yuyouli@tzc.edu.cn

Received 7 October 2011; Accepted 22 October 2011

Academic Editor: Yonghong Yao

Copyright © 2012 Youli Yu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present a projection algorithm for finding a solution of a variational inclusion problem in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm converges strongly to a solution of the variational inclusion problem which also solves some variational inequality.

1. Introduction

Let H be a real Hilbert space. Let $B : H \rightarrow H$ be a single-valued nonlinear mapping and $R : H \rightarrow 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \quad (1.1)$$

where θ is the zero vector in H . The set of solutions of problem (1.1) is denoted by $I(B, R)$. If $H = \mathbb{R}^m$, then problem (1.1) becomes the generalized equation introduced by Robinson [1]. If $B = 0$, then problem (1.1) becomes the inclusion problem introduced by Rockafellar [2]. It is known that (1.1) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so forth. Also various types of variational inclusions problems have been extended and generalized. Recently, Zhang et al. [3] introduced a new iterative scheme for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [4] introduced another iterative scheme

by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping. For some related works, see [5–28] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we present a projection algorithm for finding a solution of a variational inclusion problem in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm converges strongly to a solution of the variational inclusion problem which also solves some variational inequality.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H . Recall that a mapping $B : C \rightarrow C$ is said to be α -inverse strongly monotone if there exists a constant $\alpha > 0$ such that $\langle Bx - By, x - y \rangle \geq \alpha \|Bx - By\|^2$, for all $x, y \in C$. A mapping A is strongly positive on H if there exists a constant $\mu > 0$ such that $\langle Ax, x \rangle \geq \mu \|x\|^2$ for all $x \in H$.

For any $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C. \quad (2.1)$$

Such a P_C is called the metric projection of H onto C . We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \iff \langle x - x^*, x^* - y \rangle \geq 0 \quad \text{for all } y \in C. \quad (2.2)$$

A set-valued mapping $T : H \rightarrow 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \rightarrow 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let the set-valued mapping $R : H \rightarrow 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}(x), \quad x \in H, \quad (2.3)$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single-valued, nonexpansive, and 1-inverse strongly monotone, and that a solution of problem (1.1) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see for instance [29].

Lemma 2.1 (see [30]). *Let $R : H \rightarrow 2^H$ be a maximal monotone mapping and $B : H \rightarrow H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \rightarrow 2^H$ is maximal monotone.*

Lemma 2.2 (see [8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.*

Lemma 2.3 (see [31]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Result

In this section, we will prove our main result. First, we give some assumptions on the operators and the parameters. Subsequently, we introduce our iterative algorithm for finding solutions of the variational inclusion (1.1). Finally, we will show that the proposed algorithm has strong convergence.

In the sequel, we will assume that

- (A1) C is a nonempty closed convex subset of a real Hilbert space H ;
- (A2) A is a strongly positive bounded linear operator with coefficient $0 < \mu < 1$, $R : H \rightarrow 2^H$ is a maximal monotone mapping and $B : C \rightarrow C$ is an α -inverse strongly monotone mapping;
- (A3) $\lambda > 0$ is a constant satisfying $\lambda < 2\alpha$.

Now we introduce the following iteration algorithm.

Algorithm 3.1. For given $x_0 \in C$ arbitrarily, compute the sequence $\{x_n\}$ as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)x_n], \quad n \geq 0, \quad (3.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$.

Now we study the strong convergence of the algorithm (3.1)

Theorem 3.2. Suppose that $I(B, R) \neq \emptyset$. Assume the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $\tilde{x} \in I(B, R)$ which solves the following variational inequality:

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in I(B, R). \quad (3.2)$$

Proof. Take $x^* \in I(B, R)$. It is clear that

$$J_{R,\lambda}(x^* - \lambda Bx^*) = x^*. \quad (3.3)$$

We divide our proofs into the following five steps:

- (1) the sequence $\{x_n\}$ is bounded.
- (2) $\|x_{n+1} - x_n\| \rightarrow 0$.
- (3) $\|Bx_n - Bx^*\| \rightarrow 0$.
- (4) $\limsup_{n \rightarrow \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle \geq 0$ where $\tilde{x} = P_{I(B,R)}(I - A)(\tilde{x})$.
- (5) $x_n \rightarrow \tilde{x}$.

□

Proof of (1.1). Since B is α -inverse strongly monotone, we have

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Bx - By\|^2. \quad (3.4)$$

It is clear that if $0 \leq \lambda \leq 2\alpha$, then $(I - \lambda B)$ is nonexpansive. Set $y_n = J_{R,\lambda}(x_n - \lambda Bx_n)$, $n \geq 0$. It follows that

$$\begin{aligned} \|y_n - x^*\| &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\| \\ &\leq \|(x_n - \lambda Bx_n) - (x^* - \lambda Bx^*)\| \\ &\leq \|x_n - x^*\|. \end{aligned} \quad (3.5)$$

Since A is linear bounded self-adjoint operator on H , then

$$\|A\| = \sup\{|\langle Au, u \rangle| : u \in H, \|u\| = 1\}. \quad (3.6)$$

Observe that

$$\begin{aligned} \langle (I - \alpha_n A)u, u \rangle &= 1 - \alpha_n \langle Au, u \rangle \\ &\geq 1 - \alpha_n \|A\| \\ &\geq 0, \end{aligned} \quad (3.7)$$

that is to say $I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(I - \alpha_n A)\| &= \sup\{\langle (I - \alpha_n A)u, u \rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n \langle Au, u \rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - \alpha_n \mu. \end{aligned} \quad (3.8)$$

From (3.1), we deduce that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|(1 - \beta_n)x_n + \beta_n P_C[(I - \alpha_n A)y_n] - x^*\| \\
 &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \|[(I - \alpha_n A)(y_n - x^*)] - \alpha_n A x^*\| \\
 &\leq (1 - \beta_n)\|x_n - x^*\| + (1 - \alpha_n \mu)\beta_n \|y_n - x^*\| + \alpha_n \beta_n \|A x^*\| \\
 &\leq (1 - \alpha_n \beta_n \mu)\|x_n - x^*\| + \alpha_n \beta_n \|A x^*\| \\
 &\leq \max \left\{ \|x_0 - x^*\|, \frac{\|A x^*\|}{\mu} \right\}.
 \end{aligned} \tag{3.9}$$

Therefore, $\{x_n\}$ is bounded. □

Proof of (3.1). Set $z_n = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)x_n]$ for all $n \geq 0$. Then, we have

$$\begin{aligned}
 \|z_n - z_{n-1}\| &= \|P_C[(I - \alpha_n A)y_n] - P_C[(I - \alpha_{n-1} A)y_{n-1}]\| \\
 &\leq \|[(I - \alpha_n A)y_n] - [(I - \alpha_{n-1} A)y_{n-1}]\| \\
 &= \|(I - \alpha_n A)(y_n - y_{n-1}) + (\alpha_{n-1} - \alpha_n)A y_{n-1}\| \\
 &\leq \|(I - \alpha_n A)(y_n - y_{n-1})\| + \|(\alpha_{n-1} - \alpha_n)A y_{n-1}\| \\
 &\leq (1 - \alpha_n \mu)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|A y_{n-1}\|.
 \end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned}
 \|y_n - y_{n-1}\| &= \|J_{R,\lambda}(x_n - \lambda B x_n) - J_{R,\lambda}(x_{n-1} - \lambda B x_{n-1})\| \\
 &\leq \|(x_n - \lambda B x_n) - (x_{n-1} - \lambda B x_{n-1})\| \\
 &\leq \|x_n - x_{n-1}\|.
 \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we get

$$\|z_n - z_{n-1}\| \leq (1 - \alpha_n \mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|A y_{n-1}\|. \tag{3.12}$$

Therefore,

$$\limsup_{n \rightarrow \infty} (\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0. \tag{3.13}$$

This together with Lemma 2.2 imply that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{3.14}$$

Hence,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|z_n - x_n\| = 0. \quad (3.15)$$

□

Proof of (3.4). From (3.4), we get

$$\begin{aligned} \|y_n - x^*\|^2 &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\ &\leq \|(x_n - \lambda Bx_n) - (x^* - \lambda Bx^*)\|^2 \\ &\leq \|x_n - x^*\|^2 + \lambda(\lambda - 2\alpha)\|Bx_n - Bx^*\|^2. \end{aligned} \quad (3.16)$$

By (3.1), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n \|P_C[(I - \alpha_n A)y_n] - x^*\|^2 \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n \|(I - \alpha_n A)y_n - x^*\|^2 \\ &= (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n (\|y_n - x^* - \alpha_n Ay_n\|^2) \\ &= (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n (\|y_n - x^*\|^2 - 2\alpha_n \langle y_n - x^*, Ay_n \rangle + \alpha_n^2 \|Ay_n\|^2) \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n (\|y_n - x^*\|^2 + \alpha_n (2\|y_n - x^*\| \|Ay_n\| + \|Ay_n\|^2)) \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n (\|y_n - x^*\|^2 + \alpha_n M), \end{aligned} \quad (3.17)$$

where $M > 0$ is some constant satisfying $\sup_n \{2\|y_n - x^*\| \|Ay_n\| + \|Ay_n\|^2\} \leq M$. From (3.16) and (3.17), we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \lambda(\lambda - 2\alpha)\beta_n \|Bx_n - Bx^*\|^2 + \alpha_n M. \quad (3.18)$$

Thus,

$$\begin{aligned} \lambda(2\alpha - \lambda)\beta_n \|Bx_n - Bx^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M \\ &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + \alpha_n M, \end{aligned} \quad (3.19)$$

which imply that

$$\lim_{n \rightarrow \infty} \|Bx_n - Bx^*\| = 0. \quad (3.20)$$

□

Proof of (3.10). Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|J_{R,\lambda}(x_n - \lambda Bx_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\leq \langle x_n - \lambda Bx_n - (x^* - \lambda Bx^*), y_n - x^* \rangle \\
&= \frac{1}{2} \left(\|x_n - \lambda Bx_n - (x^* - \lambda Bx^*)\|^2 + \|y_n - x^*\|^2 \right. \\
&\quad \left. - \|x_n - \lambda Bx_n - (x^* - \lambda Bx^*) - (y_n - x^*)\|^2 \right) \\
&\leq \frac{1}{2} \left(\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - \|x_n - y_n - \lambda(Bx_n - Bx^*)\|^2 \right) \\
&= \frac{1}{2} \left(\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 \right. \\
&\quad \left. + 2\lambda \langle Bx_n - Bx^*, x_n - y_n \rangle - \lambda^2 \|Bx_n - Bx^*\|^2 \right),
\end{aligned} \tag{3.21}$$

which implies that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\lambda \|Bx_n - Bx^*\| \|x_n - y_n\|. \tag{3.22}$$

Substitute (3.22) into (3.17) to get

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \beta_n \|x_n - y_n\|^2 + 2\lambda \|Bx_n - Bx^*\| \|x_n - y_n\| + \alpha_n M. \tag{3.23}$$

Then we derive

$$\begin{aligned}
\beta_n \|x_n - y_n\|^2 &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\
&\quad + 2\lambda \|Bx_n - Bx^*\| \|x_n - y_n\| + \alpha_n M.
\end{aligned} \tag{3.24}$$

So, we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{3.25}$$

We note that $P_{I(B,R)}(I - A)$ is a contraction. As a matter of fact,

$$\begin{aligned}
\|P_{I(B,R)}(I - A)x - P_{I(B,R)}(I - A)y\| &\leq \|(I - A)x - (I - A)y\| \\
&\leq \|I - A\| \|x - y\| \\
&\leq (1 - \mu) \|x - y\|,
\end{aligned} \tag{3.26}$$

for all $x, y \in H$. Hence $P_{I(B,R)}(I - A)$ has a unique fixed point, say $\tilde{x} \in I(B, R)$. That is $\tilde{x} = P_{I(B,R)}(I - A)(\tilde{x})$. This implies that $\langle A\tilde{x}, y - \tilde{x} \rangle \geq 0$ for all $y \in I(B, R)$. Next, we prove that

$$\limsup_{n \rightarrow \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle \geq 0. \quad (3.27)$$

First, we note that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle = \lim_{j \rightarrow \infty} \langle A\tilde{x}, x_{n_j} - \tilde{x} \rangle. \quad (3.28)$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to w . Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$.

Next, we show that $w \in I(B, R)$. In fact, since B is α -inverse strongly monotone, B is Lipschitz-continuous monotone mapping. It follows from Lemma 2.1 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$, that is, $g - Bv \in R(v)$. Again since $y_{n_i} = J_{R,\lambda}(x_{n_i} - \lambda Bx_{n_i})$, we have $x_{n_i} - \lambda Bx_{n_i} \in (I + \lambda R)(y_{n_i})$, that is, $(1/\lambda)(x_{n_i} - y_{n_i} - \lambda Bx_{n_i}) \in R(y_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda Bx_{n_i}) \right\rangle \geq 0, \quad (3.29)$$

and so

$$\begin{aligned} \langle v - y_{n_i}, g \rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(x_{n_i} - y_{n_i} - \lambda Bx_{n_i}) \right\rangle \\ &= \left\langle v - y_{n_i}, Bv - By_{n_i} + By_{n_i} - Bx_{n_i} + \frac{1}{\lambda}(x_{n_i} - y_{n_i}) \right\rangle \\ &\geq \langle v - y_{n_i}, By_{n_i} - Bx_{n_i} \rangle + \left\langle v - y_{n_i}, \frac{1}{\lambda}(x_{n_i} - y_{n_i}) \right\rangle. \end{aligned} \quad (3.30)$$

It follows from $\|x_n - y_n\| \rightarrow 0$, $\|Bx_n - By_n\| \rightarrow 0$ and $y_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \rightarrow \infty} \langle v - y_{n_i}, g \rangle = \langle v - w, g \rangle \geq 0. \quad (3.31)$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is, $w \in I(B, R)$. Therefore, $w \in I(B, R)$. It follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle A\tilde{x}, x_n - \tilde{x} \rangle &= \lim_{j \rightarrow \infty} \langle A\tilde{x}, x_{n_j} - \tilde{x} \rangle \\ &= \langle A\tilde{x}, w - \tilde{x} \rangle \\ &\geq 0. \end{aligned} \quad (3.32)$$

□

Proof of (3.11). First, we note that $z_n = P_C[(I - \alpha_n A)y_n]$, then for all $x \in C$, we have $\langle z_n - (I - \alpha_n A)y_n, z_n - x \rangle \leq 0$. Thus,

$$\begin{aligned}
 \|z_n - \tilde{x}\|^2 &= \langle z_n - \tilde{x}, z_n - \tilde{x} \rangle \\
 &= \langle z_n - (I - \alpha_n A)y_n + (I - \alpha_n A)y_n - \tilde{x}, z_n - \tilde{x} \rangle \\
 &= \langle z_n - (I - \alpha_n A)y_n, z_n - \tilde{x} \rangle + \langle (I - \alpha_n A)y_n - \tilde{x}, z_n - \tilde{x} \rangle \\
 &\leq \langle (I - \alpha_n A)(y_n - \tilde{x}) - \alpha_n A\tilde{x}, z_n - \tilde{x} \rangle \\
 &= \langle (I - \alpha_n A)(y_n - \tilde{x}), z_n - \tilde{x} \rangle + \alpha_n \langle -A\tilde{x}, z_n - \tilde{x} \rangle \\
 &\leq \|(I - \alpha_n A)(y_n - \tilde{x})\| \|z_n - \tilde{x}\| + \alpha_n \langle -A\tilde{x}, z_n - \tilde{x} \rangle \\
 &\leq \frac{(1 - \alpha_n \mu)}{2} (\|x_n - \tilde{x}\|^2 + \|z_n - \tilde{x}\|^2) + \alpha_n \langle -A\tilde{x}, z_n - \tilde{x} \rangle,
 \end{aligned} \tag{3.33}$$

that is,

$$\|z_n - \tilde{x}\|^2 \leq (1 - \alpha_n \mu) \|x_n - \tilde{x}\|^2 + \frac{\alpha_n}{1 + \alpha_n \mu} \langle -A\tilde{x}, z_n - \tilde{x} \rangle. \tag{3.34}$$

So,

$$\begin{aligned}
 \|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|z_n - x^*\|^2 \\
 &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n (1 - \alpha_n \mu) \|x_n - \tilde{x}\|^2 + \frac{\alpha_n \beta_n}{1 + \alpha_n \mu} \langle -A\tilde{x}, z_n - \tilde{x} \rangle \\
 &= (1 - \alpha_n \beta_n \mu) \|x_n - x^*\|^2 + \frac{\alpha_n \beta_n}{1 + \alpha_n \mu} \langle -A\tilde{x}, z_n - \tilde{x} \rangle \\
 &= (1 - \delta_n) \|x_n - x^*\|^2 + \delta_n \sigma_n,
 \end{aligned} \tag{3.35}$$

where $\delta_n = \alpha_n \beta_n \mu$ and $\sigma_n = (1/(1 + \alpha_n \mu) \mu) \langle -A\tilde{x}, z_n - \tilde{x} \rangle$. It is easy to see that $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$. Hence, by Lemma 2.3, we conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{x}$. This completes the proof. $\square$

4. Conclusion

The results proved in this paper may be extended for multivalued variational inclusions and related optimization problems.

Acknowledgment

This research was partially supported by Youth Foundation of Taizhou University (2011QN11).

References

- [1] S. M. Robinson, "Generalized equations and their solutions. I: basic theory," *Mathematical Programming Study*, no. 10, pp. 128–141, 1979.
- [2] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.
- [3] S. S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.
- [4] J. W. Peng, Y. Wang, D. S. Shyu, and J. C. Yao, "Common solutions of an iterative scheme for variational inclusions, equilibrium problems, and fixed point problems," *Journal of Inequalities and Applications*, vol. 2008, Article ID 720371, 15 pages, 2008.
- [5] M. A. Noor, "Equivalence of variational inclusions with resolvent equations," *Nonlinear Analysis*, vol. 41, no. 7-8, pp. 963–970, 2000.
- [6] M. A. Noor and T. M. Rassias, "Projection methods for monotone variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 237, no. 2, pp. 405–412, 1999.
- [7] M. A. Noor and Z. Huang, "Some resolvent iterative methods for variational inclusions and nonexpansive mappings," *Applied Mathematics and Computation*, vol. 194, no. 1, pp. 267–275, 2007.
- [8] M. A. Noor, K. I. Noor, and E. Al-Said, "Some resolvent methods for general variational inclusions," *Journal of King Saud University—Science*, vol. 23, no. 1, pp. 53–61, 2011.
- [9] M. A. Noor and K. I. Noor, "Sensitivity analysis for quasi-variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 236, no. 2, pp. 290–299, 1999.
- [10] M. A. Noor, "Three-step iterative algorithms for multivalued quasi variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 255, no. 2, pp. 589–604, 2001.
- [11] M. A. Noor, "Generalized set-valued variational inclusions and resolvent equations," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 206–220, 1998.
- [12] Y. Yao, Y. J. Cho, and Y. C. Liou, "Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems," *European Journal of Operational Research*, vol. 212, no. 2, pp. 242–250, 2011.
- [13] Y. Yao and N. Shahzad, "New methods with perturbations for non-expansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2011, article ID 79, 2011.
- [14] Y. Yao and N. Shahzad, "Strong convergence of a proximal point algorithm with general errors," *Optimization Letters*. In press.
- [15] Y. Yao, Y. C. Liou, and C. P. Chen, "Algorithms construction for nonexpansive mappings and inverse-strongly monotone mappings," *Taiwanese Journal of Mathematics*, vol. 15, pp. 1979–1998, 2011.
- [16] Y. Yao, R. Chen, and Y. C. Liou, "A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem," *Mathematical & Computer Modelling*. In press.
- [17] S. Adly, "Perturbed algorithms and sensitivity analysis for a general class of variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.
- [18] S. S. Chang, "Set-valued variational inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 438–454, 2000.
- [19] S. S. Chang, "Existence and approximation of solutions of set-valued variational inclusions in Banach spaces," *Nonlinear Analysis*, vol. 47, pp. 583–594, 2001.
- [20] X. P. Ding, "Perturbed Ishikawa type iterative algorithm for generalized quasivariational inclusions," *Applied Mathematics and Computation*, vol. 141, no. 2-3, pp. 359–373, 2003.
- [21] N. J. Huang, "Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 1–7, 1998.
- [22] L. J. Lin, "Variational inclusions problems with applications to Ekeland's variational principle, fixed point and optimization problems," *Journal of Global Optimization*, vol. 39, no. 4, pp. 509–527, 2007.
- [23] M. A. Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
- [24] M. A. Noor, "Extended general variational inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–186, 2009.
- [25] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–121, 1988.
- [26] Y. Yao, Y. C. Liou, and S. M. Kang, "Two-step projection methods for a system of variational inequality problems in Banach spaces," *Journal of Global Optimization*. In press.

- [27] Y. Yao, M. A. Noor, and Y. C. Liou, "Strong convergence of a modied extra-gradient method to the minimum-norm solution of variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 817436, 9 pages, 2012.
- [28] Y. Yao, Y. C. Liou, C. L. Li, and H. T. Lin, "Extended extra-gradient methods for generalized variational inequalities," *Journal of Applied Mathematics*, vol. 2012, Article ID 237083, 14 pages, 2012.
- [29] B. Lemaire, "Which fixed point does the iteration method select?" in *Recent Advances in Optimization*, vol. 452 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1997.
- [30] H. Brezis, *Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland Publishing, Amsterdam, The Netherlands, 1973.
- [31] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.