Research Article

The Existence of Fixed Points for Nonlinear Contractive Maps in Metric Spaces with w -Distances

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Received 20 November 2011; Accepted 4 December 2011

Academic Editor: Yeong-Cheng Liou

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Some fixed point theorems for (φ, ψ, p) -contractive maps and (φ, k, p) -contractive maps on a complete metric space are proved. Presented fixed point theorems generalize many results existing in the literature.

1. Introduction and Preliminaries

Branciari [1] established a fixed point result for an integral type inequality, which is a generalization of Banach contraction principle. Kada et al. [2] introduced and studied the concept of w -distance on a metric space. They give examples of w -distances and improved Caristi's fixed point theorem, Ekeland's ϵ -variational's principle, and the nonconvex minimization theorem according to Takahashi (see many useful examples and results on w -distance in [2–5] and in references therein). Kada et al. [2] defined the concept of w -distance in a metric space as follows.

Definition 1.1 (see [2]). Let X be a metric space endowed with a metric d . A function $p : X \times X \rightarrow [0, \infty)$ is called a w -distance on X if it satisfies the following properties:

- (1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$,
- (2) p is lower semicontinuous in its second variable, that is, if $x \in X$ and $y_n \rightarrow y$ in X then $p(x, y) \leq \liminf_{n \rightarrow \infty} p(x, y_n)$,
- (3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

We denote by Φ the set of functions $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following hypotheses:

- (c1) φ is continuous and nondecreasing,
- (c2) $\varphi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the set of functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following hypotheses:

- (h1) ψ is right continuous and nondecreasing,
- (h2) $\psi(t) < t$ for all $t > 0$.

Let p be a w -distance on metric space (X, d) , $\varphi \in \Phi$ and $\psi \in \Psi$. A map T from X into itself is a (φ, ψ, p) -contractive map on X if for each $x, y \in X$, $\varphi p(Tx, Ty) \leq \psi \varphi p(x, y)$.

The following lemmas are used in the next section.

Lemma 1.2 (see [3]). *If $\psi \in \Psi$, then $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for each $t > 0$, and if $\varphi \in \Phi$, $\{a_n\} \subseteq [0, \infty)$ and $\lim_{n \rightarrow \infty} \varphi(a_n) = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

Lemma 1.3 (see [2]). *Let (X, d) be a metric space and let p be a w -distance on X .*

- (i) *If $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$, then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$, then $x = y$.*
- (ii) *If $p(x_n, y_n) \leq \alpha_n p(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0, then $\{y_n\}$ converges to y .*
- (iii) *Let p be a w -distance on metric space (X, d) and $\{x_n\}$ a sequence in X such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ (or $\lim_{m, n \rightarrow \infty} p(x_n, x_m) = 0$), then $\{x_n\}$ is a Cauchy sequence.*

Note that if $p(a, b) = p(b, a) = 0$ and $p(a, a) \leq p(a, b) + p(b, a) = 0$, then $p(a, a) = 0$ and, by Lemma 1.3, $a = b$.

In [3], Razani et al. proved a fixed point theorem for (φ, ψ, p) -contractive mappings, which is a new version of the main theorem in [1], by considering the concept of the w -distance.

The main aim of this paper is to present some generalization fixed point Theorems by Kada et al. [2], Hicks and Rhoades [6] and several other results with respect to (φ, ψ, p) -contractive maps on a complete metric space.

2. (φ, ψ, p) -Contractive Maps

In the next theorem we state one of the main results of this paper generalizing Theorem 4 of [2]. In what follows, we use φp to denote the composition of φ with p .

Theorem 2.1. *Let p be a w -distance on complete metric space (X, d) , $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose $T : X \rightarrow X$ is a map that satisfies*

$$\varphi p(Tx, T^2x) \leq \psi(\varphi p(x, Tx)), \quad (2.1)$$

for each $x \in X$ and that

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0 \quad (2.2)$$

for every $y \in X$ with $y \neq Ty$. Then there exists $u \in X$ such that $u = Tu$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. Fix $x \in X$. Set $x_{n+1} = Tx_n$ with $x_0 = x$. Then by (2.1)

$$\begin{aligned} \varphi p(x_n, x_{n+1}) &\leq \psi \varphi p(x_{n-1}, x_n) \\ &\leq \psi^2 \varphi p(x_{n-2}, x_{n-1}) \\ &\leq \cdots \leq \psi^n (\varphi p(x_0, x_1)), \end{aligned} \quad (2.3)$$

thus $\lim_n \varphi p(x_n, x_{n+1}) = 0$ and Lemma 1.2 implies

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0, \quad (2.4)$$

and similarly

$$\lim_{n \rightarrow \infty} p(x_{n+1}, x_n) = 0. \quad (2.5)$$

Now we proof that $\{x_n\}$ is a Cauchy sequence. By triangle inequality, continuity of φ and (2.4), we have

$$\varphi p(x_n, x_{n+2}) \leq \psi \varphi [p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})] \longrightarrow 0, \quad (2.6)$$

as $n \rightarrow \infty$ and so $\lim_{n \rightarrow \infty} \varphi p(x_n, x_{n+2}) = 0$ which concludes

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+2}) = 0. \quad (2.7)$$

By induction, for any $k > 0$ we have

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+k}) = 0. \quad (2.8)$$

So, by Lemma 1.3, $\{x_n\}$ is a Cauchy sequence, and since X is complete, there exists $u \in X$ such that $x_n \rightarrow u$ in X .

Now we prove that u is a fixed point of T .

From (2.8), for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon$ implies $p(x_{N_\varepsilon}, x_n) < \varepsilon$ but $x_n \rightarrow u$ and $p(x, \cdot)$ is lower semicontinuous, thus

$$p(x_{N_\varepsilon}, u) \leq \liminf_{n \rightarrow \infty} p(x_{N_\varepsilon}, x_n) \leq \varepsilon. \quad (2.9)$$

Therefore, $p(x_{N_\varepsilon}, u) \leq \varepsilon$. Set $\varepsilon = 1/k$, $N_\varepsilon = n_k$ and we have

$$\lim_{k \rightarrow \infty} p(x_{n_k}, u) = 0. \quad (2.10)$$

□

Now, assume that $u \neq Tu$. Then by hypothesis, we have

$$0 < \inf \{p(x, u) + p(x, Tx) : x \in X\} \leq \inf \{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} \rightarrow 0 \quad (2.11)$$

as $n \rightarrow \infty$ by (2.4) and (2.10). This is a contradiction. Hence $u = Tu$.

If $v = Tv$, we have

$$\varphi p(v, v) = \varphi p(Tv, T^2v) \leq \varphi \varphi p(v, Tv) = \varphi \varphi p(v, v) < \varphi p(v, v). \quad (2.12)$$

This is a contradiction. So $\varphi p(v, v) = 0$, and by hypothesis $p(v, v) = 0$.

Here we give a simple example illustrating Theorem 2.1. In this example, we will show that Theorem 4 in [2] cannot be applied.

Example 2.2. Let $X = \{(1/n) \mid n \in \mathbb{N}\} \cup \{0\}$, which is a complete metric space with usual metric d of reals. Moreover, by defining $p(x, y) = y$, p is a w -distance on (X, d) . Let $T : X \rightarrow X$ be a map as $T(1/n) = 1/(n+1)$, $T0 = 0$. Suppose $\varphi(t) = t^{1/t}$ is a continuous and strictly nondecreasing map and $\psi(t) = (1/3)t$, for any $t > 0$. We have

$$\sup_{x \in X} \frac{p(Tx, T^2x)}{p(x, Tx)} = 1, \quad (2.13)$$

and so there is not any $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$, and hence Theorem 4 in [2] dose not work. But

$$\begin{aligned} \varphi p(Tx, T^2x) &= p(Tx, T^2x)^{1/p(Tx, T^2x)} = \left(\frac{1}{n+2}\right)^{n+2} \leq \frac{1}{3} \left(\frac{1}{n+1}\right)^{n+1} \\ &= \frac{1}{3} p(x, Tx)^{1/p(x, Tx)} = \psi \varphi p(x, Tx), \end{aligned} \quad (2.14)$$

because for any $n \in \mathbb{N}$ we have $((n+1)/(n+2))^{n+1} 1/(n+2) \leq 1/3$. Also for any $n \in \mathbb{N}$ we have $1/n \neq T(1/n)$. So for arbitrary $n \in \mathbb{N}$, $\inf \{p(1/m, 1/n) + p(1/m, 1/(m+1)) : m \in \mathbb{N}\} = 1/n > 0$, hence T is satisfied in Theorem 2.1. We note that 0 is a fixed point for T .

The next examples show the role of the conditions (2.1) and (2.2).

Example 2.3. Let $X = [-1, 1]$, $d(x, y) = |x - y|$, and define $p : X \rightarrow X$ by $p(x, y) = |3x - 3y|$, where $x, y \in X$. Set $\varphi(t) = rt$ and $\psi(t) = t$ for all $t \in [0, \infty)$. Let us define $T : X \rightarrow X$ by $T0 = 1$ and $Tx = x/10$ if $x \neq 0$. We have

$$\varphi p(T0, T^20) = p(T0, T^20) = p\left(1, \frac{1}{10}\right) = 3 - \frac{3}{10} \leq 3 = \frac{1}{3} p(0, T0) = \psi \varphi p(0, T0). \quad (2.15)$$

If $x \neq 0$, then

$$\varphi p(Tx, T^2x) = p(Tx, T^2x) = p\left(\frac{x}{10}, \frac{x}{100}\right) = \frac{1}{10} \left| 3x - \frac{3x}{10} \right| \leq \frac{1}{3} p(x, Tx) = \varphi p p(x, Tx) \quad (2.16)$$

and hence (2.1) holds.

Now, we remark that $0 \neq T(0)$, and

$$\inf_{n \in \mathbb{N}} p(T^n(x), 0) + p(T^n(x), TT^n(x)) = 0 \quad \text{for every } x \in X. \quad (2.17)$$

Thus, the condition (2.2) is not satisfied, and there is no $z \in X$ with $Tz = z$. In this case we observe that Theorem 2.1 is invalid without condition (2.2).

Example 2.4. Let $X = [2, \infty) \cup \{0, 1\}$, $d(x, y) = |x - y|$, $x, y \in X$, and set $p = d$. Let φ, ψ be as Example 2.3. Let us define $T : X \rightarrow X$ by $T0 = 1$ and $Tx = 0$ if $x \neq 0$. Clearly, T has no fixed point in X . Now, for each $x \in X$ and that

$$\inf\{d(x, y) + d(x, Tx) : x \in X\} > 0 \quad (2.18)$$

for every $y \in X$ with $y \neq Ty$, so condition (2.2) is satisfied. But, for $x = 0$, $d(Tx, T^2x) > rd(x, Tx)$ for any $r \in [0, 1)$. Hence, condition (2.1) does not hold. We note that Theorem 2.1 does not work without condition (2.1).

Suppose $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is Lebesgue-integrable mapping which is summable and $\int_0^\varepsilon \theta(\eta) d\eta > 0$, for each $\varepsilon > 0$. Now, in the next corollary, set $\varphi(t) = \int_0^t \theta(\eta) d\eta$ and $\psi(t) = ct$, where $c \in [0, 1[$. Then, $\varphi \in \Phi$ and $\psi \in \Psi$. Hence we can conclude the following corollary as a special case.

Corollary 2.5. *Let T be a selfmap of a complete metric space (X, d) satisfying*

$$\int_0^{d(Tx, T^2x)} \theta(t) dt \leq c \int_0^{d(x, Tx)} \theta(t) dt \quad (2.19)$$

for all $x \in X$. Suppose that

$$\inf\{d(x, y) + d(x, Tx) : x \in X\} > 0 \quad \text{for every } y \in X \quad (2.20)$$

with $y \neq Ty$. Then there exists a $u \in X$ such that $Tu = u$.

Note that Corollary 2.5 is invalid without condition (2.20). For example, take $X = \{0\} \cup \{1/2^n : n \geq 1\}$, which is a complete metric space with usual metric d of reals. Define $T : X \rightarrow X$ by $T(0) = 1/2$ and $T(1/2^n) = 1/2^{n-1}$ for $n \geq 1$. Set $\varphi(t) \equiv 1$. It is easy to check that $\int_0^{d(Tx, T^2x)} \varphi(t) dt \leq (1/2) \int_0^{d(x, Tx)} \varphi(t) dt$, for any $x \in X$; however, $y \neq Ty$ for any $y \in X$ and $\inf\{d(x, y) + d(x, Tx) : x \in X\} = 0$. Clearly, T has got no fixed point in X .

Remark 2.6. From Theorem 2.1, we can obtain Theorem 4 in [2] as a special case. For this, in the hypotheses of Theorem 2.1, set $\varphi(t) = rt$ and $\varphi(t) = t$ for all $t \in [0, \infty)$.

Corollary 2.7. *Let p be a w -distance on complete metric space (X, d) , $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose T is a continuous mapping for X into itself such that (2.1), is satisfied. Then there exists $u \in X$ such that $u = Tu$. Moreover, if $v = Tv$, then $p(v, v) = 0$.*

Proof. Assume that there exists $y \in X$ with $y \neq Ty$ and $\inf\{p(x, y) + p(x, Tx) : x \in X\} = 0$. Then there exists a sequence $\{x_n\}$ such that

$$p(x_n, y) + p(x_n, Tx_n) \rightarrow 0 \quad (2.21)$$

as $n \rightarrow \infty$. Hence $p(x_n, y) \rightarrow 0$ and $p(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Lemma 1.3 implies that $Tx_n \rightarrow y$ as $n \rightarrow \infty$. Now by assumption

$$\varphi p(Tx_n, T^2x_n) \leq \psi(\varphi p(x_n, Tx_n)) \quad (2.22)$$

and so $\varphi p(Tx_n, T^2x_n) \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 1.2, $p(Tx_n, T^2x_n) \rightarrow 0$ as $n \rightarrow \infty$. We also have

$$p(x_n, T^2x_n) \leq p(x_n, Tx_n) + p(Tx_n, T^2x_n), \quad (2.23)$$

hence $p(x_n, T^2x_n) \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 1.3, we conclude that $\{T^2x_n\}$ converges to y . Since T is continuous, we have

$$Ty = T\left(\lim_{n \rightarrow \infty} Tx_n\right) = \lim_{n \rightarrow \infty} T^2x_n = y. \quad (2.24)$$

This is a contradiction. Therefore, if $y \neq Ty$, then $\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0$. So, Theorem 2.1 gives desired result. $\square$

In Example 2.3, T is satisfied in condition (2.1), but it is not continuous. So, the hypotheses in Corollary 2.7 are not satisfied. We note that T has no fixed point.

It is an obvious fact that, if $f : X \rightarrow X$ is a map which has a fixed point $x \in X$, then x is also a fixed point of f^n for every natural number n . However, the converse is false. If a map satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ denotes a set of all fixed points of f , then it is said to have property P [7, 8]. The following theorem extends and improves Theorem 2 of [7].

Theorem 2.8. *Let (X, d) be a complete metric space with w -distance p on X . Suppose $T : X \rightarrow X$ satisfies*

(i)

$$\varphi p(Tx, T^2x) \leq \psi \varphi p(x, Tx), \quad \forall x \in X, \quad (2.25)$$

or

(ii) *with strict inequality, $\psi \equiv 1$ and for all $x \in X$, $x \neq Tx$. If $F(T) \neq \emptyset$, then T has property P .*

Proof. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $u \in F(T^n)$. Suppose that T satisfies (i). Then,

$$\varphi p(u, Tu) = \varphi p(T^n u, TT^n u) \leq \varphi \varphi p(T^{n-1} u, TT^{n-1} u) \leq \cdots \leq \varphi^n \varphi p(u, Tu), \quad (2.26)$$

and so $p(u, Tu) = 0$. Now from

$$\varphi p(u, u) = \varphi \varphi p(u, T^n u) \leq \sum_{i=0}^{n-1} \varphi \varphi p(T^i u, T^{i+1} u) = 0, \quad (2.27)$$

we have $p(u, u) = 0$. Hence, by Lemma 1.3, we have $u = Tu$, and $u \in F(T)$. Suppose that T satisfies (ii). If $Tu = u$, then there is nothing to prove. Suppose, if possible, that $Tu \neq u$. Then a repetition of the argument for case (i) leads to $\varphi p(u, Tu) < \varphi \varphi p(u, Tu)$, that is a contradiction. Therefore, in all cases, $u = Tu$ and $F(T^n) = F(T)$. $\square$

The following theorem extends Theorem 2.1 of [6]. A function G mapping X into the real is T -orbitally lower semicontinuous at z if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $x_n \rightarrow z$ implies that $G(p) \leq \liminf_{n \rightarrow \infty} G(x_n)$.

Theorem 2.9. *Let (X, d) be a complete metric space with w -distance p on X . Suppose $T : X \rightarrow X$ and there exists an x such that*

$$\varphi p(Ty, T^2 y) \leq \varphi \varphi p(y, Ty), \quad \forall y \in O(x, \infty). \quad (2.28)$$

Then,

- (i) $\lim T^n x = z$ exists,
- (ii)

$$\varphi p(T^n x, z) \leq \frac{\varphi^n}{1 - \varphi} \varphi p(x, Tx) \quad \text{for } n \geq 1, \quad (2.29)$$

- (iii) $p(z, Tz) = 0$ if and only if $G(x) = p(x, Tx)$ is T -orbitally lower semicontinuous at z .

Proof. Observe that (i) and (ii) are immediate from the proof of Theorem 2.1. We prove (iii). It is clear that $p(z, Tz) = 0$ implying $G(x)$ is T -orbitally lower semicontinuous at z .

$x_n = T^n x \rightarrow z$ and G is T -orbitally lower semicontinuous at x implies

$$0 \leq \varphi p(z, Tz) = \varphi G(z) \leq \liminf_{n \rightarrow \infty} \varphi G(x_n) = \liminf_{n \rightarrow \infty} \varphi \varphi p(x_n, Tx_n) \leq \liminf_{n \rightarrow \infty} \varphi^n \varphi p(x, Tx) = 0. \quad (2.30)$$

So, $p(z, Tz) = 0$. $\square$

The mapping T is orbitally lower semicontinuous at $u \in X$ if $\lim_{k \rightarrow \infty} T^{n_k} x = u$ implies that $\lim_{k \rightarrow \infty} T^{n_k+1} x = Tu$. In the following, we improve Theorem 2 of [9] that it is correct form Theorem 1 of [7].

Theorem 2.10. Let p be a w -distance on complete metric space (X, d) , $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose $T : X \rightarrow X$ is orbitally lower semicontinuous map on X that satisfies

$$\varphi p(Tx, T^2x) \leq \psi(\varphi p(x, Tx)) \quad (2.31)$$

for each $x \in X$. Then there exists $u \in X$ such that $u \in F(T)$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. Observe that the sequence $\{x_n\}$ is a Cauchy sequence immediate from the proof of Theorem 2.1 and so there exists a point u in X such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Since T is orbitally lower semicontinuous at u , we have $p(u, Tu) \leq \liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$. Now, we have

$$\varphi p(u, Tu) \leq \varphi \liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = \varphi(0) = 0, \quad (2.32)$$

and so $p(u, Tu) = 0$. Similarly, $p(Tu, u) = 0$. Hence, $u \in F(T)$. By Theorem 2.1 we can conclude that if $v = Tv$, then $p(v, v) = 0$. $\square$

The following example shows that Theorem 2 in [9] cannot be applicable. So our generalization is useful.

Example 2.11. Let $X = [0, \infty)$ be a metric space with metric d defined by $d(x, y) = (40/3)|x - y|$, $x, y \in X$, which is complete. We define $p : X \rightarrow X$ by $p(x, y) = (1/3)|y|$. Let φ be as defined before in Corollary 2.5 and $\psi(t) = (1/10)t$, $t > 0$. Assume that $T : X \rightarrow X$ by $Tx = x/10$ for any $x \in X$. We have, $d(Tx, T^2x) = (4/3)d(x, Tx)$, $x \in X$, and so Theorem 2 in [9] does not work. But

$$\varphi p(Tx, T^2x) \leq \psi(\varphi p(x, Tx)) \quad (2.33)$$

for each $x \in X$. Hence by Theorem 2.10 there exists a fixed point for T . We note that 0 is fixed point for T .

3. (φ, k, p) -Contractive Maps

In this section we obtain fixed points for (φ, k, p) -contractive maps (i.e., (φ, ψ, p) -contractive maps that $\psi(t) = k$ for all $t \in [0, \infty)$, where $k \in [0, 1)$).

In 1969, Kannan [10] proved the following fixed point theorem. Contractions are always continuous and Kannan maps are not necessarily continuous.

Theorem 3.1 (see [10]). Let (X, d) be a complete metric space. Let T be a Kannan mapping on X , that is, there exists $k \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \quad (3.1)$$

for all $x, y \in X$. Then, T has a unique fixed point in X . For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

In the next theorem, we generalize this theorem as follows.

Theorem 3.2. *Let (X, d) be a complete metric space. Let T be a (φ, k) -Kannan mapping on X , that is, there exists $k \in [0, 1/2)$ such that*

$$\varphi d(Tx, Ty) \leq k(\varphi d(x, Tx) + \varphi d(y, Ty)) \quad (3.2)$$

for all $x, y \in X$. Then, T has a unique fixed point in X . For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Proof. Let $x \in X$ and define $x_{n+1} = T^n x$ for any $n \in \mathbb{N}$, and set $r = k/(1 - k)$. Then, $r \in [0, 1)$,

$$\varphi d(Tx, T^2x) \leq k(\varphi d(x, Tx) + \varphi d(Tx, T^2x)) \quad (3.3)$$

and so

$$\varphi d(Tx, T^2x) \leq r\varphi d(x, Tx). \quad (3.4)$$

Then, from the proof of Theorem 2.1, $\lim T^n x = z$ exists. From (3.4), we have

$$\varphi d(T^n x, Tz) \leq r\varphi d(T^{n-1}x, z) \leq \frac{r^n}{1-r}\varphi d(x, Tx) \quad \text{for } n \geq 1. \quad (3.5)$$

Thus, $\lim T^n x = Tz$, and so $z = Tz$. Clearly, z is unique. This completes the proof. $\square$

The set of all subadditive functions φ in Φ is denoted by Φ' . In the following theorems, we generalize Theorems 3.4 and 3.5 due to Suzuki and Takahashi [4].

Theorem 3.3. *Let p be a w -distance on complete metric space (X, d) , $\varphi \in \Phi'$ and T be a selfmap. Suppose there exists $k \in [0, 1/2)$ such that*

- (i) $\varphi p(Tx, T^2x) \leq k\varphi p(x, T^2x)$ for each $x \in X$,
- (ii) $\inf\{p(x, z) + p(x, Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$.

Then T has a fixed point in X . Moreover, if v is a fixed point of T , then $p(v, v) = 0$.

Proof. Fix $x \in X$. Define $x_0 = x$ and $x_n = T^n x_0$ for every $n \in \mathbb{N}$. Put $r = k/(1 - k)$. Then, $0 \leq r < 1$. By hypothesis, since $\varphi \in \Phi'$, we have

$$\varphi p(x_n, x_{n+1}) \leq k\varphi p(x_{n-1}, x_{n+1}) \leq k\varphi p(x_{n-1}, x_n) + k\varphi p(x_n, x_{n+1}), \quad (3.6)$$

for all $n \in \mathbb{N}$. It follows that

$$\varphi p(x_n, x_{n+1}) \leq r\varphi p(x_{n-1}, x_n) \leq \cdots \leq r^n \varphi p(x_0, x_1), \quad (3.7)$$

for all $n \in \mathbb{N}$. Using the similar argument as in the proof of Theorem 2.1, we can prove that the sequence $\{u_n\}$ is Cauchy and so there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Also, we have $u \in F(T)$. Since

$$\varphi p(v, v) = \varphi p(Tv, T^2v) \leq k\varphi p(v, T^2v) = k\varphi p(v, v), \quad (3.8)$$

we have $\varphi p(v, v) = 0$ and so $p(v, v) = 0$. The proof is completed. $\square$

Corollary 3.4. *Let p be a w -distance on complete metric space (X, d) , $\varphi \in \Phi'$ and let T be a continuous map. Suppose there exists $k \in [0, 1/2)$ such that*

$$\varphi p(Tx, T^2x) \leq k\varphi p(x, T^2x), \quad (3.9)$$

for each $x \in X$.

Then T has a fixed point in X . Moreover, if v is a fixed point of T , then $p(v, v) = 0$.

Proof. It suffices to show that $\inf\{p(x, z) + p(x, Tx) : x \in X\} > 0$ for every $u \in X$ with $u \neq Tu$. Assume that there exists $u \in X$ with $u \neq Tu$ and $\inf\{p(x, u) + p(x, Tx) : x \in X\} = 0$. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} [p(x_n, u) + p(x_n, Tx_n)] = 0$. It follows that $p(x_n, u) \rightarrow 0$ and $p(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, $Tx_n \rightarrow u$. On the other hand, since $\varphi \in \Phi'$ and (3.9), we have

$$\varphi p(x_n, T^2x_n) \leq \varphi p(x_n, Tx_n) + \varphi p(Tx_n, T^2x_n) \leq \varphi p(x_n, Tx_n) + k\varphi p(x_n, T^2x_n), \quad (3.10)$$

and hence

$$\varphi p(x_n, T^2x_n) \leq \frac{1}{1-k} \varphi p(x_n, Tx_n), \quad (3.11)$$

for all $n \in \mathbb{N}$. Thus, $p(x_n, T^2x_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $T^2x_n \rightarrow u$. Since $T : X \rightarrow X$ is continuous, we have

$$T(u) = T\left(\lim_{n \rightarrow \infty} Tx_n\right) = \lim_{n \rightarrow \infty} T^2x_n = u, \quad (3.12)$$

which is a contradiction. Therefore, using Theorem 3.3, $p(v, v) = 0$. This completes the proof. $\square$

Question 1. Can we generalize Theorems 3.2, 3.3, and Corollary 3.4 for (φ, ψ, p) -contractive maps?

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