Review Article

Three-Step Fixed Point Iteration for Generalized Multivalued Mapping in Banach Spaces

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Received 19 September 2011; Revised 2 December 2011; Accepted 8 December 2011

Academic Editor: Nazim I. Mahmudov

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The convergence of three-step fixed point iterative processes for generalized multivalued nonexpansive mapping was considered in this paper. Under some different conditions, the sequences of three-step fixed point iterates strongly or weakly converge to a fixed point of the generalized multivalued nonexpansive mapping. Our results extend and improve some recent results.

1. Introduction

Let X be a Banach space and K a nonempty subset of X . The set K is called proximal if for each $x \in X$, there exists an element $y \in K$ such that $\|x - y\| = d(x, K)$, where $d(x, K) = \inf\{\|x - z\| : z \in K\}$. Let $CB(K), C(K), P(K), F(T)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, nonempty proximal bounded subsets of K , and the set of fixed points, respectively. A multivalued mapping $T : K \rightarrow CB(K)$ is said to be nonexpansive (quasi-nonexpansive) if

$$\begin{aligned} H(Tx, Ty) &\leq \|x - y\|, \quad x, y \in K, \\ (H(Tx, Tp) &\leq \|x - p\|, x \in K, p \in F(T)), \end{aligned} \quad (1.1)$$

where $H(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}, \quad A, B \in CB(X). \quad (1.2)$$

A point x is called a fixed point of T if $x \in Tx$. Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler in 1969 (see [1]), many authors have studied the fixed point theory for multivalued mappings (e.g., see [2]). For single-valued nonexpansive mappings, Mann [3] and Ishikawa [4], respectively, introduced a new iteration procedure for approximating its fixed point in a Banach space as follows:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n y_n, \quad y_n = (1 - b_n)x_n + b_n Tx_n, \end{aligned} \quad (1.3)$$

where $\{\alpha_n\}$ and $\{b_n\}$ are sequences in $[0, 1]$. Obviously, Mann iteration is a special case of Ishikawa iteration. Recently Song and Wang in [5, 6] introduce the following algorithms for multivalued nonexpansive mapping:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n s_n, \quad (1.4)$$

where $s_n \in Tx_n$, $\gamma_n \in (0, +\infty)$ such that $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\|s_{n+1} - s_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n r_n, \quad y_n = (1 - b_n)x_n + b_n s_n, \quad (1.5)$$

where $\|s_n - r_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|s_{n+1} - r_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $s_n \in Tx_n$ and $r_n \in Ty_n$. They show some strong convergence results of the above iterates for multivalued nonexpansive mapping T under some appropriate conditions. However, the iteration scheme constructed by Song and Wang involves the following estimates,

$$\|s_n - r_n\| \leq H(Tx_n, Ty_n) + \gamma_n, \quad \|s_{n+1} - r_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n, \quad (1.6)$$

which are not easy to be computed and the scheme is more time consuming. It is observed that Song and Wang [6] did not use the above estimates in their proofs and the assumption on T , namely, $T(p) = \{p\}$ for any $p \in F(T)$ is quite strong. It is noted that the domain of T is compact, which is a strong condition. The aim of this paper is to construct an three iteration scheme for a generalized multivalued mappings, which removes the restriction of T , namely, $T(p) = \{p\}$ for any $p \in F(T)$ and also relax compactness of the domain of T . The generalized multivalued mappings was introduced in [7], if

$$\frac{1}{2}d(x, Tx) \leq \|x - y\| \text{ implies } H(Tx, Ty) \leq \|x - y\| \quad \forall x, y \in K, \quad (1.7)$$

where d is induced by the norm. Obviously, the condition is weaker than nonexpansiveness and stronger than quasinonexpansiveness, furthermore, there are some examples of a generalized nonexpansive multivalued mapping which is not a nonexpansive multivalued mapping (see [7, 8]).

Let $T : K \rightarrow P(K)$ be a generalized nonexpansive multivalued mapping and $P_T(x) = \{y \in T(x) : \|x - y\| = d(x, T(x))\}$. The three-step mean multivalued iterative scheme is defined by $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - a_n)x_n + a_ns_n, \\ y_n &= (1 - b_n - c_n)x_n + b_nt_n + c_ns_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_nr_n + \beta_nt_n + \gamma_ns_n, \end{aligned} \quad (1.8)$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{b_n + c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\alpha_n + \beta_n + \gamma_n\}$ are appropriate sequence in $[0, 1]$, furthermore $s_n \in P_T(x_n), t_n \in P_T(z_n), r_n \in P_T(y_n)$. If $a_n = c_n = \beta_n = \gamma_n \equiv 0$ or $a_n = b_n = c_n = \beta_n = \gamma_n \equiv 0$, then iterative scheme (1.8) reduces to the Ishikawa and Mann multivalued iterative scheme. In fact let $\gamma_n \equiv 0$ or $c_n = \beta_n = \gamma_n \equiv 0$ or $b_n = c_n = \alpha_n = \gamma_n \equiv 0$, we also have the other three algorithms.

The mapping $T : K \rightarrow CB(K)$ is called hemicompact if, for any sequence x_n in K such that $d(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \rightarrow p \in K$. We note that if K is compact, then every multivalued mapping $T : K \rightarrow CB(K)$ is hemicompact. The following definition was introduced in [9].

Definition 1.1. A multivalued mapping $T : K \rightarrow CB(K)$ is said to satisfy Condition (A) if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(x) > 0$ for $x \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \forall x \in K. \quad (1.9)$$

where $F(T) \neq \emptyset$ is the fixed point set of the multivalued mapping T . From now on, $F(T)$ stands for the fixed point set of the multivalued mapping T .

2. Preliminaries

A Banach space X is said to be satisfy Opial's condition [10] if, for any sequence $\{x_n\}$ in X , $x_n \rightharpoonup x (n \rightarrow \infty)$ implies the following inequality:

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad (2.1)$$

for all $y \in X$ with $y \neq x$. It is known that Hilbert spaces and $l_p (1 < p < \infty)$ have the Opial's condition.

Lemma 2.1 (see [7, 11]). Let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be sequence in uniformly convex Banach space X . Suppose that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequence in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1, \limsup_n \|x_n\| \leq d, \limsup_n \|y_n\| \leq d, \limsup_n \|z_n\| \leq d$, and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.

Lemma 2.2 (see [7, 11]). Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} & \|\lambda x + \mu y + \xi z + \vartheta \omega\|^2 \\ & \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|\omega\|^2 - \frac{1}{3} \vartheta (\lambda g(\|x - \omega\|) + \mu g(\|y - \omega\|) + \xi g(\|z - \omega\|)), \end{aligned} \quad (2.2)$$

for all $x, y, z, \omega \in B_r$ and $\lambda, \mu, \xi, \vartheta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

3. Main Results

Lemma 3.1. Let X be a real Banach space and K be a nonempty convex subset of X , $T : K \rightarrow P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by (1.8), then one has the following conclusion:

$$\lim_n \|x_n - p\| \text{ exists for any } p \in F(T). \quad (3.1)$$

Proof. Let $p \in F(T)$, then $p \in P_T(p) = \{p\}$. Since T is quasi-nonexpansive, thus we obtain

$$\begin{aligned} \|z_n - p\| & \leq (1 - a_n) \|x_n - p\| + a_n \|s_n - p\| \\ & \leq (1 - a_n) \|x_n - p\| + a_n d(s_n, P_T(p)) \\ & \leq (1 - a_n) \|x_n - p\| + a_n H(P_T(x_n), P_T(p)) \\ & \leq (1 - a_n) \|x_n - p\| + a_n \|x_n - p\| \\ & \leq \|x_n - p\|, \end{aligned} \quad (3.2)$$

similarly $\|y_n - p\| \leq \|x_n - p\|$, then we have

$$\begin{aligned} \|x_{n+1} - p\| & \leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - p\| + \alpha_n \|r_n - p\| \\ & \quad + \beta_n \|t_n - p\| + \gamma_n \|s_n - p\| \\ & \leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - p\| + \alpha_n H(P_T(y_n), P_T(p)) \\ & \quad + \beta_n H(P_T(z_n), P_T(p)) + \gamma_n H(P_T(x_n), P_T(p)) \\ & \leq \|x_n - p\|. \end{aligned} \quad (3.3)$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and hence $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$. $\square$

Lemma 3.2. Let X be a uniformly convex Banach space and K be a nonempty convex subset of X , $T : K \rightarrow P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by (1.8), if the coefficient satisfy one of the following control conditions:

(i) $\liminf_n \alpha_n > 0$ and one of the following holds:

(a) $\limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n (b_n + c_n) < 1$,

$$(b) \ 0 < \liminf_n \beta_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1 \text{ and } \limsup_n c_n < 1,$$

$$(c) \ 0 < \liminf_n b_n \leq \limsup_n (b_n + c_n) < 1 \text{ and } \limsup_n a_n < 1,$$

$$(d) \ 0 < \liminf_n c_n \leq \limsup_n (b_n + c_n) < 1;$$

$$(ii) \ 0 < \liminf_n \beta_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1 \text{ and } \limsup_n a_n < 1;$$

$$(iii) \ 0 < \liminf_n \gamma_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1;$$

$$(iv) \ 0 < \liminf_n (\alpha_n b_n + \beta_n) \text{ and } 0 < \liminf_n a_n \leq \limsup_n a_n < 1;$$

then we have $\lim_n d(x_n, Tx_n) = 0$.

Proof. By Lemma 3.1, we know that $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$, then it follows that $\{s_n - p\}$, $\{t_n - p\}$, and $\{r_n - p\}$ are all bounded. We may assume that these sequences belong to B_r where $r > 0$. Note that $p \in P_T(p) = \{p\}$ for any fixed point $p \in F(T)$ and T is quasi-nonexpansive. By Lemma 2.2, we get

$$\begin{aligned} \|z_n - p\|^2 &\leq (1 - a_n) \|x_n - p\|^2 + a_n \|s_n - p\|^2 \\ &\leq (1 - a_n) \|x_n - p\|^2 + a_n H(P_T(x_n), P_T(p))^2 \\ &\leq \|x_n - p\|^2, \\ \|y_n - p\|^2 &\leq (1 - b_n - c_n) \|x_n - p\|^2 + b_n \|t_n - p\|^2 + c_n \|s_n - p\|^2 \\ &\quad - \frac{1}{3} (1 - b_n - c_n) (b_n g(\|t_n - x_n\|) + c_n g(\|s_n - x_n\|)) \\ &\leq (1 - b_n - c_n) \|x_n - p\|^2 + b_n H(P_T(z_n), P_T(p))^2 + c_n H(P_T(x_n), P_T(p))^2 \\ &\quad - \frac{1}{3} (1 - b_n - c_n) b_n g(\|t_n - x_n\|) \\ &\leq \|x_n - p\|^2 - \frac{1}{3} (1 - b_n - c_n) b_n g(\|t_n - x_n\|), \end{aligned} \tag{3.4}$$

and therefore we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - p\|^2 + \alpha_n \|r_n - p\|^2 + \beta_n \|t_n - p\|^2 + \gamma_n \|s_n - p\|^2 \\ &\quad - \frac{1}{3} (1 - \alpha_n - \beta_n - \gamma_n) [\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)] \\ &\leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - p\|^2 + \alpha_n H(P_T(y_n), P_T(p))^2 + \beta_n H(P_T(z_n), P_T(p))^2 \\ &\quad + \gamma_n H(P_T(x_n), P_T(p))^2 \\ &\quad - \frac{1}{3} (1 - \alpha_n - \beta_n - \gamma_n) [\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)] \end{aligned}$$

$$\begin{aligned} &\leq \|x_n - p\|^2 - \frac{\alpha_n}{3}(1 - b_n - c_n)b_n g(\|t_n - x_n\|) - \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n) \\ &\quad \times [\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)]. \end{aligned} \quad (3.5)$$

Then

$$(1 - \alpha_n - \beta_n - \gamma_n)\alpha_n g(\|x_n - r_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \quad (3.6)$$

$$(1 - \alpha_n - \beta_n - \gamma_n)\beta_n g(\|x_n - t_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \quad (3.7)$$

$$(1 - \alpha_n - \beta_n - \gamma_n)\gamma_n g(\|x_n - s_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \quad (3.8)$$

$$\alpha_n(1 - b_n - c_n)b_n g(\|t_n - x_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2). \quad (3.9)$$

Since $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$, it follows from (3.6) that $\lim_n (1 - \alpha_n - \beta_n - \gamma_n)\alpha_n g(\|x_n - r_n\|) = 0$. From g is continuous strictly increasing with $g(0) = 0$ and $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$, then

$$\lim_n \|x_n - r_n\| = 0. \quad (3.10)$$

Using a similarly method together with inequalities (3.7) and $0 < \liminf_n \beta_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$, then

$$\lim_n \|x_n - t_n\| = 0. \quad (3.11)$$

Similarly, from (3.8) and $0 < \liminf_n \gamma_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$, we have $\lim_n \|x_n - s_n\| = 0$, since $s_n \in Tx_n$, then $0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0$, thus we get (iii). In the sequence we prove (i) (a). From iterative scheme (1.8), we have

$$\begin{aligned} \|s_n - x_n\| &\leq \|s_n - r_n\| + \|r_n - x_n\| \leq H(P_T(x_n), P_T(y_n)) + \|r_n - x_n\| \\ &\leq \|x_n - y_n\| + \|r_n - x_n\| \\ &\leq b_n \|x_n - t_n\| + c_n \|x_n - s_n\| + \|r_n - x_n\|. \end{aligned} \quad (3.12)$$

To show that $\lim_n \|x_n - s_n\| = 0$, it suffices to show that there exist a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{n_j} \|x_{n_j} - s_{n_j}\| = 0$. If $\liminf_j b_{n_j} > 0$, it follows from (3.9) that

$$\alpha_{n_j}(1 - b_{n_j} - c_{n_j})b_{n_j} g(\|t_{n_j} - x_{n_j}\|) \leq 3(\|x_{n_j} - p\|^2 - \|x_{n_j+1} - p\|^2). \quad (3.13)$$

Since $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$, we have

$$\lim_{n_j} \alpha_{n_j}(1 - b_{n_j} - c_{n_j})b_{n_j} g(\|t_{n_j} - x_{n_j}\|) = 0. \quad (3.14)$$

From g is continuous strictly increasing with $g(0) = 0$, $\liminf_j \alpha_{n_j} > 0$ and $0 < \liminf_{n_j} b_{n_j} \leq \limsup_{n_j} (b_{n_j} + c_{n_j}) < 1$, we have

$$\lim_{n_j} \|t_{n_j} - x_{n_j}\| = 0. \quad (3.15)$$

This together with (3.10), (3.12), (3.15) gives

$$\lim_j (1 - c_{n_j}) \|s_{n_j} - x_{n_j}\| = 0. \quad (3.16)$$

Since $\liminf_{n_j} (1 - c_{n_j}) = 1 - \limsup_{n_j} c_{n_j} > 0$, we have $\lim_j \|s_{n_j} - x_{n_j}\| = 0$. On the other hand, if $\liminf_j b_{n_j} = 0$, then we may extract a subsequence $\{b_{n_k}\}$ of $\{b_{n_j}\}$ so that $\lim_k b_{n_k} = 0$. This together with (i) (a) and (3.10), (3.12) gives

$$\lim_k (1 - c_{n_k}) \|s_{n_k} - x_{n_k}\| = 0, \text{ and so } \lim_k \|s_{n_k} - x_{n_k}\| = 0. \quad (3.17)$$

By Double Extract Subsequence Principle, we obtain the result.

If $0 < \liminf_n \beta_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$, we will prove (ii),

$$\begin{aligned} \|s_n - x_n\| &\leq \|s_n - t_n\| + \|t_n - x_n\| \leq H(P_T(x_n), P_T(z_n)) + \|t_n - x_n\| \\ &\leq \|x_n - z_n\| + \|t_n - x_n\| \\ &\leq a_n \|x_n - s_n\| + \|t_n - x_n\|. \end{aligned} \quad (3.18)$$

Since $\limsup_n a_n < 1$, then

$$\liminf_n (1 - a_n) = 1 - \limsup_n a_n > 0. \quad (3.19)$$

This together with (3.11), (3.18), we obtain the result.

We will prove (i) (b), let $p \in F(T)$. By Lemma 3.1, we let $\lim_n \|x_n - p\| = d$ for some $d \geq 0$. From iterative scheme (1.8), we know

$$d = \lim_n \|x_{n+1} - p\| = \lim_n \|(1 - \alpha_n - \beta_n - \gamma_n)(x_n - p) + \alpha_n(r_n - p) + \beta_n(t_n - p) + \gamma_n(s_n - p)\|. \quad (3.20)$$

From Lemma 3.1, we have known that $\|z_n - p\| \leq \|x_n - p\|$ and $\|y_n - p\| \leq \|x_n - p\|$, then

$$\begin{aligned} \limsup_n \|r_n - p\| &\leq \limsup_n H(P_T(y_n), P_T(p)) \leq \limsup_n \|y_n - p\| \leq d, \\ \limsup_n \|t_n - p\| &\leq \limsup_n H(P_T(z_n), P_T(p)) \leq \limsup_n \|z_n - p\| \leq d, \\ \limsup_n \|s_n - p\| &\leq \limsup_n H(P_T(x_n), P_T(p)) \leq \limsup_n \|x_n - p\| \leq d. \end{aligned} \quad (3.21)$$

From (3.20) and Lemma 2.1, we have

$$\lim_n \|x_n - t_n\| = \lim_n \|r_n - x_n\| = 0. \quad (3.22)$$

Notice that

$$\begin{aligned}\|x_n - s_n\| &\leq \|x_n - r_n\| + \|r_n - s_n\| \leq \|x_n - r_n\| + H(P_T(y_n), P_T(x_n)) \\ &\leq \|x_n - y_n\| + \|x_n - r_n\| \\ &\leq b_n\|x_n - t_n\| + c_n\|x_n - s_n\| + \|x_n - r_n\|.\end{aligned}\quad (3.23)$$

Since $\limsup_n c_n < 1$, we have $\lim_n \|s_n - x_n\| = 0$, therefore $0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0$.

We will prove (i) (c). From iterative scheme (1.8) and Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\|.\end{aligned}\quad (3.24)$$

which implies

$$\|x_{n+1} - p\| - \|x_n - p\| + \alpha_n\|x_n - p\| \leq \alpha_n\|y_n - p\|. \quad (3.25)$$

Notice that $\liminf_n \alpha_n > 0$ and $\lim_n \|x_n - p\|$ exists. Hence from (3.25) we have

$$d = \lim_n \|x_n - p\| \leq \liminf_n \|y_n - p\| \leq \limsup_n \|y_n - p\| \leq d. \quad (3.26)$$

Therefore, from iterative scheme (1.8) we have

$$d = \lim_n \|y_n - p\| = \lim_n \|(1 - b_n - c_n)(x_n - p) + b_n(t_n - p) + c_n(s_n - p)\|. \quad (3.27)$$

From Lemma 2.1, we have

$$\lim_n \|x_n - t_n\| = 0. \quad (3.28)$$

Notice that

$$\begin{aligned}\|s_n - x_n\| &\leq \|s_n - t_n\| + \|t_n - x_n\| \leq H(P_T(x_n), P_T(z_n)) + \|t_n - x_n\| \\ &\leq \|x_n - z_n\| + \|t_n - x_n\| \\ &\leq a_n\|x_n - s_n\| + \|t_n - x_n\|.\end{aligned}\quad (3.29)$$

Since $\limsup_n a_n < 1$, then $0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0$.

By (3.27) and Lemma 2.1, we can similarly prove (i) (d).

Finally, we will prove (iv). From iterative scheme (1.8) and Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|r_n - p\| + \beta_n\|t_n - p\| + \gamma_n\|s_n - p\| \\ &\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\leq (1 - \alpha_n - \beta_n)\|x_n - p\| + \alpha_n[(1 - b_n)\|x_n - p\| + b_n\|z_n - p\|] + \beta_n\|z_n - p\|,\end{aligned}\quad (3.30)$$

which implies

$$\|x_{n+1} - p\| - \|x_n - p\| + (\alpha_n b_n + \beta_n)\|x_n - p\| \leq (\alpha_n b_n + \beta_n)\|z_n - p\|. \quad (3.31)$$

Notice that

$$0 < \liminf_n (\alpha_n b_n + \beta_n), \quad \lim_n \|x_n - p\| \text{ exists.} \quad (3.32)$$

Hence we have

$$d = \lim_n \|x_n - p\| \leq \liminf_n \|z_n - p\| \leq \limsup_n \|z_n - p\| \leq d. \quad (3.33)$$

Thus, we have

$$d = \lim_n \|z_n - p\| = \lim_n (1 - a_n)\|x_n - p\| + a_n\|s_n - p\|. \quad (3.34)$$

By Lemma 2.1 and $0 < \liminf_n a_n \leq \limsup_n a_n < 1$, we have $0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0$. $\square$

Theorem 3.3. Let X be a uniformly convex Banach space and K be a nonempty convex subset of X , $T : K \rightarrow P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by (1.8), the coefficient satisfy the control conditions in Lemma 3.2 and T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By Lemma 3.2, we have $\lim_n d(x_n, Tx_n) = 0$. Since T satisfies Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leq d(x_n, Tx_n) \longrightarrow 0. \quad (3.35)$$

Thus, we get $\lim_n d(x_n, F(T)) = 0$. The remainder of the proof is the same as in [6, Theorem 2.4], we omit it. $\square$

Theorem 3.4. Let X be a uniformly convex Banach space and K be a nonempty convex subset of X , $T : K \rightarrow P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by (1.8), the coefficient satisfy the control conditions in Lemma 3.2 and T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By Lemma 3.2, we have $\lim_n d(x_n, Tx_n) = 0$. Since T is hemicompact, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \rightarrow \infty} \|x_{n_k} - q\| = 0$ for some $q \in K$. Thus,

$$\begin{aligned} d(q, Tq) &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq 2\|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \rightarrow 0. \end{aligned} \quad (3.36)$$

Hence, q is a fixed point of T . Now on take on q in place of p , we get that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. It follows that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. $\square$

Theorem 3.5. Let X, T and $\{x_n\}$ be the same as in Lemma 3.2. If K be a nonempty weakly compact convex subset of a Banach space X and X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a fixed point of T .

Proof. The proof of the Theorem is the same as in [6, Theorem 2.5], we omit it. $\square$

Remark 3.6. From the definition of iterative scheme (1.8), Theorems 3.3, 3.4, and 3.5 extend some results in [6, 12], and also give some new results are different from the [5]. In fact, we can present an example of a multivalued map T for which P_T is nonexpansive. A multivalued map $T : D \rightarrow CB(X)$ is $*$ -nonexpansive [13] if for all $x, y \in D$ and $u_x \in T(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in T(x)\}$, there exists $u_y \in T(y)$ with $d(y, u_y) = \inf\{d(y, w) : w \in T(y)\}$ such that

$$d(u_x, u_y) \leq d(x, y). \quad (3.37)$$

It is clear that if T is $*$ -nonexpansive, then P_T is nonexpansive. It is known that $*$ -nonexpansiveness is different from nonexpansiveness for multivalued maps. Let $D = [0, \infty)$ and T be defined by $Tx = [x, 2x]$ for $x \in D$ [14]. Then $P_T(x) = x$ for $x \in D$ and thus it is nonexpansive. Note that T is $*$ -nonexpansive but not nonexpansive (see [14]).

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