Research Article

Successive Matrix Squaring Algorithm for Computing the Generalized Inverse $A_{T,S}^{(2)}$

Xiaoji Liu$^{1,2}$ and Yonghui Qin$^1$

$^1$ College of Science, Guangxi University for Nationalities, Nanning 530006, China
$^2$ Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis, Nanning 530006, China

Correspondence should be addressed to Xiaoji Liu, liuxiaoji.2003@yahoo.com.cn

Received 12 June 2012; Accepted 29 November 2012

Academic Editor: J. Biazar

Copyright © 2012 X. Liu and Y. Qin. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate successive matrix squaring (SMS) algorithms for computing the generalized inverse $A_{T,S}^{(2)}$ of a given matrix $A \in \mathbb{C}^{m \times n}$.

1. Introduction

Throughout this paper, the symbol $\mathbb{C}^{m \times n}$ denotes a set of all $m \times n$ complex matrices. Let $A \in \mathbb{C}^{m \times n}$, and the symbols $R(A), N(A), \rho(A)$, and $\| \cdot \|$ stand for the range, the null space, the spectrum of matrix $A$, and the matrix norm, respectively.

A matrix $B$ is called a $\{2\}$-inverse of matrix $A$ if $BAB = B$ holds. The symbols $A^\dagger$, Ind($A$), and $A^D$ denote, respectively, the Moore-Penrose inverse, the index, and the Drazin inverse of $A$, and, obviously, $\text{rank}(A^\dagger) = \text{rank}(A)$ (see [1] for details). Let $A \in \mathbb{C}^{m \times n}$, $T \subset \mathbb{C}^n$, $S \subset \mathbb{C}^n$, and $\text{dim}(T) = t \leq r$ and $\text{dim}(S) = m - t$, and there exists and unique matrix $B \in \mathbb{C}^{n \times m}$ such that

$$BAB = B, \quad R(B) = T, \quad N(B) = S$$

(1.1)

then $B \in \mathbb{C}^{n \times m}$ is called $\{2\}$-inverse of $A$ with the prescribed range $T$ and null space $S$ of $A$, denoted by $A_{T,S}^{(2)}$.

In [1], it is well known that the generalized inverse $A_{T,S}^{(2)}$ of a given matrix $A \in \mathbb{C}^{m \times n}$ with the prescribed range $T$ and null space $S$ is very important in applications of many mathematics branches such as stable approximations of ill-posed problems, linear and
nonlinear problems involving rank-deficient generalized, and the applications to statistics [2]. In particular, the generalized inverse $A_{T,S}^{(2)}$ plays an important role for the iterative methods for solving nonlinear equations [1, 2].

In recent years, successive matrix squaring algorithms are investigated for computing the generalized inverse of a given matrix $A \in \mathbb{C}^{m \times n}$ in [3–7]. In [3], the authors exhibit a deterministic iterative algorithm for linear system solution and matrix inversion based on a repeated matrix squaring scheme. Wei derives a successive matrix squaring (SMS) algorithm to approximate the Drazin inverse in [4]. Wei et al. in [5] derive a successive matrix squaring (SMS) algorithm to approximate the weighted generalized inverse $A_{M,N}^\dagger$, which can be expressed in the form of successive squaring of a composite matrix $T$. Stanimirović and Cvetković-Ilić derive a successive matrix squaring (SMS) algorithm to approximate an outer generalized inverse with prescribed range and null space of a given matrix $A \in \mathbb{C}^{m \times n}$ in [6]. In [7], authors introduce a new algorithm based on the successive matrix squaring (SMS) method and this algorithm uses the strategy of $\epsilon$-displacement rank in order to find various outer inverses with prescribed ranges and null spaces of a square Toeplitz matrix.

In this paper, based on [3–5], we investigate successive matrix squaring algorithms for computing the generalized inverse $A_{T,S}^{(2)}$ of a matrix $A$ in Section 2 and also give a numerical example for illustrating our results in Section 3.

The following lemma suggests that the generalized inverse $A_{T,S}^{(2)}$ is unique.

**Lemma 1.1** (see [1, Theorem 2.14]). Let $A \in \mathbb{C}^{m \times n}$ with rank $r$, let $T$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let $S$ be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then, $A$ has a [2]-inverse $X$ such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if

$$AT \oplus S = \mathbb{C}^m$$

(1.2)

in which case $X$ is unique.

The following statements are equivalent:

(i) $A$ has a [2]-inverse $B \in K, H$ such that $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$,

(ii) $T$ is a complemented subspace of $H, A|_T : T \rightarrow A(T)$ is invertible and $A(T) \oplus S = K$.

**Lemma 1.2** (see [8, Section 4]). Let $A \in \mathcal{B}(H, K)$ and $T$ and $S$, respectively, closed subspaces of $H$ and $K$. Then the following statements are equivalent:

(i) $A$ has a [2]-inverse $B \in K, H$ such that $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$, 

(ii) $T$ is a complemented subspace of $H, A|_T : T \rightarrow A(T)$ is invertible and $A(T) \oplus S = K$.

**Lemma 1.3** (see [9, Section 3]). Suppose that the conditions of Lemma 1.2 are satisfied. If we take $T_1 = N(A_{T,S}^{(2)}), then$ $H = T \oplus T_1$ holds and $A$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \left[ \begin{array}{c} T \\ T_1 \end{array} \right] \rightarrow \left[ \begin{array}{c} A(T) \\ S \end{array} \right],$$

(1.3)
where $A_1$ is invertible. Moreover, $A_{T,S}^{(2)}$ has the matrix following form:

\[ A_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \] (1.4)

From (1.5), we obtain the following projections (see [9]):

\[ P_{A(T),S} = AA_{T,S}^{(2)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}, \] (1.5)

\[ P_{T,T_1} = A_{T,S}^{(2)}A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \]

2. Main Result

In this section, we consider successive matrix squaring (SMS) algorithms for computing the generalized inverse $A_{T,S}^{(2)}$.

Let $A \in C^{mxn}$ and the sequence $\{X_{k}\}$ in $C^{nxn}$, and we can define the iterative form as follows ([10, Theorem 2.2] for computing the generalized inverse $A_{T,S}^{(2)}$ in the infinite space case):

\[ R_k = P_{A(T),S} - P_{A(T),S}AX_k, \] (2.1)

\[ X_{k+1} = X_0R_k + X_k, \quad k = 0, 1, 2, \ldots. \]

From [10], the authors have proved that the iteration (2.1) converges to the generalized inverse $A_{T,S}^{(2)}$ if and only if $R(X_0) \subset T$, $\rho(R_0) < 1$, where $T \subset C^n$ and $P_{A(T),S} = AA_{T,S}^{(2)}$ (for the proof see [11] and [10, Theorem 2.1] when $p = 2$).

In the following, we give the algorithm for computing the generalized inverse $A_{T,S}^{(2)}$ of a matrix $A \in C^{mxn}$.

Let $P = R_0 = P_{A(T),S} - P_{A(T),S}AX_0$ and $Q = X_0$. It is not difficult to see that the above fact can be written as follows:

\[ M = \begin{bmatrix} R_0 & 0 \\ X_0 & I \end{bmatrix} = \begin{bmatrix} P & 0 \\ Q & I \end{bmatrix}. \] (2.2)

From (2.2) and letting $X_k = Q \sum_{i=0}^{k} P^i$, we have

\[ M^k = \begin{bmatrix} p^{k} \\ Q \sum_{i=0}^{k-1} P^i \end{bmatrix} = \begin{bmatrix} p^{k} & 0 \\ X_{k-1} & I \end{bmatrix}. \] (2.3)

By (2.3), we prove that the iterative (2.1) $X_k$ is equal to the right upper block in the matrix $M^k$. Note that we defined the new iterative form $\{M_k\}$ as follows:

\[ M_0 = M, \quad M_{k+1} = M_k^2, \quad k = 0, 1, 2, \ldots. \] (2.4)
Algorithm 1: SMS algorithm for computing the generalized inverse $A_{T,S}^{(2)}$.

From the new iterative form (2.4), we arrive at

$$M_k = M^{2^k} = \begin{bmatrix} P^{2^k} & 0 \\ Q \sum_{i=0}^{2^k-1} P^i I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ X_{2^{k-1}} & I \end{bmatrix}. \tag{2.5}$$

Assume that $X_{2^{k-1}} = \tilde{X}_k$, and by (2.5), we have

$$M_k = \begin{bmatrix} P^{2^k} & 0 \\ \tilde{X}_k & I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ X_{2^{k-1}} & I \end{bmatrix} = \begin{bmatrix} P^{2^k} & 0 \\ 2^{k-1} \sum_{i=0}^{2^k-1} P^i I \end{bmatrix}. \tag{2.6}$$

By (2.4)–(2.6), we have Algorithm 1.

From (2.4)–(2.6) and Algorithm 1, we obtain the following result.

**Theorem 2.1.** Let $A \in C^{m \times n}$, and the sequence $\{\tilde{X}_k\}$ converges to the generalized inverse $A_{T,S}^{(2)}$ if and only if $R(X_0) \subset T$, $\rho(R_0) < 1$. In this case

$$\|A_{T,S}^{(2)} - \tilde{X}_k\| \leq q^{2^k+1} (1-q)^{-1}\|X_0\|, \tag{2.7}$$

where $q = \|R_0\|$ and

$$T \subset C^n, \quad P_{A(T),S} = AA_{T,S}^{(2)}. \tag{2.8}$$

**Proof.** From the proof in [11] and [10, Theorem 2.1] when $p = 2$ and according to (2.4), (2.5) and (2.6), we easily finish the proof of the former of the theorem. In the following, we only prove the last section, that is, prove that the inequality (2.7) holds.

By applying (2.5) and (2.6), we obtain

$$\tilde{X}_k = X_{2^{k-1}} = \sum_{i=0}^{2^k-1} P^i Q. \tag{2.9}$$
By the iteration (2.4) and (2.9), we arrive at

\[
\| A_{T,S}^{(2)} - \tilde{X}_k \| = \left\| X_0 (I - R_0)^{-1} - X_0 \sum_{i=2}^{k-1} R_i^0 \right\| = \left\| X_0 \sum_{i=0}^{\infty} R_i^0 - X_0 \sum_{i=0}^{k-1} R_i^0 \right\|
\]

(2.10)

\[
\leq q^{k+1} (1 - q^{-1})^{-1} \| X_0 \|.
\]

The following corollary given the result is the same as theorem in [6, Theorem 2.3]. It also presents an explicit representation of the the generalized inverse \(A_{T,S}^{(2)}\) and the sequence (2.4) converges to a \([2]\)-inverse of a given matrix \(A\) by its full-rank decomposition.

**Corollary 2.2.** Let \(A \in \mathbb{C}^{m \times n}\), \(A = FG\) be full rank decomposition, and the sequence \(\{\tilde{X}_k\}\) converges to the \([2]\)-inverse \(X = F(GAF)^{-1}G\) if and only if \(\rho(R_0) < 1\). In this case

\[
\left\| X - \tilde{X}_k \right\| \leq q^{k+1} (1 - q^{-1})^{-1} \| X_0 \|,
\]

(2.11)

where \(q = \| R_0 \|\) and

\[
F \in \mathbb{C}^{m \times s}, \quad G \in \mathbb{C}^{s \times n}, \quad P_{R(AX),N(AX)} = AX.
\]

(2.12)

**Proof.** From Theorem 2.5 and by [6, Theorem 2.3], we have the result.

In the following, we consider the improvement of the iterative form (2.1) (see [11] for computing the Moore-Penrose inverse and the Drazin inverse of the matrix case and [10, Theorem 2.2] for computing the generalized inverse \(A_{T,S}^{(2)}\) in the infinite space case):

\[
R_k = P_{A(T),S} - P_{A(T),S}AX_k,
\]

\[
X_{k+1} = X_k \left( I + R_k + \cdots + R_k^{p-1} \right), \quad p \geq 2, \quad k = 0, 1, 2, \ldots
\]

(2.13)

Let \(M\) be a \(m \times m\) block matrix and

\[
M = \begin{bmatrix}
p^{m-1} & 0 & \cdots & 0 \\
p^{m-2} & 0 & \cdots & 0 \\
ap & 0 & \cdots & 0 \\
Q & Q & \cdots & I
\end{bmatrix}
\]

(2.14)
then

\[ M^2 = \begin{bmatrix}
  p^{2m-1} & 0 & \cdots & 0 \\
p^{2m-2} & 0 & \cdots & 0 \\
\ast & 0 & & \\
p^m & 0 & \cdots & 0 \\
\end{bmatrix}, \] (2.15)

By induction if \( M^{k-1} \) has the following form:

\[ M^{k-1} = \begin{bmatrix}
p^{(k-1)m-1} & 0 & \cdots & 0 \\
p^{(k-1)m-2} & 0 & \cdots & 0 \\
\ast & 0 & & \\
p^{(k-2)m} & 0 & \cdots & 0 \\
\end{bmatrix}, \] (2.16)

then

\[ M^k = \begin{bmatrix}
p^{km-1} & 0 & \cdots & 0 \\
p^{km-2} & 0 & \cdots & 0 \\
\ast & 0 & & \\
p^{(k-1)m} & 0 & \cdots & 0 \\
\end{bmatrix}. \] (2.17)

Similarly to the iterative form (2.4), we also define the new iterative scheme \( \{ M_k \} \)

\[ M_0 = M, \quad M_{k+1} = M_k^p, \quad k = 0, 1, 2, \ldots \] (2.18)

Note that from (2.18)

\[ M_k = M_p^k = \begin{bmatrix}
p^{p^k m-1} & 0 & \cdots & 0 \\
p^{p^k m-2} & 0 & \cdots & 0 \\
\ast & 0 & & \\
p^{(p^k-1)m} & 0 & \cdots & 0 \\
\end{bmatrix} = \begin{bmatrix}
p^{p^k m-1} & 0 & \cdots & 0 \\
p^{p^k m-2} & 0 & \cdots & 0 \\
\ast & 0 & & \\
p^{(p^k-1)m} & 0 & \cdots & 0 \\
\end{bmatrix}. \] (2.19)

Let \( X_{(p^k-1)m-1} = \hat{X}_k \), and by (2.18), and (2.19), we arrive at

\[ M_k = \begin{bmatrix}
\ast & 0 \\
\ast & \hat{X}_k \\
\end{bmatrix}. \] (2.20)
Analogous to the proof of Theorem 2.5, we finish the proof of the theorem.

Similarly the proof in

where

we also obtain Algorithm 2.

Theorem 2.3. Let \( A \in C^{m \times n} \), and the sequence \( \{ \hat{X}_k \} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( R(X_0) \subset T, \ \rho(R_0) < 1 \). In this case

\[
\left\| A_{T,S}^{(2)} - \hat{X}_k \right\| \leq q^{(p^k-1)m+1}(1-q)^{-1}\|X_0\|, \tag{2.21}
\]

where \( q = \|X_0\| \) and

\[
T \subset C^n, \quad P_{A(T),S} = AA_{T,S}^{(2)}. \tag{2.22}
\]

Proof. Similarly the proof in [10, Theorem 2.1], we can prove the former of this theorem. Analogous to the proof of Theorem 2.5, we finish the proof of the theorem.

In the following, we extend the sequence (2.4) to

\[
M_0 = M, \quad M_{k+1} = M_k^t, \quad k = 0, 1, 2, \ldots, \text{ for any } t \geq 2. \tag{2.23}
\]
By (2.26) and by induction, we have
\[ M_k = M^k = \begin{bmatrix} P^k & 0 \\ Q \sum_{i=0}^{k-1} P^i & I \end{bmatrix}. \] (2.24)

Assume that \( X_{k-1} = \hat{X}_k \), we easily have
\[ M_k = \begin{bmatrix} P^k & 0 \\ \hat{X}_k & I \end{bmatrix} = \begin{bmatrix} P^k & 0 \\ X_{k-1} & I \end{bmatrix} = \begin{bmatrix} P^k & 0 \\ Q \sum_{i=0}^{k-1} P^i & I \end{bmatrix}. \] (2.25)

Similarly, from (2.23) and (2.25), we obtain the following result.

**Theorem 2.4.** Let \( A \in C^{m \times n} \), and the sequence \( \{ \hat{X}_k \} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( R(X_0) \subset T, \rho(R_0) < 1 \). In this case
\[ \| A_{T,S}^{(2)} - \hat{X}_k \| \leq q^{k+1} (1 - q)^{-1} \| X_0 \|, \] (2.26)
where \( q = \| R_0 \| \) and
\[ T \subset C^n, \quad P_{A(T),S} = AA_{T,S}^{(2)}. \] (2.27)

**Proof.** From (2.25) and only using \( t \) instead of 2 in Theorem 2.1, we easily have that \( \{ \hat{X}_k \} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( R(X_0) \subset T, \rho(R_0) < 1 \). Similarly to the formula (2.29), we obtain that
\[ \| A_{T,S}^{(2)} - \hat{X}_k \| \leq q^{k+1} (1 - q)^{-1} \| X_0 \|, \] (2.28)
where \( q, T, \) and \( P_{A(T),S} \) are the same as Theorem 2.5. \( \square \)

In the following, we consider the dually iterative form.

Let \( A \in C^{m \times n} \) and the sequence \( \{ X_n \} \) in \( C^{m \times n} \), and we can define the iterative form as follows (see [11] and [10, Theorem 2.3]):
\[ R_k = P_{T,T_1} - AX_k P_{T,T_1}, \]
\[ X_{k+1} = R_k X_0 + X_k, \quad k = 0, 1, 2, \ldots \] (2.29)

Let \( P = R_0 = P_{T,T_1} - X_0 AP_{T,T_1} \) and \( Q = X_0 \). It is not difficult to see that the above fact can be written as follows:
\[ M = \begin{bmatrix} R_0 & 0 \\ X_0 & I \end{bmatrix} = \begin{bmatrix} P & 0 \\ Q & I \end{bmatrix}. \] (2.30)

From iterative forms (2.26) and (2.29), we have the following theorem.
**Theorem 2.5.** Let \( A \in \mathbb{C}^{m \times n} \), and the sequence \( \{\hat{X}_k\} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( R(X_0) \subset T, \rho(R_0) < 1 \). In this case

\[
\left\| A_{T,S}^{(2)} - \hat{X}_k \right\| \leq q^{k+1} (1 - q)^{-1} \|X_0\|, \tag{2.31}
\]

where \( q = \|R_0\| \) and

\[
T \subset \mathbb{C}^n, \quad P_{A(T),S} = AA_{T,S}^{(2)}. \tag{2.32}
\]

Similarly to Corollary 2.2, we have the result as follows.

**Corollary 2.6.** Let \( A \in \mathbb{C}^{m \times n} \), \( A = FG \) full rank decomposition, and the sequence \( \{\hat{X}_k\} \) converges to the \( (2) \)-inverse \( X = F(GAF)^{-1}G \) if and only if \( \rho(R_0) < 1 \). In this case

\[
\left\| X - \hat{X}_k \right\| \leq q^{k+1} (1 - q)^{-1} \|X_0\|, \tag{2.33}
\]

where \( q = \|R_0\| \) and

\[
F \in \mathbb{C}_s^{m \times s}, \quad G \in \mathbb{C}_s^{s \times n}, \quad P_{R(XA),N(XA)} =XA. \tag{2.34}
\]

In the following, we consider the improvement of the iterative form (2.29) (see [11] for computing the Moore-Penrose inverse and the Drazin inverse of the matrix case and [10, Theorem 2.3] for computing the generalized inverse \( A_{T,S}^{(2)} \) in the infinite space case):

\[
R_k = P_{T,T_1} - AX_kP_{T,T_1},
\]

\[
X_{k+1} = \left( I + R_k + \cdots + R_k^{p-1} \right)X_k, \quad p \geq 2, \quad k = 0, 1, 2, \ldots \tag{2.35}
\]

It is similar to (2.14), and we have

\[
M = \begin{bmatrix}
  P^m & P^{m-1} & \cdots & P & Q \\
  0 & 0 & \cdots & 0 & Q \\
  * & & & 0 \\
  0 & 0 & \cdots & 0 & Q \\
  0 & 0 & \cdots & 0 & I \\
\end{bmatrix}. \tag{2.36}
\]

Analogous to Theorem 2.5 by Algorithm 2 and from (2.36), we obtain the theorem in the following.

**Theorem 2.7.** Let \( A \in \mathbb{C}^{m \times n} \), and the sequence \( \{\hat{X}_k\} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( R(X_0) \subset T, \rho(R_0) < 1 \). In this case

\[
\left\| A_{T,S}^{(2)} - \hat{X}_k \right\| \leq q^{(p-1)m+1} (1 - q)^{-1} \|X_0\|, \tag{2.37}
\]
where \( q = \|X_0\| \) and

\[
T \subset \mathbb{C}^n, \quad P_{T,T_1} = A_{T,S}^{(2)} A.
\] (2.38)

Dually, we give the SMS algorithm for computing the generalized inverse \( A_{T,S}^{(2)} \) which are analogous to the iterative form (2.23) as follows and omit their proofs:

\[
M_k = \begin{bmatrix}
P_t^k & \hat{X}_k \\
0 & I
\end{bmatrix} = \begin{bmatrix}
P_t^k & X_{t-1} \\
0 & I
\end{bmatrix} = \begin{bmatrix}
P_t^k & \sum_{i=0}^{k-1} P_i \\
0 & I
\end{bmatrix}.
\] (2.39)

Similarly Theorem 2.4, from (2.35) and (2.39), we obtain the following result.

**Theorem 2.8.** Let \( A \in \mathbb{C}^{m \times n} \), and the sequence \( \{\hat{X}_k\} \) converges to the generalized inverse \( A_{T,S}^{(2)} \) if and only if \( \mathbb{R}(X_0) \subset T, \; \rho(R_0) < 1 \). In this case

\[
\left\| A_{T,S}^{(2)} - \hat{X}_k \right\| \leq q^{k+1} (1 - q)^{-1} \|X_0\|,
\] (2.40)

where \( q = \|R_0\| \) and

\[
T \subset \mathbb{C}^n, \quad P_{T,T_1} = A_{T,S}^{(2)} A.
\] (2.41)

**3. Example**

Here is an example to verify the effectiveness of the SMS method.

**Example 3.1.** Let

\[
A = \begin{bmatrix}
2 & 1 \\
0 & 2 \\
0 & 0
\end{bmatrix}.
\] (3.1)

Let \( T \in \mathbb{C}^2; \; e = (0;0;1)^T \in \mathbb{C}^3, \; S = \text{span}\{e\} \).

Take

\[
X_0 = \begin{bmatrix}
0.4 & 0 & 0 \\
0 & 0.4 & 0
\end{bmatrix}.
\] (3.2)

By (2.2), we have

\[
R_0 = \begin{bmatrix}
0.2 & -0.4 & 0 \\
0 & 0.2 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (3.3)
From [10, 12], we easily have the generalized inverse $A_{T,S}^{(2)}$ in

$$A_{T,S}^{(2)} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}. \tag{3.4}$$

Then, from Algorithm 1, we obtain

$$X_1 = \begin{bmatrix} 0.4800 & -0.1600 & 0 \\ 0 & 0.4800 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.5600 & -0.3200 & 0 \\ 0 & 0.5600 & 0 \end{bmatrix}. \tag{3.5}$$

But by the iteration (2.1), we get

$$X_1 = \begin{bmatrix} 0.4800 & -0.1600 & 0 \\ 0 & 0.4800 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.4960 & -0.2240 & 0 \\ 0 & 0.4960 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0.4992 & -0.2432 & 0 \\ 0 & 0.4992 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0.4998 & -0.2483 & 0 \\ 0 & 0.4998 & 0 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} 0.5000 & -0.2496 & 0 \\ 0 & 0.5000 & 0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0.5000 & -0.2499 & 0 \\ 0 & 0.5000 & 0 \end{bmatrix}. \tag{3.6}$$

From the data in (3.5) and (3.6), we obtain Table 1.

From the above in (3.5), (3.6), and Table 1, we know that we only need two steps by Algorithm 1, but five steps by using iterative form (2.1).

**Acknowledgments**

X. Liu is supported by the National Natural Science Foundation of China (11061005), College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, China, and Y. Qin is supported by the Innovation Project of Guangxi University for Nationalities (gxun-chx2011075), College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, China.

**References**


