Research Article

The System of Mixed Equilibrium Problems for Quasi-Nonexpansive Mappings in Hilbert Spaces

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Received 19 February 2012; Accepted 23 March 2012

Academic Editor: Rudong Chen

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We first introduce the iterative procedure to approximate a common element of the fixed-point set of two quasinonexpansive mappings and the solution set of the system of mixed equilibrium problem SMEP in a real Hilbert space. Next, we prove the weak convergence for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed points of asymptotic nonspreading mapping and asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let $C$ be a nonempty closed convex subset of $H$, and let $T$ be a mapping of $C$ into $H$, then $T : C \to H$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasinonexpansive if $\| Tx - y \| \leq \| x - y \|$ for all $x \in C$ and $y \in F(T) := \{ x \in C : Tx = x \}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is a closed and convex set [1]. A mapping $T : C \to C$ is said to be firmly nonexpansive [2] if

$$
\| Tx - Ty \|^2 \leq \langle x - y, Tx - Ty \rangle,
$$

(1.1)

for all $x, y \in C$, and it is an important example of nonexpansive mappings in a Hilbert space.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, and let $F : C \times C \to \mathbb{R}$ be an equilibrium bifunction, that is, $F(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x \in C$ such that

$$
F(x, y) + \varphi(y) - \varphi(x) \geq 0 \quad \forall y \in C.
$$

(1.2)
Denote the set of solution of (1.2) by MEP\((F, \varphi)\). In particular, if \(\varphi = 0\), this problem reduces to the equilibrium problem, which is to find \(x \in C\) such that

\[
F(x, y) \geq 0 \quad \forall y \in C. \tag{1.3}
\]

The set of solution of (1.3) is denoted by EP\(F\).

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min-Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [3] and Moudafi [4]. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3).

Let \(F_1, F_2 : C \times C \rightarrow \mathbb{R}\) be two monotone bifunctions and \(\mu > 0\) is constant. In 2009, Moudafi [5] introduced an alternating algorithm for approximating a solution of the system of equilibrium problems, finding \((x', y') \in C \times C\) such that

\[
\begin{cases}
F_1(x', z) + \frac{1}{\mu} \langle y' - x', x' - z \rangle \geq 0, & \forall z \in C, \\
F_2(y', z) + \frac{1}{\mu} \langle x' - y', y' - z \rangle \geq 0, & \forall z \in C.
\end{cases} \tag{1.4}
\]

For such mappings \(F_1\) and \(F_2\) and two given positive constants \(\lambda, \mu > 0\), Plubtieng and Sombut [6] considered the following system of mixed equilibrium problem, finding \((x', y') \in C \times C\) such that

\[
\begin{cases}
F_1(x', z) + \varphi(z) - \varphi(x') + \frac{1}{\lambda} \langle y' - x', x' - z \rangle \geq 0, & \forall z \in C, \\
F_2(y', z) + \varphi(z) - \varphi(y') + \frac{1}{\mu} \langle x' - y', y' - z \rangle \geq 0, & \forall z \in C.
\end{cases} \tag{1.5}
\]

In particular, if \(\lambda = \mu\) and \(\varphi \equiv 0\), then problem (SMEP) reduces to (SEP). Furthermore, Plubtieng and Sombut [6] introduced the following iterative procedure to approximate a common element of the fixed-point set of a quasi-nonexpansive mapping \(T\) and the solution set of (SMEP) in a Hilbert space \(H\). Let \(\{x_n\}, \{y_n\}\), and \(\{u_n\}\) be given by

\[
x_1 \in C \text{ chosen arbitrary},
\]

\[
u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\mu} \langle z - u_n, u_n - x_n \rangle \geq 0, \quad \forall z \in C, \tag{1.6}
\]

\[
y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - u_n \rangle \geq 0, \quad \forall z \in C,
\]

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Ty_n, \quad \forall n \in \mathbb{N},
\]

where \(\{\alpha_n\} \subseteq [a, b]\) for some \(a, b \in (0, 1)\) and satisfying appropriate conditions. The weak convergence theorems are obtained in a real Hilbert space.
On the other hand, in 1953, Mann [7] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping $T$ in a Hilbert space $H$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

(1.7)

where the initial point $x_0$ is taken in $C$ arbitrarily, and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

For two nonexpansive mappings $T_1, T_2$ of $C$ into itself, Moudafi [4] studied weak convergence theorems in the following iterative process:

$$x_0 \in C \text{ chosen arbitrary},$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n T_1 x_n + (1 - \beta_n)T_2 x_n),$$

(1.8)

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$ and $F(T_1) \cap F(T_2) \neq \emptyset$.

Recently, Iemoto and Takahashi [8] also considered this iterative procedure for $T_1$ is a nonexpansive mapping and $T_2 : C \to C$ is a nonspreading mapping. Very recently, Kim [9] studied the weak and strong convergence for the Moudafi’s iterative scheme (1.8) of two quasi-nonexpansive mappings.

In this paper, inspired and motivated by Plubtieng and Sombut [6], Moudafi [4], Iemoto and Takahashi [8], and Kim [9], we first introduce the iterative procedure to approximate a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of SMEP in a real Hilbert space. Next, we prove the weak convergence theorem for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed point of asymptotic nonspreading mapping and asymptotic $TJ$ mapping and the solution set of SMEP in a real Hilbert space.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{R}$ be the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let $C$ be a closed convex subset of $H$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

From [10], for each $x, y \in H$ and $\lambda \in [0, 1]$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$  

(2.1)

For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C.$$  

(2.2)

$P_C$ is called the metric projection of $H$ onto $C$. It is well know that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \quad \forall x, y \in H.$$  

(2.3)
Moreover, \( P_C x \) is characterized by the following properties: \( P_C x \in C \),

\[
\langle x - P_C x, P_C y - y \rangle \geq 0, \tag{2.4}
\]

\[
\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C y \|^2 \quad \forall x \in H, \ y \in C.
\]

Further, for all \( x \in H \) and \( y \in C \), \( y = P_C x \) if and only if \( \langle x - y, y - z \rangle \geq 0 \), for all \( z \in C \).

**Lemma 2.1** (see [11]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( P \) be the metric projection of \( H \) onto \( C \), and let \( \{x_n\}_{n \in \mathbb{N}} \) be in \( H \). If

\[
\| x_{n+1} - u \| \leq \| x_n - u \|, \tag{2.5}
\]

for all \( u \in C \) and \( n \in \mathbb{N} \). Then, \( \{P_C x_n\} \) converges strongly to an element of \( C \).

**Theorem 2.2** (Opial’s theorem, [10]). Let \( H \) be a real Hilbert space, and suppose that \( x_n \rightharpoonup x \), then

\[
\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|, \tag{2.6}
\]

for all \( y \in H \) with \( x \neq y \).

All Hilbert space and \( L^p \) \((1 < p < \infty)\) satisfy Opial’s condition, while \( L^p \) with \( 1 < p \neq 2 < \infty \) do not.

For solving the mixed equilibrium problem for an equilibrium bifunction \( F : C \times C \to \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:

(A1) \( F(x, x) = 0 \) for all \( x \in C \),

(A2) \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in C \),

(A3) for each \( y \in C \), \( x \mapsto F(x, y) \) is weakly upper semicontinuous,

(A4) for each \( x \in C \), \( y \mapsto F(x, y) \) is convex and semicontinuous.

The following lemma appears implicitly in [3, 12].

**Lemma 2.3** (see [3]). Let \( C \) be a nonempty closed convex subset of \( H \), and let \( F : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4). Let \( r > 0 \) and \( x \in H \), then there exists \( z \in C \) such that

\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \tag{2.7}
\]

The following lemma was also given in [12].

**Lemma 2.4** (see [12]). Let \( C \) be a nonempty closed convex subset of \( H \) and let \( F : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4), then, for any \( r > 0 \) and \( x \in H \), define a mapping \( T_r \) as follows:

\[
T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall \ y \in C \right\}, \tag{2.8}
\]
for all \( z \in H, r \in \mathbb{R} \). Then the following hold:

(i) \( T_r \) is single valued,

(ii) \( T_r \) is firmly nonexpansive, that is,

\[
\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H;
\]

(iii) \( F(T_r) = \text{EP}(F) \),

(iv) \( \text{EP}(F) \) is closed and convex.

We note that Lemma 2.4 is equivalent to the following lemma.

Lemma 2.5 (see [6]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( F : C \times C \to \mathbb{R} \) be an equilibrium bifunction satisfying (A1)–(A4) and let \( \varphi : C \to \mathbb{R} \) be a lower semicontinuous and convex functional. For each \( r > 0 \) and \( x \in H \), define a mapping

\[
S_r(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle y - x, z - y \rangle \geq 0, \forall z \in C \right\}, \quad \forall x \in H.
\]

Then, the following results hold:

(i) for each \( x \in C \), \( S_r(x) \neq \emptyset \),

(ii) \( S_r \) is single valued,

(iii) \( S_r \) is firmly nonexpansive, that is, for any \( x, y \in H \),

\[
\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle,
\]

(iv) \( F(S_r) = \text{MEP}(F, \varphi) \),

(v) \( \text{MEP}(F, \varphi) \) is closed and convex.

3. Main Results

In this section, we prove the weak convergence for approximating a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of the system of mixed equilibrium problems in a Hilbert space.

To begin with, let us state and proof the following characterizations of the solution set of GMEP.

Lemma 3.1. Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( F_1 \) and \( F_2 \) be two mappings from \( C \times C \to \mathbb{R} \) satisfying (A1)–(A4), and let \( S_{1, \lambda} \) and \( S_{2, \mu} \) be defined as in Lemma 2.5 associated to \( F_1 \) and \( F_2 \), respectively. For given \( x', y' \in C \), \( (x', y') \) is a solution of problem (1.5) if and only if \( x' \) is a fixed point of the mapping \( G : C \to C \) defined by

\[
G(x) = S_{1, \lambda}(S_{2, \mu}x), \quad \forall x \in C,
\]

where \( y' = S_{2, \mu}x' \).
Throughout this paper, we denote the set of fixed points of $G$ by $\Omega$.

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F_1$ and $F_2$ be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $r, \lambda > 0$ and $S_{1,r}$ and $S_{2,\lambda}$ be defined as in Lemma 2.5 associated to $F_1$ and $F_2$, respectively. Let $T_i : C \to C$, $i = 1, 2$, be two quasi-nonexpansive mappings such that $I - T_i$ are demiclosed at zero, that is, if $\{w_n\} \subset C$, $w_n \to w$, and $(I - T_i)w_n \to 0$, then $w \in F(T_i)$, with $F(T_1) \cap F(T_2) \cap \Omega \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$, and $\{u_n\}$, be given by

\[ x_1 \in C \text{ chosen arbitrary}, \]

\[ u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\lambda}(z - u_n, u_n - x_n) \geq 0, \quad \forall z \in C, \]

\[ y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}(z - y_n, y_n - u_n) \geq 0, \quad \forall z \in C, \]

\[ x_{n+1} = a_n x_n + (1 - a_n) (bnT_1 y_n + (1 - b_n) T_2 y_n), \quad \forall n \in \mathbb{N}, \]

where $\{a_n\}, \{b_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and satisfy

\[ \liminf_{n \to \infty} a_n (1 - a_n) > 0, \quad \liminf_{n \to \infty} b_n (1 - b_n) > 0, \]

then $x_n \to \bar{x} := \lim_{n \to \infty} P_{F(T_1) \cap F(T_2) \cap \Omega} x_n$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.5), where $\bar{y} = S_{2,\lambda} \bar{x}$.

**Proof.** Let $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, then $x^* = T_1 x^* = T_2 x^*$ and $x^* = S_{1,r} (S_{2,\lambda} x^*)$.

Putting $y^* = S_{2,\lambda} x^*$, $y_n = S_{1,r} u_n$, and $u_n = S_{2,\lambda} x_n$, we have

\[ \|y_n - x^*\| = \|S_{1,r} u_n - S_{1,r} y^*\| \leq \|u_n - y^*\| = \|S_{2,\lambda} x_n - S_{2,\lambda} x^*\| \leq \|x_n - x^*\|. \]
Next, we prove that
\[
\lim_{n \to \infty} \|x_n - x^*\| \text{ exists.} \tag{3.6}
\]

Since \(T_1\) and \(T_2\) are quasi-nonexpansive, we obtain that
\[
\|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 = \|b_n (T_1 y_n - x^*) + (1 - b_n) (T_2 y_n - x^*)\|^2
\]
\[
= b_n \|T_1 y_n - x^*\|^2 + (1 - b_n) \|T_2 y_n - x^*\|^2 - b_n (1 - b_n) \|T_1 y_n - T_2 y_n\|^2
\]
\[
\leq b_n \|T_1 y_n - x^*\|^2 + (1 - b_n) \|T_2 y_n - x^*\|^2
\]
\[
\leq b_n \|y_n - x^*\|^2 + (1 - b_n) \|y_n - x^*\|^2
\]
\[
= \|y_n - x^*\|^2
\]
\[
\leq \|x_n - x^*\|^2,
\]
which gives that
\[
\|x_{n+1} - x^*\|^2 = \|a_n x_n + (1 - a_n) (b_n T_1 y_n + (1 - b_n) T_2 y_n) - x^*\|^2
\]
\[
= \|a_n (x_n - x^*) + (1 - a_n) (b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*)\|^2
\]
\[
= a_n \|x_n - x^*\|^2 + (1 - a_n) \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 - a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2
\]
\[
\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|x_n - x^*\|^2
\]
\[
- a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2
\]
\[
= \|x_n - x^*\|^2 - a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2
\]
\[
\leq \|x_n - x^*\|^2.
\]

Hence, \(\{\|x_n - x^*\|\}\) is a nonincreasing sequence, and hence, \(\lim_{n \to \infty} \|x_n - x^*\|\) exists. This implies that \(\{x_n\}, \{y_n\}, \{u_n\}, \{T_1 y_n\},\) and \(\{T_2 y_n\}\) are bounded. From (3.8), we have
\[
a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.9}
\]

Since \(\liminf_{n \to \infty} a_n (1 - a_n) > 0\), this implies that
\[
\lim_{n \to \infty} \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\| = 0. \tag{3.10}
\]
Furthermore, since $0 < a \leq a_n \leq b < 1$, we have
\begin{equation}
\|x_{n+1} - x_n\| = \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n) - x_n\|
= \|(1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n) - (1 - a_n)x_n\|
= (1 - a_n)\|b_n T_1 y_n + (1 - b_n)T_2 y_n - x_n\|
\leq (1 - a)\|b_n T_1 y_n + (1 - b_n)T_2 y_n - x_n\|.
\end{equation}

From (3.10), we conclude that
\begin{equation}
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\end{equation}

From (3.7), we have
\begin{align*}
b_n(1 - b_n)\|T_1 y_n - T_2 y_n\|^2 &= b_n\|T_1 y_n - x^*\|^2 + (1 - b_n)\|T_2 y_n - x^*\|^2 \\
&- \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|^2 \\
&\leq b_n\|y_n - x^*\|^2 + (1 - b_n)\|y_n - x^*\|^2 \\
&- \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|^2 \\
&= \|y_n - x^*\|^2 - \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|^2 \\
&= (\|x_n - x^*\| - \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|) \\
&\times (\|x_n - x^*\| + \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|) \\
&\leq M(\|x_n - x^*\| - \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|) \\
&\leq M(\|x_n - (b_n T_1 y_n + (1 - b_n)T_2 y_n)\|)
\end{align*}

where $M$ is a constant satisfying $M \geq \sup_{n \geq 1} (\|x_n - x^*\| + \|b_n T_1 y_n + (1 - b_n)T_2 y_n - x^*\|)$. Again from (3.10), we conclude that
\begin{equation}
\lim_{n \to \infty} b_n(1 - b_n)\|T_1 y_n - T_2 y_n\| = 0.
\end{equation}

Using $\lim \inf_{n \to \infty} b_n(1 - b_n) > 0$, we have
\begin{equation}
\lim_{n \to \infty} \|T_1 y_n - T_2 y_n\| = 0.
\end{equation}

Now, we prove that
\begin{equation}
\lim_{n \to \infty} \|x_{n+1} - T_1 y_n\| = 0, \quad \lim_{n \to \infty} \|x_{n+1} - T_2 y_n\| = 0.
\end{equation}
We observe that

\[
\|x_{n+1} - T_1 y_n\| = \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n) - T_1 y_n\|
\]

\[
= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n) - (a_n + (1 - a_n)T_1 y_n)\|
\]

\[
= \|a_n(x_n - T_1 y_n) + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n - T_1 y_n)\|
\]

\[
= \|a_n(x_n - T_1 y_n) + (1 - a_n)(b_n T_1 y_n + T_2 y_n - b_n T_2 y_n - T_1 y_n)\|
\]

\[
= \|a_n(x_n - T_1 y_n) + (1 - a_n)((1 - b_n)T_2 y_n - (1 - b_n)T_1 y_n)\| \tag{3.17}
\]

\[
\leq a_n\|x_n - T_1 y_n\| + (1 - a_n)(1 - b_n)\|T_2 y_n - T_1 y_n\|
\]

\[
= a_n\|x_n - x_{n+1} + x_{n+1} - T_1 y_n\| + (1 - a_n)(1 - b_n)\|T_2 y_n - T_1 y_n\|
\]

\[
\leq a_n\|x_n - x_{n+1}\| + a_n\|x_{n+1} - T_1 y_n\| + (1 - a_n)(1 - b_n)\|T_2 y_n - T_1 y_n\|,
\]

which gives that

\[
(1 - a_n)\|x_{n+1} - T_1 y_n\| \leq a_n\|x_n - x_{n+1}\| + (1 - a_n)(1 - b_n)\|T_2 y_n - T_1 y_n\|. \tag{3.18}
\]

Since \(0 < a \leq a_n \leq b < 1\), we have

\[
a_n(1 - a_n)\|x_{n+1} - T_1 y_n\| \leq (1 - a_n)\|x_{n+1} - T_1 y_n\|
\]

\[
\leq a_n\|x_n - x_{n+1}\| + (1 - a_n)(1 - b_n)\|T_2 y_n - T_1 y_n\|. \tag{3.19}
\]

Using (3.12) and (3.15), we conclude that

\[
\lim_{n \to \infty} a_n(1 - a_n)\|x_{n+1} - T_1 y_n\| = 0, \tag{3.20}
\]

which gives that

\[
\lim_{n \to \infty}\|x_{n+1} - T_1 y_n\| = 0, \tag{3.21}
\]

since \(\lim \inf_{n \to \infty} a_n(1 - a_n) > 0\). Similarly, we have

\[
\|x_{n+1} - T_2 y_n\| = \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n) - T_2 y_n\|
\]

\[
= \|a_n(x_n - T_2 y_n) + (1 - a_n)(b_n T_1 y_n + (1 - b_n)T_2 y_n - T_2 y_n)\|
\]

\[
= \|a_n(x_n - T_2 y_n) + (1 - a_n)b_n(T_1 y_n - T_2 y_n)\|
\]

\[
\leq a_n\|x_n - T_2 y_n\| + (1 - a_n)b_n\|T_1 y_n - T_2 y_n\| \tag{3.22}
\]

\[
= a_n\|x_n - x_{n+1} + x_{n+1} - T_2 y_n\| + (1 - a_n)b_n\|T_2 y_n - T_1 y_n\|
\]

\[
\leq a_n\|x_n - x_{n+1}\| + a_n\|x_{n+1} - T_2 y_n\| + (1 - a_n)b_n\|T_2 y_n - T_1 y_n|,
\]
which implies that

\[(1 - a_n) \|x_{n+1} - T_2 y_n\| \leq a_n \|x_n - x_{n+1}\| + (1 - a_n)b_n \|T_2 y_n - T_1 y_n\|. \tag{3.23}\]

Thus, we have

\[a_n(1 - a_n) \|x_{n+1} - T_2 y_n\| \leq (1 - a_n) \|x_{n+1} - T_2 y_n\| \]
\[\leq a_n \|x_n - x_{n+1}\| + (1 - a_n)b_n \|T_2 y_n - T_1 y_n\| \]
\[\leq b \|x_n - x_{n+1}\| + (1 - a_n) \|T_2 y_n - T_1 y_n\|. \tag{3.24}\]

Hence, \(\lim_{n \to \infty} a_n(1 - a_n) \|x_{n+1} - T_2 y_n\| = 0\). Since \(\liminf_{n \to \infty} a_n(1 - a_n) > 0\), we have

\[\lim_{n \to \infty} \|x_{n+1} - T_2 y_n\| = 0. \tag{3.25}\]

Next, we prove that

\[\lim_{n \to \infty} \|y_n - T_1 y_n\| = 0, \quad \lim_{n \to \infty} \|y_n - T_2 y_n\| = 0. \tag{3.26}\]

Since \(S_{1,r}\) and \(S_{2,1}\) are firmly nonexpansive, it follows that

\[\|u_n - y^*\|^2 = \|S_{2,1} x_n - S_{2,1} x^*\|^2 \leq \langle S_{2,1} x_n - S_{2,1} x^*, x_n - x^* \rangle = \langle u_n - y^*, x_n - x^* \rangle, \tag{3.27}\]

which gives that

\[\|(u_n - y^*) - (x_n - x^*)\|^2 = \|u_n - y^*\|^2 - 2\langle u_n - y^*, x_n - x^* \rangle + \|x_n - x^*\|^2 \]
\[\leq \|u_n - y^*\|^2 - 2\|u_n - y^*\|^2 + \|x_n - x^*\|^2 \]
\[= -\|u_n - y^*\|^2 + \|x_n - x^*\|^2. \tag{3.28}\]

This implies that

\[\|u_n - y^*\|^2 \leq \|x_n - x^*\|^2 - \|(u_n - y^*) - (x_n - x^*)\|^2. \tag{3.29}\]

By the convexity of \(\| \cdot \|^2\), we have

\[\|x_{n+1} - x^*\|^2 = \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x^*\|^2 \]
\[= \|a_n(x_n - x^*) + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*)\|^2. \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|b_nT_1y_n + (1 - b_n)T_2y_n - x^*\|^2 \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|y_n - x^*\|^2 \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|u_n - y^*\|^2 \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\left(\|x_n - x^*\|^2 - \|u_n - y^*\|^2 - \|u_n - y^*\|^2 - \|x_n - x^*\|^2\right) \]
\[ = \|x_n - x^*\|^2 - (1 - a_n)\|u_n - y^*\|^2 - \|x_n - x^*\|^2. \]
\[ (3.30) \]

Thus,
\[ (1 - a_n)\|u_n - y^*\| - \|x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \]
\[ (3.31) \]

Since 0 < \( a \leq a_n \leq b < 1 \), we have
\[ (1 - b)\|u_n - y^*\| - \|x_n - x^*\|^2 \leq (1 - a_n)\|u_n - y^*\| - \|x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \]
\[ (3.32) \]

Since \( \lim_{n \to \infty} \|x_n - x^*\| \) exists, we have
\[ \lim_{n \to \infty} \|u_n - x_n + x^* - y^*\| = 0. \]
\[ (3.33) \]

Similarly, we have
\[ \|y_n - x^*\|^2 = \|S_{1,r_n}u_n - S_{1,r_n}y^*\|^2 \leq \langle S_{1,r_n}u_n - S_{1,r_n}y^*, u_n - y^* \rangle = \langle y_n - x^*, u_n - x^* \rangle, \]
\[ (3.34) \]

which gives that
\[ \| (y_n - x^*) - (u_n - y^*) \|^2 = \|y_n - x^*\|^2 - 2\langle y_n - x^*, u_n - x^* \rangle + \|u_n - y^*\|^2 \]
\[ \leq \|y_n - x^*\|^2 - 2\|y_n - x^*\|^2 + \|u_n - y^*\|^2 \]
\[ = -\|y_n - x^*\|^2 + \|u_n - y^*\|^2. \]
\[ (3.35) \]

This implies that
\[ \|y_n - x^*\|^2 \leq \|u_n - y^*\|^2 - \|y_n - u_n - x^* + y^*\|^2. \]
\[ (3.36) \]

By the convexity of \( \| \cdot \|^2 \), we have
\[ \|x_{n+1} - x^*\|^2 = \|a_nx_n + (1 - a_n)(b_nT_1y_n + (1 - b_n)T_2y_n) - x^*\|^2 \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|y_n - x^*\|^2 \]
\[ \leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|y_n - x^*\|^2 \]
\[ \begin{align*}
&\leq a_n\|x_n - x^*\|^2 + (1 - a_n)\left(\|u_n - y^*\|^2 - \|y_{n-1} - u_n - x^* + y^*\|^2\right) \\
&\leq a_n\|x_n - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2 - (1 - a_n)\|y_{n-1} - u_n - x^* + y^*\|^2 \\
&= \|x_n - x^*\|^2 - (1 - a_n)\|y_n - x^*\| - (u_n - y^*)\|^2. \\
&\text{Thus,} \\
&(1 - a_n)\|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \\
&\text{Since } 0 < a \leq a_n \leq b < 1, \text{ we have} \\
&(1 - b)\|y_n - u_n - x^* + y^*\|^2 \leq (1 - a_n)\|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \\
&\text{Since } \lim_{n \to \infty}\|x_n - x^*\| \text{ exists, we have} \\
&\lim_{n \to \infty}\|y_n - u_n - x^* + y^*\| = 0. \\
&\text{Hence,} \\
&\| (b_nT_1y_n + (1 - b_n)T_2y_n) - y_n \| \\
&= \| (b_nT_1y_n + (1 - b_n)T_2y_n) - x_n + x_n - u_n + u_n - y^* + y^* - x^* + x^* - y_n \| \\
&\leq \| (b_nT_1y_n + (1 - b_n)T_2y_n) - x_n \| + \|x_n - u_n - x^* + y^*\| + \|u_n - y_n + x^* - y^*\|. \\
&\text{It follows from (3.10), (3.33), and (4.40) that} \\
&\lim_{n \to \infty}\| (b_nT_1y_n + (1 - b_n)T_2y_n) - y_n \| = 0, \\
&\text{from which it follows that} \\
&\|y_n - x_n\| = \|y_n - (b_nT_1y_n + (1 - b_n)T_2y_n) + (b_nT_1y_n + (1 - b_n)T_2y_n) - x_n\| \\
&\leq \|y_n - (b_nT_1y_n + (1 - b_n)T_2y_n)\| + \| (b_nT_1y_n + (1 - b_n)T_2y_n) - x_n\| \to 0 \text{ as } n \to \infty, \\
&\text{that is,} \\
&\lim_{n \to \infty}\|y_n - x_n\| = 0. \\
&\text{Thus,} \\
&\|y_n - T_1y_n\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - T_1y_n\| \to 0. \\
\end{align*} \]
Similarly, we have \(\|y_n - T_2y_n\| \to 0\). Since \(\{y_n\}\) is bounded sequence, there exists a subsequence \(\{y_{n_i}\}\) of \(\{y_n\}\) such that \(y_{n_i} \to \bar{x}\) as \(i \to \infty\). Since \(T_1\) and \(T_2\) are demiclosed at 0, we conclude that \(\bar{x} \in F(T_1) \cap F(T_2)\). Let \(G\) be a mapping which is defined as in Lemma 3.1. Thus, we have
\[
\|y_n - G(y_n)\| = \|S_{1,T}x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \leq \|x_n - y_n\|, 
\] (3.46)
and hence,
\[
\|x_n - G(x_n)\| = \|x_n - y_n + y_n - G(y_n) + G(y_n) - G(x_n)\|
\leq \|x_n - y_n\| + \|y_n - G(y_n)\| + \|G(y_n) - G(x_n)\|
\leq \|x_n - y_n\| + \|x_n - y_n\| + \|y_n - x_n\|
= 3\|x_n - y_n\| \to 0.
\] (3.47)

This together with \(x_n \rightharpoonup \bar{x}\) implies that \(\bar{x} \in F(G) := \Omega\), if \(\{y_{n_i}\}\) is another subsequence of \(\{y_n\}\) such that \(y_{n_i} \to \hat{x}\) as \(i \to \infty\). Since \(T_1\) and \(T_1\) are demiclosed at 0, we conclude that \(\hat{x} \in F(T_1) \cap F(T_2) \cap \Omega\). From \(x_n \rightharpoonup \bar{x}\) and \(x_{n_i} \rightharpoonup \hat{x}\), we will show that \(\bar{x} = \hat{x}\). Assume that \(\bar{x} \neq \hat{x}\).

Since \(\lim_{n \to \infty} \|x_n - x^*\|\) exists for all \(x^* \in F(T_1) \cap F(T_2) \cap \Omega\), by Opial’s Theorem 2.2, we have
\[
\lim_{n \to \infty} \|x_n - \bar{x}\| = \liminf_{i \to \infty} \|x_{n_i} - \bar{x}\| < \liminf_{i \to \infty} \|x_{n_i} - \hat{x}\| = \lim_{n \to \infty} \|x_n - \hat{x}\| = \liminf_{j \to \infty} \|x_{n_j} - \hat{x}\| < \liminf_{j \to \infty} \|x_{n_j} - \bar{x}\| = \lim_{n \to \infty} \|x_n - \bar{x}\|. 
\] (3.48)

This is a contradiction. Thus, we have \(\bar{x} = \hat{x}\). This implies that \(y_n \rightharpoonup \bar{x} \in F(T_1) \cap F(T_2) \cap \Omega\).

Since \(\|x_n - y_n\| \to 0\), we have \(x_n \rightharpoonup \bar{x}\). Put \(z_n = P_{F(T_1) \cap F(T_2) \cap \Omega}x_n\). Finally, we show that \(\bar{x} = \lim_{n \to \infty} z_n\). Now from (2.4) and \(\bar{x} \in F(T_1) \cap F(T_2) \cap \Omega\), we have
\[
\langle \bar{x} - z_n, z_n - x_n \rangle \geq 0. 
\] (3.49)

Since \(\|x_n - x^*\|\) is nonnegative and nonincreasing for all \(x^* \in F(T_1) \cap F(T_2) \cap \Omega\), it follows by Lemma 2.1 that \(\{z_n\}\) converges strongly to some \(\hat{x} \in F(T_1) \cap F(T_2) \cap \Omega\). By (3.49), we have
\[
\langle \bar{x} - \hat{x}, \bar{x} - \hat{x} \rangle \geq 0. 
\] (3.50)

Therefore, \(\bar{x} = \hat{x}\). \(\square\)
Setting $T := T_1 = T_2$ in Theorem 3.2, we have the following result.

**Corollary 3.3** (see [6]). Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $F_1$ and $F_2$ be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda, \mu > 0$, and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.5 associated to $F_1$ and $F_2$, respectively. Let $T : C \to C$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero and $F(T) \cap \Omega \neq \emptyset$. Suppose that $x_0 = x \in C$ and \{x_n, y_n, z_n\} are chosen so that
\[ z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C, \]
\[ y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, \quad \forall z \in C, \]
\[ x_{n+1} = a_n x_n + (1 - a_n) Ty_n, \]
for all $n \in \mathbb{N}$, where \{a_n\} $\subset$ [a, b] for some $a, b \in (0, 1)$, and satisfy $\liminf_{n \to \infty} a_n (1 - a_n) > 0$, then \{x_n\} converges weakly to $\bar{x} = \lim_{n \to \infty} P_{F(T) \cap \Omega} x_n$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.5), where $\bar{y} = S_{2,\mu} \bar{x}$.

Setting $F_1 = F_2 \equiv 0$, $\varphi \equiv 0$ in Theorem 3.2, we have the following result.

**Corollary 3.4** (see [9]). Let $H$ be a Hilbert space, let $C$ be a nonempty, closed, and convex subset of $H$, and let $T_1, T_2$ be two quasi-nonexpansive mappings of $C$ into itself such that $I - T_1, I - T_2$ are demiclosed at zero with $F(T_1) \cap F(T_2) \neq \emptyset$. For any $x_1 \in C$, let \{x_n\} be defined by
\[ x_{n+1} = (1 - a_n)x_n + a_n(b_n T_1 x_n + (1 - b_n) T_2 x_n), \]
where \{a_n\} and \{b_n\} are chosen so that
\[ \liminf_{n \to \infty} a_n (1 - a_n) > 0 \quad \liminf_{n \to \infty} b_n (1 - b_n) > 0, \]
then $x_n \rightharpoonup p \in F(T_1) \cap F(T_2)$.

### 4. Applications

In this section, we apply our results to approximate a common element of the set of common fixed points of an asymptotic nonspreading mapping and an asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space. We recall the following definitions. A mapping $T : C \to C$ is called nonspreading [13] if
\[ 2 \|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C. \]  

(4.1)

Furthermore, Takahashi and Yao [14] also introduced two nonlinear mappings in Hilbert spaces. A mapping $T : C \to C$ is called a $TJ - 1$ mapping [14] if
\[ 2 \|Tx - Ty\|^2 = \|x - y\|^2 + \|Tx - y\|^2, \]

(4.2)
for all \( x, y \in C \). A mapping \( T : C \to C \) is called a \( TJ - 2 \) \([14]\) mapping if

\[
3\|Tx - Ty\|^2 = 2\|Tx - y\|^2 + \|Ty - x\|^2,
\]

for all \( x, y \in C \). For these two nonlinear mappings, \( TJ - 1 \) and \( TJ - 2 \) mappings, Takahashi and Yao \([14]\) studied the existence results of fixed points in Hilbert spaces. Very recently, Lin et al. \([15]\) introduced the following definitions of new mappings.

**Definition 4.1.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). We say that \( T : C \to C \) is an asymptotic nonspreading mapping if there exist two functions \( \alpha : C \to [0, 2) \) and \( \beta : C \to [0, k], k < 2 \), such that

\[(A1) \ 2\|Tx - Ty\|^2 \leq \alpha(x)\|Tx - y\|^2 + \beta(x)\|Ty - x\|^2 \quad \text{for all} \ x, y \in C, \]

\[(A2) \ 0 < \alpha(x) + \beta(x) \leq 2 \quad \text{for all} \ x \in C. \]

**Remark 4.2.** The class of asymptotic nonspreading mappings contains the class of nonspreading mappings and the class of \( TJ - 2 \) mappings in a Hilbert space. Indeed, in Definition 4.1, we know that

(i) if \( \alpha(x) = \beta(x) = 1 \) for all \( x \in C \), then \( T \) is a nonspreading mapping,

(ii) if \( \alpha(x) = 4/3 \) and \( \beta(x) = 2/3 \) for all \( x \in C \), then \( T \) is a \( TJ - 2 \) mapping.

**Definition 4.3.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). We say \( T : C \to C \) is an asymptotic \( TJ \) mapping if there exist two functions \( \alpha : C \to [0, 2) \) and \( \beta : C \to [0, k], k < 2 \), such that

\[(B1) \ 2\|Tx - Ty\|^2 \leq \alpha(x)\|x - y\|^2 + \beta(x)\|Tx - y\|^2 \quad \text{for all} \ x, y \in C, \]

\[(B2) \ \alpha(x) + \beta(x) \leq 2 \quad \text{for all} \ x \in C. \]

**Remark 4.4.** The class of asymptotic \( TJ \) mappings contains the class of \( TJ - 1 \) mappings and the class of nonexpansive mappings in a Hilbert space. Indeed, in Definition 4.3, we know that

(i) if \( \alpha(x) = 2 \) and \( \beta(x) = 0 \) for each \( x \in C \), then \( T \) is a nonexpansive mapping,

(ii) if \( \alpha(x) = \beta(x) = 1 \) for each \( x \in C \), then \( T \) is a \( TJ - 1 \) mapping.

It is well known that the set \( F(T) \) of fixed points of a quasi-nonexpansive mapping \( T \) is a closed and convex set \([1]\). Hence, if \( T : C \to C \) is an asymptotic nonspreading mapping (resp., asymptotic \( TJ \) mapping) with \( F(T) \neq \emptyset \), then \( T \) is a quasi-nonexpansive mapping, and this implies that \( F(T) \) is a nonempty closed convex subset of \( C \).

**Theorem 4.5** (see \([15]\)). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and let \( T : C \to C \) be an asymptotic nonspreading mapping, then \( I - T \) is demiclosed at 0.

**Theorem 4.6** (see \([15]\)). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and let \( T : C \to C \) be an asymptotic \( TJ \) mapping, then \( I - T \) is demiclosed at 0.

Applying the above results, we have the following theorem.
Theorem 4.7. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F_1$ and $F_2$ be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $r, \lambda > 0$ and $S_{1,r}$ and $S_{2,\lambda}$ be defined as in Lemma 2.5 associated to $F_1$ and $F_2$, respectively. Let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $F(T_1) \cap F(T_2) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$, and $\{u_n\}$ be given by

$$
x_1 \in C \text{ chosen arbitrary},
$$

$$
u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\lambda} \langle z - u_n, u_n - x_n \rangle \geq 0, \quad \forall z \in C,
$$

$$
y_n \in C, \quad 2F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r} \langle z - y_n, y_n - u_n \rangle \geq 0, \quad \forall z \in C,
$$

$$
x_{n+1} = a_n x_n + (1 - a_n) (b_n T_1 y_n + (1 - b_n) T_2 y_n), \quad \forall n \in \mathbb{N},
$$

where $\{a_n\}, \{b_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and satisfying

$$
\liminf_{n \to \infty} a_n (1 - a_n) > 0, \quad \liminf_{n \to \infty} b_n (1 - b_n) > 0,
$$

then $x_n \rightharpoonup \bar{x} := \lim_{n \to \infty} P_{F(T_1) \cap F(T_2) \cap \Omega} x_n$, and $(\bar{x}, \bar{y})$ is a solution of problem (1.5), where $\bar{y} = S_{2,\lambda} \bar{x}$.

Setting $F_1 = F_2 := F$ and $\varphi \equiv 0$ in the above theorem, we have the following result.

Corollary 4.8. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be the bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $\mathcal{F} := F(T_1) \cap F(T_2) \cap EP(F) \neq \emptyset$. For given $u \in C$ and $r > 0$, let the sequences $\{x_n\}$ and $\{u_n\}$ be defined by

$$
x_1 \in C \text{ chosen arbitrary},
$$

$$
u_n \in C, \quad F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,
$$

$$
x_{n+1} = a_n x_n + (1 - a_n) (b_n T_1 u_n + (1 - b_n) T_2 u_n) \quad \forall n \in \mathbb{N},
$$

where $\{a_n\}, \{b_n\}$ are two sequences in $(0, 1)$ satisfying

$$
\liminf_{n \to \infty} a_n (1 - a_n) > 0, \quad \liminf_{n \to \infty} b_n (1 - b_n) > 0,
$$

then $x_n \rightharpoonup w$ for some $w \in \mathcal{F}$.

Setting $F_1 = F_2 \equiv 0$ and $\varphi \equiv 0$ in Theorem 4.7, we have the following result.
Corollary 4.9 (see [15]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mappings and asymptotic $T_i$-mappings. Let $\mathcal{F} := F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be defined by

\begin{align*}
x_1 & \in C \text{ chosen arbitrary,} \\
x_{n+1} & = a_n x_n + (1 - a_n)(b_n T_1 x_n + (1 - b_n) T_2 x_n). \quad (4.8)
\end{align*}

Assume that $\lim \inf_{n \to \infty} a_n (1 - a_n) > 0$ and $\lim \inf_{n \to \infty} b_n (1 - b_n) > 0$, then $x_n \rightharpoonup w$ for some $w \in \mathcal{F}$.

Acknowledgment

This paper is supported by the Centre of Excellence in Mathematics under the Commission on Higher Education, Ministry of Education, Thailand.

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