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Research Article

Some Properties of the (p, q)-Fibonacci and (p, q)-Lucas Polynomials

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Riordan arrays are useful for solving the combinatorial sums by the help of generating functions. Many theorems can be easily proved by Riordan arrays. In this paper we consider the Pascal matrix and define a new generalization of Fibonacci polynomials called (p, q)-Fibonacci polynomials. We obtain combinatorial identities and by using Riordan method we get factorizations of Pascal matrix involving (p, q)-Fibonacci polynomials.

1. Introduction

Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers [1]. Such polynomials, called the Fibonacci polynomials, were studied in 1883 by the Belgian Mathematician Eugene Charles Catalan and the German Mathematician E. Jacobsthal. The polynomials $f_n(x)$ studied by Catalan are defined by the recurrence relation

$$f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \tag{1.1}$$

where $f_1(x) = 1,\ f_2(x) = x$, and $n \geq 3$. Notice that $f_n(1) = F_n$, the nth Fibonacci number. The Fibonacci polynomials studied by Jacobsthal were defined by

$$J_n(x) = J_{n-1}(x) + x J_{n-2}(x), \tag{1.2}$$

where $J_1(x) = 1 = J_2(x)$ and $n \geq 3$. The Pell polynomials $p_n(x)$ are defined by

$$p_n(x) = 2x p_{n-1}(x) + p_{n-2}(x), \tag{1.3}$$

where $p_0(x) = 0$, $p_1(x) = 1$ and $n \geq 2$. The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{1.4}$$

where $L_0(x) = 2$, $L_1(x) = x$, and $n \geq 2$.

Horadam [2] introduced the polynomial sequence $\{w_n(x)\}$ defined recursively by

$$w_n(x) = p(x)w_{n-1}(x) + q(x)w_{n-2}(x), \quad (n \geq 2), \tag{1.5}$$

where

$$w_0(x) = c_0, \qquad w_1(x) = c_1x^d, \qquad p(x)c_2x^d, \qquad q(x) = c_3x^d \tag{1.6}$$

in which c_0, c_1, c_2, c_3 are constants and $d = 0$ or 1. Special cases of the $w(x)$ with given initial conditions are given in Table 1.

For a fixed n, Brawer and Pirovino [3] defined the $n \times n$ lower triangular Pascal as matrix $P_n = [p_{i,j}]_{i,j=1,2,\ldots,n}$

$$P_{i,j} = \begin{cases} \dbinom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

The Pascal matrices have many applications in probability, numerical analysis, surface reconstruction, and combinatorics. In [4] the relationships between the Pascal matrix and the Vandermonde, Frobenius, Stirling matrices are studied. Also in [4] other applications in stability properties of numerical methods for solving ordinary differential equations are shown. In [5–8] the binomial coefficients, fibonomial coefficients, Pascal matrix, and its generalizations are studied. The authors in [9] factorized the Pascal matrix involving the Fibonacci matrix.

Lee et al. [10] defined the $n \times n$ Fibonacci matrix as follows:

$$\mathcal{F}_n = [f_{ij}] = \begin{cases} F_{i-j+1} & \text{if } i-j+1 \geq 0, \\ 0 & \text{if } i-j+1 < 0. \end{cases} \tag{1.8}$$

Also in [10] factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices are studied. The inverse of this matrix is given as follows;

$$\mathcal{F}_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix}. \tag{1.9}$$

Table 1: Special cases of the $w(x)$ with given initial conditions are given.

$p(x)$	$q(x)$	$w_0(x) = 0, w_1(x) = 1$	$w_0(x) = 2, w_1(x) = x$
x	1	Fibonacci Polynomial	Lucas Polynomial
$2x$	1	Pell Polynomial	Pell-Lucas Polynomial
1	$2x$	Jacobsthal Poly	Jacobsthal-Lucas Poly.
$3x$	-2	Fermat Polynomial	Fermat-Lucas Polynomial
$2x$	-1	Cheby. Poly. 2nd kind	Cheby. Poly. first kind

The Riordan group was introduced by Shapiro et al. in [6] as follows.

Let $R = [r_{ij}]_{i,j\geq 0}$ be an infinite matrix with complex entries. Let $c_i(t) = \sum_{n\geq 0}^{\infty} r_{n,i}t^n$ be the generating function of the ith column of R. We call R a Riordan matrix if $c_i(t) = g(t)[f(t)]^i$, where

$$g(t) = 1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots, \qquad f(t) = t + f_2 t^2 + f_3 t^3 + \cdots. \tag{1.10}$$

In this case we write $R = (g(t), f(t))$ and we denote by $\mathcal{R}$ the set of Riordan matrices. Then the set $\mathcal{R}$ is a group under matrix multiplication $*$, with the following properties:

(R1) $(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t)))$,

(R2) $I = (1, t)$ is the identity element,

(R3) the inverse of R is given by $R^{-1} = (1/g(\overline{f}(t)), \overline{f}(t))$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$, that is, $f(\overline{f}(t)) = \overline{f}(f(t)) = t$,

(R4) if $(a_0, a_1, a_2, \ldots)^T$ is a column vector with the generating function $A(t)$, then multiplying $R = (g(t), f(t))$ on the right by this column vector yields a column vector with the generating function $B(t) = g(t)A(f(t))$.

This group has many applications. Three of them are given in [6] such as Euler's problem of the King walk, binomial and inverse identities, and Bessel-Neumann expansion.

Riordan arrays are also useful for solving the combinatorial sums by the help of generating functions. For example, in [11], Cheon, Kim, and Shapiro have many results including a generalized Lucas polynomial sequences from Riordan array and combinatorial interpretations for a pair of generalized Lucas polynomial sequences.

2. The (p, q)-Fibonacci and (p, q)-Lucas Polynomials with Some Properties

In [12], the authors introduced the $h(x)$-Fibonacci polynomials, where $h(x)$ is a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1, \tag{2.1}$$

with initial conditions $F_{h,0}(x) = 0$, $F_{h,1}(x) = 1$.

In this paper, we introduce a generalization of the $h(x)$-Fibonacci polynomials.

Definition 2.1. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p,q)-Fibonacci polynomials $\{F_{p,q,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \quad n \geq 1 \tag{2.2}$$

with the initial conditions $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$.

For later use $F_{p,q,2}(x) = p(x)$, $F_{p,q,3}(x) = p^2(x) + q(x)$, $F_{p,q,4}(x) = p^3(x) + 2p(x)q(x)$, $F_{p,q,5}(x) = p^4(x) + 3p^2(x)q(x) + q^2(x)\cdots$.

Now, we introduce (p,q)-Lucas polynomials $\{L_{p,q,n}(x)\}_{n=0}^{\infty}$ as the following definition.

Definition 2.2. The (p,q)-Lucas polynomials $\{L_{p,q,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$L_{p,q,n}(x) = F_{p,q,n+1}(x) + q(x)F_{p,q,n-1}(x). \tag{2.3}$$

Also for later use $L_{p,q,0}(x) = 2$, $L_{p,q,1}(x) = p(x)$, $L_{p,q,2}(x) = p^2(x) + 2q(x)$, $L_{p,q,3}(x) = p^3(x) + 3p(x)q(x)$, $L_{p,q,4}(x) = p^4(x) + 4p^2(x)q(x) + 2q^2(x)\cdots$.

In [12], the authors defined $h(x)$-Lucas polynomials as follows:

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geq 1, \tag{2.4}$$

with initial conditions $L_{h,0}(x) = 2$, $L_{h,1}(x) = h(x)$. However, we defined (p,q)-Lucas polynomials in the Definition 2.2 which is different from $h(x)$-Lucas polynomials. From the Definition 2.2, for $p(x) = 1$ and $q(x) = 1$, we obtain the usual Lucas numbers. And, for $p(x) = h(x)$ and $q(x) = 1$, we obtain the $h(x)$-Lucas polynomials.

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given in Table 1.

The generating function $g_F(t)$ of the sequence $\{F_{p,q,n}(x)\}$ is defined by

$$g_F(t) = \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n. \tag{2.5}$$

We know that the generating function $g_F(t)$ is a convergence formal series.

Theorem 2.3. *Let $g_F(t)$ be the generating function of the (p,q)-Fibonacci polynomial sequence $F_{p,q,n}(x)$. Then*

$$g_F(t) = \frac{t}{1 - p(x)t - q(x)t^2}. \tag{2.6}$$

Proof. Let $g_F(t)$ be the generating function of the (p,q)-Fibonacci polynomial sequence $F_{p,q,n}(x)$, then

$$\begin{aligned}
g_F(t) &= \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n \\
&= F_{p,q,1}(x)t + F_{p,q,2}(x)t^2 + \sum_{n=3}^{\infty} F_{p,q,n}(x)t^n \\
&= t + t^2 p(x) + \sum_{n=3}^{\infty} \left[p(x)F_{p,q,n-1}(x) + q(x)F_{p,q,n-2}(x)\right]t^n \\
&= t + t^2 p(x) + \sum_{n=3}^{\infty} p(x)F_{p,q,n-1}(x)t^n + \sum_{n=3}^{\infty} q(x)F_{p,q,n-2}(x)t^n \\
&= t + t^2 p(x) + t\sum_{n=3}^{\infty} p(x)F_{p,q,n-1}(x)t^{n-1} + t^2\sum_{n=3}^{\infty} q(x)F_{p,q,n-2}(x)t^{n-2} \\
&= t + t^2 p(x) + tp(x)\sum_{n=2}^{\infty} F_{p,q,n}(x)t^n + t^2 q(x)\sum_{n=1}^{\infty} F_{p,q,n}(x)t^n \\
&= t + t^2 p(x) + tp(x)\left[\sum_{n=1}^{\infty} F_{p,q,n}(x)t^n - t\right] + t^2 q(x)\sum_{n=1}^{\infty} F_{p,q,n}(x)t^n \\
&= t + t^2 p(x) + tp(x)\left[g_F(t) - t\right] + t^2 q(x) g_F(t).
\end{aligned} \tag{2.7}$$

By taking $g_F(t)$ parenthesis we get

$$g_F(t) = \frac{t}{1 - p(x)t - q(x)t^2}. \tag{2.8}$$

The proof is completed. □

Corollary 2.4. *Let $g_L(t)$ be the generating function of the (p,q)-Lucas polynomial sequence $L_{p,q,n}(x)$. Then*

$$g_L(t) = \frac{2 - p(x)t}{1 - p(x)t - q(x)t^2}. \tag{2.9}$$

The Binet formula is also very important in Fibonacci numbers theory. Now we can get the Binet formula of (p,q)-Fibonacci polynomials. Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation

$$t^2 - p(x)t - q(x) = 0 \tag{2.10}$$

of the recurrence relation (2.2). Then

$$\alpha(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}, \qquad \beta(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}. \tag{2.11}$$

Note that $\alpha(x)+\beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Now we can give the Binet formula for the (p, q)-Fibonacci and (p, q)-Lucas polynomials.

Theorem 2.5. *For $n \geq 0$*

$$\begin{aligned} F_{p,q,n}(x) &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \\ L_{p,q,n}(x) &= \alpha^n(x) + \beta^n(x). \end{aligned} \tag{2.12}$$

Proof. The theorem can be proved by mathematical induction on n. □

Lemma 2.6. *For $n \geq 1$,*

$$t^n = F_{p,q,n}(x)t + q(x)F_{p,q,n-1}(x). \tag{2.13}$$

Proof. From the characteristic equation of the (p, q)-Fibonacci polynomials we have

$$\begin{aligned} t^2 &= p(x)t + q(x) \\ &= F_{p,q,2}(x)t + q(x)F_{p,q,1}(x). \end{aligned} \tag{2.14}$$

By induction on n we get

$$\begin{aligned} t^{n+1} &= t^n t \\ &= \left(F_{p,q,n}(x)t + q(x)F_{p,q,n-1}(x)\right)t \\ &= F_{p,q,n}(x)t^2 + q(x)tF_{p,q,n-1}(x) \\ &= F_{p,q,n}(x)\left(p(x)t + q(x)\right) + q(x)tF_{p,q,n-1}(x) \\ &= \left(F_{p,q,n}(x)p(x) + q(x)F_{p,q,n-1}(x)\right)t + q(x)F_{p,q,n}(x) \\ &= F_{p,q,n+1}(x)t + q(x)F_{p,q,n}(x). \end{aligned} \tag{2.15}$$

Thus we have

$$t^n = F_{p,q,n}(x)t + q(x)F_{p,q,n-1}(x). \tag{2.16}$$

□

Theorem 2.7. *Let $L_{p,q,n}(x) = F_{p,q,n+1}(x) + q(x)F_{p,q,n-1}(x)$. Then for $n \geq 3$,*

$$L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x). \tag{2.17}$$

Proof. If $n = 3$ then

$$\begin{aligned} L_{p,q,3} &= p^3(x) + 3p(x)q(x) \\ &= p(x)L_{p,q,2}(x) + q(x)L_{p,q,1}(x). \end{aligned} \tag{2.18}$$

By induction on n we have

$$\begin{aligned} p(x)L_{p,q,n}(x) + q(x)L_{p,q,n-1}(x) &= p(x)\left(F_{p,q,n+1}(x) + q(x)F_{p,q,n-1}(x)\right) \\ &\quad + q(x)\left(F_{p,q,n}(x) + q(x)F_{p,q,n-2}(x)\right) \\ &= p(x)F_{p,q,n+1}(x) + p(x)q(x)F_{p,q,n-1}(x) \\ &\quad + q(x)F_{p,q,n}(x) + q^2(x)F_{p,q,n-2}(x) \\ &= p(x)F_{p,q,n+1}(x) + q(x)F_{p,q,n}(x) \\ &\quad + q(x)\left(p(x)F_{p,q,n-1}(x) + q(x)F_{p,q,n-2}(x)\right) \\ &= F_{p,q,n+2}(x) + q(x)F_{p,q,n}(x) \\ &= L_{p,q,n+1}(x). \end{aligned} \tag{2.19}$$

□

In [12], the author introduced the matrix $Q_h(x)$ that plays the role of the Q-matrix. The Q-matrix is associated with the Fibonacci numbers and is defined as

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.20}$$

Actually, in [12], Nalli and Pentti defined the matrix $Q_h(x)$ as follows

$$Q_h(x) = \begin{bmatrix} h(x) & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.21}$$

We now introduce the matrix $Q_{\mathrm{p},q}(x)$ which is a generalization of the $Q_h(x)$.

Definition 2.8. Let $Q_{p,q}(x)$ denote the 2×2 matrix defined as

$$Q_{p,q}(x) = \begin{bmatrix} p(x) & q(x) \\ 1 & 0 \end{bmatrix}. \tag{2.22}$$

Theorem 2.9. *Let $n \geq 1$. Then*

$$Q_{p,q}^{n}(x) = \begin{bmatrix} F_{p,q,n+1}(x) & q(x)F_{p,q,n}(x) \\ F_{p,q,n}(x) & q(x)F_{p,q,n-1}(x) \end{bmatrix}. \tag{2.23}$$

Proof. We can prove the theorem by induction on n. The result holds for $n = 1$. Suppose that it holds for $n = m$ $(m \geq 1)$. Then

$$\begin{aligned} Q_{p,q}^{m+1}(x) &= Q_{p,q}^{m}(x) \cdot Q_{p,q}(x) \\ &= \begin{bmatrix} F_{p,q,m+1}(x) & q(x)F_{p,q,m}(x) \\ F_{p,q,m}(x) & q(x)F_{p,q,m-1}(x) \end{bmatrix} \begin{bmatrix} p(x) & q(x) \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} F_{p,q,m+2}(x) & q(x)F_{p,q,m+1}(x) \\ F_{p,q,m+1}(x) & q(x)F_{p,q,m}(x) \end{bmatrix} \end{aligned} \tag{2.24}$$

which completes the proof. □

Corollary 2.10. *Let $m, n \geq 0$. Then*

$$F_{p,q,m+n+1}(x) = F_{p,q,m+1}(x)F_{p,q,n+1}(x) + q(x)F_{p,q,m}(x)F_{p,q,n}(x). \tag{2.25}$$

If an integer $a \neq 0$ divides an integer b, we denote $a \mid b$.

Corollary 2.11. *For $k \geq 1$,*

$$F_{p,q,n}(x) \mid F_{p,q,kn}(x). \tag{2.26}$$

Corollary 2.12. *The roots of characteristic equation of $Q_{p,q}^{n}(x)$ are $\alpha^{n}(x)$ and $\beta^{n}(x)$.*

Corollary 2.13. *For $n \geq 1$*

$$F_{p,q,n}(x) = \sum_{n=0}^{[(n-1)/2]} \binom{n-i}{i} p^{n-2i}(x) q^{i}(x). \tag{2.27}$$

The following identities of which originated from Koshy (1998) [1] are a generalization of Koshy's results.

Theorem 2.14. *For $k \geq 1$, one has*

$$\sum_{i=0}^{n} F_{p,q,ki+j}(x) = \begin{cases} \dfrac{F_{p,q,nk+j}(x)p^{k}(x) + F_{p,q,j}(x) - F_{p,q,nk+k+j}(x) + F_{p,q,k-j}(x)\left(-q(x)\right)^{j}}{p^{k}(x) - L_{p,q,k}(x) + 1}, & j < k, \\ \dfrac{F_{p,q,nk+j}(x)p^{k}(x) + F_{p,q,j}(x) - F_{p,q,nk+k+j}(x) + F_{p,q,j-k}(x)\left(-q(x)\right)^{k}}{p^{k}(x) - L_{p,q,k}(x) + 1}, & \textit{otherwise}. \end{cases} \tag{2.28}$$

Proof. We know that

$$\alpha(x) - \beta(x) = \sqrt{p^2(x) + 4q(x)}, \qquad L_{p,q,n}(x) = \alpha^n(x) + \beta^n(x). \tag{2.29}$$

Since $F_{p,q,n}(x) = (\alpha^n(x) - \beta^n(x))/\sqrt{p^2(x) + 4q(x)}$, from Theorem 2.5, we have

$$\begin{aligned}
\sum_{i=0}^{n} F_{p,q,ki+j}(x) &= \sum_{i=0}^{n} \frac{\alpha^{ki+j}(x) - \beta^{ki+j}(x)}{\sqrt{p^2(x) + 4q(x)}} \\
&= \frac{1}{\sqrt{p^2(x) + 4q(x)}} \left(\alpha^j(x) \sum_{i=0}^{n} \alpha^{ki}(x) - \beta^j(x) \sum_{i=0}^{n} \beta^{ki}(x) \right) \\
&= \frac{1}{\sqrt{p^2(x) + 4q(x)}} \left(\alpha^j(x) \frac{\alpha^{nk+k}(x) - 1}{\alpha^k(x) - 1} - \beta^j(x) \frac{\beta^{nk+k}(x) - 1}{\beta^k(x) - 1} \right) \\
&= \frac{1}{\sqrt{p^2(x) + 4q(x)}} \left(\frac{\alpha^{nk+k+j}(x) - \alpha^j(x)}{\alpha^k(x) - 1} - \frac{\beta^{nk+k+j}(x) - \beta^j(x)}{\beta^k(x) - 1} \right).
\end{aligned} \tag{2.30}$$

Set $1/\sqrt{p^2(x) + 4q(x)} = A(x)$ and

$$C(x) = \begin{cases} F_{p,q,k-j}(x)\left(-q(x)\right)^j & \text{if } j < k, \\ F_{p,q,j-k}(x)\left(-q(x)\right)^k & \text{otherwise.} \end{cases} \tag{2.31}$$

Then we have

$$\begin{aligned}
\sum_{i=0}^{n} F_{p,q,ki+j}(x) &= \frac{A(x)}{\left(\alpha(x)\beta(x)\right)^k - \left(\alpha^k(x) + \beta^k(x)\right) + 1} \\
&\quad \times \Big(\alpha^{nk+k+j}(x)\beta^k(x) - \alpha^j(x)\beta^k(x) - \alpha^{nk+k+j}(x) + \alpha^j(x) - \beta^{nk+k+j}(x)\alpha^k(x) \\
&\qquad + \alpha^k(x)\beta^j(x) + \beta^n(x) \Big) \\
&= \frac{A(x)}{p^k(x) - L_{p,q,k}(x) + 1} \\
&\quad \times \left(F_{p,q,nk+j} p^k(x) \frac{1}{A(x)} + F_{p,q,j}(x) \frac{1}{A(x)} - F_{p,q,nk+k+j}(x) \frac{1}{A(x)} \right. \\
&\qquad \left. + \alpha^k(x)\beta^j(x) - \alpha^j(x)\beta^k(x) \right) \\
&= \frac{F_{p,q,nk+j}(x) p^k(x) + F_{p,q,j}(x) - F_{p,q,nk+k+j}(x) + C(x)}{p^k(x) - L_{\mathrm{p},q,k}(x) + 1}.
\end{aligned} \tag{2.32}$$

Thus the proof is completed. □

Theorem 2.15. *For $n \geq 0$ one has*

$$\sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) F_{p,q,k}(x) = F_{p,q,2n}(x). \tag{2.33}$$

Proof. By Theorem 2.5 we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) F_{p,q,k}(x) &= \sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) \frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)} \\
&= \frac{1}{\alpha(x) - \beta(x)} \left(\sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) \left(\alpha^k(x) - \beta^k(x) \right) \right) \\
&= \frac{1}{\alpha(x) - \beta(x)} \left(\sum_{k=0}^{n} \binom{n}{k} \left(p(x)\alpha(x) \right)^k q^{n-k}(x) \right. \\
&\qquad \left. - \sum_{k=0}^{n} \binom{n}{k} \left(p(x)\beta(x) \right)^k q^{n-k}(x) \right) \\
&= \frac{1}{\alpha(x) - \beta(x)} \left(\left(p(x)\alpha(x) + q(x) \right)^n + \left(p(x)\beta(x) + q(x) \right)^n \right).
\end{aligned} \tag{2.34}$$

Since $\alpha(x)$ and $\beta(x)$ are the solutions of the equation $t^2 - p(x)t - q(x) = 0$,

$$\begin{aligned}
\alpha^2(x) &= p(x)\alpha(x) + q(x), \\
\beta^2(x) &= p(x)\beta(x) + q(x).
\end{aligned} \tag{2.35}$$

Thus we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) F_{p,q,k}(x) &= \frac{\left(\alpha^2(x) \right)^n - \left(\beta^2(x) \right)^n}{\alpha(x) - \beta(x)} \\
&= F_{p,q,2n}(x).
\end{aligned} \tag{2.36}$$

The proof is completed. □

Theorem 2.16. *For $n \geq 0$, one has*

$$\sum_{k=0}^{n} \binom{n}{k} p^k(x) \left(-q(x) \right)^{n-k} F_{p,q,k}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(-2q(x) \right)^k F_{p,q,2(n-k)}(x). \tag{2.37}$$

Proof. By Theorem 2.5 we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}p^{k}(x)(-q(x))^{n-k}F_{p,q,k}(x) &= \sum_{k=0}^{n}\binom{n}{k}p^{k}(x)(-q(x))^{n-k}\frac{\alpha^{k}(x)-\beta^{k}(x)}{\alpha(x)-\beta(x)} \\
&= \frac{1}{\alpha(x)-\beta(x)}\left(\sum_{k=0}^{n}\binom{n}{k}p^{k}(x)(-q(x))^{n-k}\left(\alpha^{k}(x)-\beta^{k}(x)\right)\right) \\
&= \frac{1}{\alpha(x)-\beta(x)}\left(\sum_{k=0}^{n}\binom{n}{k}(p(x)\alpha(x))^{k}(-q(x))^{n-k}\right. \\
&\qquad\left. -\sum_{k=0}^{n}\binom{n}{k}(p(x)\beta(x))^{k}(-q(x))^{n-k}\right) \\
&= \frac{1}{\alpha(x)-\beta(x)}\left((p(x)\alpha(x)-q(x))^{n}+(p(x)\beta(x)-q(x))^{n}\right).
\end{aligned} \tag{2.38}$$

Since $\alpha(x)$ and $\beta(x)$ are the solutions of the equation $t^2 - p(x)t - q(x) = 0$,

$$\begin{aligned}
p(x)\alpha(x) - q(x) &= \alpha^2(x) - 2q(x), \\
p(x)\beta(x) - q(x) &= \beta^2(x) - 2q(x).
\end{aligned} \tag{2.39}$$

Thus we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}p^{k}(x)(-q(x))^{n-k}F_{p,q,k}(x) &= \frac{(\alpha^2(x)-2q(x))^{n}-(\beta^2(x)-2q(x))^{n}}{\alpha(x)-\beta(x)} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-2q(x))^{k}F_{p,q,2(n-k)}(x).
\end{aligned} \tag{2.40}$$

The proof is completed. □

Corollary 2.17. *For $n \geq 0$ one has*

$$\sum_{k=0}^{n}\binom{n}{k}p^{n-k}(x)(-1)^{k}L_{p,q,k}(x) = L_{p,q,n}(x). \tag{2.41}$$

Proof. Since $p(x) - \alpha(x) = -q(x)/\alpha(x)$ and $p(x) - \beta(x) = -q(x)/\beta(x)$, we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}p^{n-k}(x)(-1)^{k}L_{p,q,k}(x) &= \sum_{k=0}^{n}\binom{n}{k}p^{n-k}(x)(-1)^{k}\left(\alpha^{k}(x)+\beta^{k}(x)\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}p^{n-k}(x)(-1)^{k}(-\alpha(x))^{k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\binom{n}{k}p^{n-k}(x)(-1)^{k}\left(-\beta(x)\right)^{k}\\
&=\left(p(x)-\alpha(x)\right)^{n}+\left(p(x)-\beta(x)\right)^{n}\\
&=\left(-\frac{q(x)}{\alpha(x)}\right)^{n}+\left(-\frac{q(x)}{\beta(x)}\right)^{n}\\
&=\left(-q(x)\right)^{n}\frac{\alpha^{n}(x)+\beta^{n}(x)}{\left(\alpha(x)\beta(x)\right)^{n}}\\
&=L_{p,q,n}(x).
\end{aligned}
\tag{2.42}
$$

□

Corollary 2.18. *For $n \geq 0$,*

$$F_{p,q,2n}(x)=F_{p,q,n}(x)L_{p,q,n}(x). \tag{2.43}$$

Corollary 2.19. *For $n \geq m$,*

$$F_{p,q,n+m}(x)-\left(-q(x)\right)^{m}F_{p,q,n-m}(x)=F_{p,q,m}(x)L_{p,q,n}(x). \tag{2.44}$$

Proof. From the Binet formula of the (p, q)-Fibonacci and Lucas polynomials we have

$$
\begin{aligned}
&F_{p,q,n+m}(x)-\left(-q(x)\right)^{m}F_{p,q,n-m}(x)\\
&\quad=\frac{\alpha^{n+m}(x)-\beta^{n+m}(x)}{\alpha(x)-\beta(x)}-\left(-q(x)\right)^{m}\frac{\alpha^{n-m}(x)-\beta^{n-m}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\frac{\alpha^{n+m}(x)-\beta^{n+m}(x)-\left(-q(x)\right)^{m}\left(\alpha^{n-m}(x)-\beta^{n-m}(x)\right)}{\alpha(x)-\beta(x)}.
\end{aligned}
\tag{2.45}
$$

Since $\alpha(x)\beta(x)=-q(x)$ then

$$
\begin{aligned}
&F_{p,q,n+m}(x)-\left(-q(x)\right)^{m}F_{p,q,n-m}(x)\\
&\quad=\frac{\alpha^{n+m}(x)-\beta^{n+m}(x)-\left(\alpha(x)\beta(x)\right)^{m}\left(\alpha^{n-m}(x)-\beta^{n-m}(x)\right)}{\alpha(x)-\beta(x)}\\
&\quad=\frac{\alpha^{n+m}(x)-\beta^{n+m}(x)-\left(\alpha^{n}(x)\beta^{m}(x)-\alpha^{m}(x)\beta^{n}(x)\right)}{\alpha(x)-\beta(x)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\alpha^m(x) - \beta^m(x)\right) - \left(\alpha^n(x) + \beta^n(x)\right)}{\alpha(x) - \beta(x)} \\
&= \frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)}\left(\left(\alpha^n(x) + \beta^n(x)\right)\right) \\
&= F_{p,q,m}(x) L_{p,q,n}(x).
\end{aligned}
\tag{2.46}
$$

□

Corollary 2.20. *For $n \geq 0$ one has*

$$
\sum_{k=0}^{n} \binom{n}{k} p^k(x) q^{n-k}(x) L_{p,q,k}(x) = L_{p,q,2n}(x). \tag{2.47}
$$

Theorem 2.21. *For $n \geq 1$, one has*

$$
F_{p,q,n-1}(x) F_{p,q,n+1}(x) - F_{p,q,n}^2(x) = (-1)^n q^{n-1}(x). \tag{2.48}
$$

Proof. We will prove the theorem by mathematical induction on n. Since

$$
\begin{aligned}
F_{p,q,0}(x) F_{p,q,1}(x) - F_{p,q,1}^2(x) &= 0 p(x) - 1^2 = 0 - 1 \\
&= (-1)^1 \left(q^0(x)\right),
\end{aligned}
\tag{2.49}
$$

the given statement is true when $n = 1$.

Now, we assume that it is true for an arbitrary positive integer k, that is,

$$
F_{p,q,k-1}(x) F_{p,q,k+1}(x) - F_{p,q,k}^2(x) = (-1)^k q^{k-1}(x). \tag{2.50}
$$

Then

$$
\begin{aligned}
F_{p,q,k}(x) F_{p,q,k+2}(x) - F_{p,q,k+1}^2(x) &= \frac{1}{p(x)}\left(F_{p,q,k+1}(x) - q(x) F_{p,q,k-1}(x)\right) \\
&\quad \times \left(p(x) F_{p,q,k+1}(x) + q(x) F_{p,q,k}(x)\right) - F_{p,q,k+1}^2(x) \\
&= \frac{1}{p(x)}\Big(q(x) F_{p,q,k+1}(x) F_{p,q,k}(x) + p(x) F_{p,q,k+1}^2(x) \\
&\quad - q^2(x) F_{p,q,k}(x) F_{p,q,k-1}(x) - p(x) q(x) F_{p,q,k-1}(x) F_{p,q,k+1}(x)\Big) \\
&\quad - F_{p,q,k+1}^2(x)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) + F^2_{p,q,k+1}(x) - \frac{q^2(x)}{p(x)} F_{p,q,k}(x) F_{p,q,k-1}(x) \\
&\quad - q(x) F_{p,q,k-1}(x) F_{p,q,k+1}(x) - F^2_{p,q,k+1}(x) \\
&= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) - \frac{q^2(x)}{p(x)} F_{p,q,k}(x) F_{p,q,k-1}(x) \\
&\quad - q(x) F_{p,q,k-1}(x) F_{p,q,k+1}(x).
\end{aligned} \tag{2.51}$$

By assumption we have

$$\begin{aligned}
F_{p,q,k}(x) F_{p,q,k+2}(x) - F^2_{p,q,k+1}(x) &= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) \\
&\quad - \frac{q^2(x)}{p(x)} F_{p,q,k}(x) F_{p,q,k-1}(x) - q(x)\left((-1)^k q^k(x) + F^2_{p,q,k}(x)\right) \\
&= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) - \frac{q^2(x)}{p(x)} F_{p,q,k}(x) F_{p,q,k-1}(x) \\
&\quad - q(x) F^2_{p,q,k}(x) - (-1)^k q^k(x) \\
&= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) - q(x) F_{p,q,k}(x) \\
&\quad \times \left(\frac{q(x)}{p(x)} F_{p,q,k-1}(x) + F_{p,q,k}(x)\right) + (-1)^{k+1} q^k(x) \\
&= \frac{q(x)}{p(x)} F_{p,q,k+1}(x) F_{p,q,k}(x) \\
&\quad - q(x) F_{p,q,k}(x) \frac{1}{p(x)} F_{p,q,k+1}(x) + (-1)^{k+1} q^k(x) \\
&= (-1)^{k+1} q^k(x).
\end{aligned} \tag{2.52}$$

Thus the formula works for $n = k + 1$. So by mathematical induction, the statement is true for every integer $n \geq 1$. □

3. The Infinite (p, q)-Fibonacci and (p, q)-Lucas Polynomial Matrix

In this section we define a new matrix which we call (p, q)-Fibonacci polynomials matrix. The infinite (p, q)-Fibonacci polynomials matrix

$$\mathcal{F}(x) = \left[F_{p,q,i,j}(x)\right] \tag{3.1}$$

is defined as follows:

$$\mathcal{F}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ p(x) & 1 & 0 & 0 & \vdots \\ p^2(x)+q(x) & p(x) & 1 & 0 & \vdots \\ p^3(x)+2p(x)q(x) & p^2(x)+q(x) & p(x) & 1 & \\ & \cdots & & & \cdots \end{bmatrix} = \left(g_{\mathcal{F}(x)}(t), f_{\mathcal{F}(x)}(t)\right). \tag{3.2}$$

The matrix $\mathcal{F}(x)$ is an element of the set of Riordan matrices. Since the first column of $\mathcal{F}(x)$ is

$$\left(1, p(x), p^2(x)+q(x), p^3(x)+2p(x)q(x), p^4(x)+3p^2(x)q(x)+q^2(x), \dots\right)^T, \tag{3.3}$$

then it is obvious that $g_{\mathcal{F}(x)}(t) = \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n = 1/(1-p(x)t-q(x)t^2)$. In the matrix $\mathcal{F}(x)$ each entry has a rule with the upper two rows, that is,

$$F_{p,q,n+1,j}(x) = p(x)F_{p,q,n,j}(x) + q(x)F_{p,q,n-1,j}(x). \tag{3.4}$$

Then $f_F(t) = t$, that is,

$$\begin{aligned} \mathcal{F}(x) &= \left(g_{\mathcal{F}(x)}(t), f_{\mathcal{F}(x)}(t)\right) \\ &= \left(\frac{1}{1-p(x)t-q(x)t^2}, t\right); \end{aligned} \tag{3.5}$$

hence $\mathcal{F}(x)$ is in $\mathcal{R}$.

Similarly we can define the (p,q)-Lucas polynomials matrix. The infinite (p,q)-Lucas polynomials matrix

$$\mathcal{L}(x) = \left[L_{p,q,i,j}(x)\right] \tag{3.6}$$

can be written as

$$\begin{aligned} \mathcal{L}(x) &= \left(g_{\mathcal{L}(x)}(t), f_{\mathcal{L}(x)}(t)\right) \\ &= \left(\frac{2-p(x)t}{1-p(x)t-q(x)t^2}, t\right). \end{aligned} \tag{3.7}$$

In this section we give two factorizations of Pascal Matrix involving the (p,q)-Fibonacci polynomial matrix. For these factorizations we need to define two matrices. Firstly we define an infinite matrix $M(x) = (m_{ij}(x))$ as follows:

$$m_{ij}(x) = \binom{i-1}{j-1} - p(x)\binom{i-2}{j-1} - q(x)\binom{i-3}{j-1}. \tag{3.8}$$

We have the infinite matrix $M(x)$ as follows:

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 1-p(x) & 1 & 0 & 0 & \\ 1-p(x)-q(x) & 2-p(x) & 1 & 0 & \vdots \\ 1-p(x)-q(x) & 3-2p(x)-q(x) & 3-p(x) & 1 & \\ & \cdots & & & \end{bmatrix}. \tag{3.9}$$

Now we can give the first factorization of the infinite Pascal matrix via the infinite (p, q)-Fibonacci polynomial matrix and the infinite matrix $M(x)$ defined in (3.8) by the following theorem.

Theorem 3.1. *Let $M(x)$ be the infinite matrix as in* (3.8) *and $\mathcal{F}(x)$ be the infinite (p, q)-Fibonacci polynomial matrix; then,*

$$P(x) = \mathcal{F}(x) * M(x), \tag{3.10}$$

where P is the usual Pascal matrix.

Proof. From the definitions of the infinite Pascal matrix and the infinite (p, q)-Fibonacci polynomial matrix we have the following Riordan representations:

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right), \qquad \mathcal{F}(x) = \left(\frac{1}{1-p(x)t-q(x)t^2}, t\right). \tag{3.11}$$

Now we can find the Riordan representation of the infinite matrix

$$M(x) = \left(g_{M(x)}(t), f_{M(x)}(t)\right) \tag{3.12}$$

as follows:

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 1-p(x) & 1 & 0 & 0 & \\ 1-p(x)-q(x) & 2-p(x) & 1 & 0 & \vdots \\ 1-p(x)-q(x) & 3-2p(x)-q(x) & 3-p(x) & 1 & \\ & \cdots & & & \end{bmatrix}. \tag{3.13}$$

From the first column of the matrix $M(x)$ we obtain

$$g_{M(x)}(t) = \frac{1-p(x)t-q(x)t^2}{1-t}. \tag{3.14}$$

From the rule of the matrix $M(x)$, $f_{M(x)}(t) = t/(1-t)$. Thus

$$M(x) = \left(\frac{1-p(x)t-q(x)t^2}{1-t}, \frac{t}{1-t}\right). \tag{3.15}$$

Finally by the Riordan representations of the matrices $\mathcal{F}(x)$ and $M(x)$ we complete the proof. □

Now we define the $n \times n$ matrix $R(x) = (r_{ij}(x))$ as follows:

$$(r_{ij}(x)) = \binom{i-1}{j-1} - p(x)\binom{i-1}{j} - q(x)\binom{i-1}{j+1}. \tag{3.16}$$

We have the infinite matrix $R(x)$ as follows.

$$R(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 1-p(x) & 1 & 0 & 0 & \\ 1-2p(x)-q(x) & 2-p(x) & 1 & 0 & \vdots \\ 1-3p(x)-3q(x) & 3-3p(x)-q(x) & 3-p(x) & 1 & \\ & \cdots & & & \end{bmatrix}. \tag{3.17}$$

Now we can give second factorization of Pascal matrix via the infinite (p,q)-Fibonacci polynomial matrix by the following corollary.

Corollary 3.2. *Let $R(x)$ be the matrix as in* (3.16)*. Then*

$$P = R(x) * \mathcal{F}(x). \tag{3.18}$$

We can find the inverses of the matrices by using the Riordan representations of the matrices easily.

Corollary 3.3. *One has*

$$\begin{gathered} \mathcal{F}^{-1}(x) = \left(1-p(x)t-q(x)t^2, t\right) \\ M^{-1}(x) = \left(\frac{1+t}{1+(2-p(x))t-(1-p(x)-q(x))t^2}, \frac{t}{1+t}\right) \\ \mathcal{L}^{-1} = \left(\frac{1-p(x)t-q(x)t^2}{2-p(x)t}, t\right). \end{gathered} \tag{3.19}$$

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