Research Article

Interval Oscillation Criteria for Super-Half-Linear Impulsive Differential Equations with Delay

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We study the following second-order super-half-linear impulsive differential equations with delay

\[ [r(t)\phi_\gamma(x'(t))']' + p(t)\phi_\gamma(x(x(t - \sigma))) + q(t)f(x(t - \sigma)) = e(t), \quad t \neq \tau_k, \]

\[ x(t^+) = a_kx(t), \quad x'(t^+) = b_kx'(t), \quad t = \tau_k, \quad k = 1, 2, \ldots, \]

where \( t \geq t_0 \in \mathbb{R} \), \( \phi_\gamma(u) = |u|^{\gamma-1}u \), \( \sigma \) is a nonnegative constant, \( \{\tau_k\} \) denotes the impulsive moments sequence with \( \tau_1 < \tau_2 < \cdots < \tau_k < \cdots \), \( \lim_{k \to \infty} \tau_k = \infty \), and \( \tau_{k+1} - \tau_k > \sigma \). By some classical inequalities, Riccati transformation, and two classes of functions, we give several interval oscillation criteria which generalize and improve some known results. Moreover, we also give two examples to illustrate the effectiveness and nonemptiness of our results.

1. Introduction

We consider the following second-order super-half-linear impulsive differential equations with delay

\[ [r(t)\phi_\gamma(x'(t))']' + p(t)\phi_\gamma(x(x(t - \sigma))) + q(t)f(x(t - \sigma)) = e(t), \quad t \neq \tau_k, \]

\[ x(t^+) = a_kx(t), \quad x'(t^+) = b_kx'(t), \quad t = \tau_k, \quad k = 1, 2, \ldots, \]

where \( t \geq t_0 \in \mathbb{R} \), \( \phi_\gamma(u) = |u|^{\gamma-1}u \), \( \sigma \) is a nonnegative constant, \( \{\tau_k\} \) denotes the impulsive moments sequence with \( \tau_1 < \tau_2 < \cdots < \tau_k < \cdots \), \( \lim_{k \to \infty} \tau_k = \infty \), and \( \tau_{k+1} - \tau_k > \sigma \).
Throughout the paper, we always assume that the following conditions hold:

1. \( r(t) \in C([t_0, \infty), (0, \infty)) \) and is nondecreasing, \( p(t), q(t), e(t) \in PLC([t_0, \infty), \mathbb{R}) \);
2. \( \gamma \) is a quotient of odd positive integers \( a_k, b_k \) are real constants satisfying \( b_k \geq a_k > 0, k = 1, 2, \ldots \);
3. \( f \in C(\mathbb{R}, \mathbb{R}) \), \( xf(x) > 0 \) and there exist some positive constants \( \eta \) and \( \alpha \) such that \( f(x)/q_{\alpha}(x) > \eta \) for all \( x \neq 0 \) with \( \alpha \geq \gamma \).

\section{Main Results}

Throughout the paper, we always assume that the following conditions hold:

\[ y'(t) + p(t) f(x(t)) = e(t), \quad t \geq t_0, \quad t \neq t_k, \quad k = 1, 2, \ldots, \]
\[ y(t_k^+) = a_k y(t_k), \quad y(t_k^-) = b_k y(t_k), \quad k = 1, 2, \ldots, \]
Lemma 2.1. Let \( k(s) := \max\{i : t_0 < \tau_i < s\} \), \( M_j := \max\{r(t) : t \in [c_j, d_j]\}, \quad j = 1, 2, \)
\[
\Omega_j(c_j, d_j) := \{w \in C^1 [c_j, d_j] \mid w(t) \neq 0, w(c_j) = w(d_j) = 0\}, \quad j = 1, 2.
\]
(2.1)

For two constants \( c, d \not\in \{\tau_k\} \) with \( c < d, k(c) < k(d) \) and a function \( \psi \in C([c, d], \mathbb{R}) \), we define an operator \( Q : C([c, d], \mathbb{R}) \to \mathbb{R} \) by
\[
Q^d[\psi] = \psi(\tau_{k(c)+1}) \frac{b_{k(c)+1}^r - a_{k(c)+1}^r}{a_{k(c)+1}^r (\tau_{k(c)+1} - c)^r} + \sum_{i=k(c)+2}^{k(d)} \psi(\tau_i) \frac{b_i^r - a_i^r}{a_i^r (\tau_i - \tau_{i-1})^r},
\]
where \( \sum_i^0 = 0 \) if \( s > t \).

In the discussion of the impulse moments of \( x(t) \) and \( x(t-\sigma) \), we need to consider the following cases for \( k(c_j) < k(d_j) \),

(S1) \( \tau_{k(c)} + \sigma < c_j \) and \( \tau_{k(d)} + \sigma > d_j \); (S2) \( \tau_{k(c)} + \sigma < c_j \) and \( \tau_{k(d)} + \sigma < d_j \);

(S3) \( \tau_{k(c)} + \sigma > c_j \) and \( \tau_{k(d)} + \sigma > d_j \); (S4) \( \tau_{k(c)} + \sigma > c_j \) and \( \tau_{k(d)} + \sigma < d_j \), and the cases for \( k(c_j) = k(d_j) \);

(\( \bar{S1} \)) \( \tau_{k(c)} + \sigma < c_j \); (\( \bar{S2} \)) \( c_j < \tau_{k(c)} + \sigma < d_j \); (\( \bar{S3} \)) \( \tau_{k(c)} + \sigma > d_j \).

Combining (S*) with (\( \bar{S} \)*), we can get 12 cases. In order to save space, throughout the paper, we study (1.1) under the case of combination of (S1) with (\( \bar{S1} \)) only. The discussions for other cases are similar and omitted.

The following preparatory lemmas will be useful to prove our theorems. The first is derived from [20] and the second from [21].

Lemma 2.1. Let \( \lambda \) and \( \delta \) be positive real numbers with \( \lambda > \delta \). Then
\[
AX^{1-\delta} + BX^{-\delta} \geq \lambda \delta^{-\delta/\lambda} (\lambda - \delta)^{(\delta/\lambda)-1} A^{\delta/\lambda} B^{1-\delta/\lambda},
\]
for all \( A, B \geq 0 \) and \( X > 0 \).

Lemma 2.2. Suppose \( X \) and \( Y \) are nonnegative, then
\[
\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1) Y^\lambda, \quad \lambda > 1,
\]
where equality holds if and if \( X = Y \).

Let \( \gamma > 0, B \geq 0, A > 0 \) and \( y \geq 0 \). Put
\[
\lambda = 1 + \frac{1}{\gamma}, \quad X = A^{\gamma/(\gamma+1)} y, \quad Y = \left( \frac{Y}{\gamma+1} \right)^{\gamma} B^\gamma A^{-\gamma^2/(\gamma+1)}.
\]
(2.5)
It follows from Lemma 2.2 that

\[ By - Ay^{(r+1)/r} \leq \frac{y^r}{(r + 1)^{r+1}} B^{r+1} A^r. \]  

(2.6)

**Lemma 2.3.** Assume that for any \( T \geq t_0 \), there exists \( c_j, d_j \notin \{ \tau_k \}, j = 1, 2 \), such that \( T < c_1 < d_1 \leq c_2 < d_2 \) and

\[ p(t), q(t) \geq 0, \quad t \in [c_1 - \sigma, d_1] \cup [c_2 - \sigma, d_2] \setminus \{ \tau_k \}, \]

\[ e(t) \leq 0, \quad t \in [c_1 - \sigma, d_1] \setminus \{ \tau_k \}, \]

\[ e(t) \geq 0, \quad t \in [c_2 - \sigma, d_2] \setminus \{ \tau_k \}. \]

If \( x(t) \) is a nonoscillatory solution of (1.1), then there exist the following estimations of \( x(t - \sigma) / x(t) \):

(a) for \( t \in (\tau_i + \sigma, \tau_{i+1}] \),

\[ \frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_i - \sigma}{t - \tau_i}, \]

(b) for \( t \in (\tau_i, \tau_{i+1}] \),

\[ \frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_i}{b_i(t + \sigma - \tau_i)}, \]

(c) for \( t \in [c_j, \tau_k(c_j)] \),

\[ \frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_k(c_j) - \sigma}{t - \tau_k(c_j)}, \]

(d) for \( t \in (\tau_k(d_j), d_j] \),

\[ \frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_k(d_j)}{b_k(d_j) (t + \sigma - \tau_k(d_j))}, \]

where \( i = k(c_j), \ldots, k(d_j) - 1, j = 1, 2 \).

*Proof.* Without loss of generality, we assume that \( x(t) > 0 \) and \( x(t - \sigma) > 0 \) for \( t \geq t_0 \). In this case the selected interval of \( t \) is \([c_1, d_1]\). From (1.1) and (2.7), we obtain

\[ [r(t) \varphi_t (x'(t))]' = e(t) - p(t) \varphi_t (x(t - \sigma)) - q(t) f(x(t - \sigma)) \leq 0. \]

(2.9)

Hence \( r(t) \varphi_t (x'(t)) \) is nonincreasing on the interval \([c_1, d_1] \setminus \{ \tau_k \}\).

*Case (a)* (if \( \tau_i + \sigma < t \leq \tau_{i+1} \), then \((t - \sigma, t) \subset (\tau_i, \tau_{i+1}]\)). Thus there is no impulsive moment in \((t - \sigma, t)\). For any \( s \in (t - \sigma, t) \), we have

\[ x(s) - x(\tau_i^+) = x'(\xi)(s - \tau_i), \quad \xi \in (\tau_i, s). \]

(2.10)
Since $x(\tau^+_i) > 0$, the function $\varphi_{\gamma}(\cdot)$ is an increasing function and $r(s)\varphi_{\gamma}(x'(s))$ is non-increasing on $(\tau_i, \tau_{i+1})$, we have

$$
\varphi_{\gamma}(x(s)) > \varphi_{\gamma}(x'(\xi_1)(s-\tau_i)) = \frac{r(\xi_1)\varphi_{\gamma}(x'(\xi_1))}{r(\xi_1)} (s-\tau_i)^+ \geq \frac{r(s)\varphi_{\gamma}(x'(s))}{r(\xi_1)} (s-\tau_i)^+, \quad \xi_1 \in (\tau_i, s).
$$

(2.11)

From (2.11) and the conditions $r(s) > 0$ and $r(s)$ is nondecreasing, we have

$$
\varphi_{\gamma}(x(s)) > \varphi_{\gamma}(x'(s))(s-\tau_i)^+ = \varphi_{\gamma}(x'(s)(s-\tau_i)), \quad \xi_1 \in (\tau_i, s).
$$

(2.12)

Thus

$$
\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i}.
$$

(2.13)

Integrating both sides of above inequality from $t - \sigma$ to $t$, we obtain

$$
\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i - \sigma}{t-\tau_i}, \quad t \in (\tau_i + \sigma, \tau_{i+1}].
$$

(2.14)

Case (b) (if $\tau_i < t < \tau_i + \sigma$, then $\tau_i - \sigma < t - \sigma < \tau_i < t < \tau_i + \sigma$). There is an impulsive moment $\tau_i$ in $(t - \sigma, t)$. For any $t \in (\tau_i, \tau_{i+1})$, we have

$$
x(t) - x(\tau_i^+) = x'(\xi_2)(t-\tau_i), \quad \xi_2 \in (\tau_i, t).
$$

(2.15)

Using the impulsive condition of (1.1) and the monotone properties of $r(t)$, $\varphi_{\gamma}(\cdot)$ and $r(t)\varphi_{\gamma}(x'(t))$, we get

$$
\varphi_{\gamma}(x(t) - a_i x(\tau_i)) = \frac{r(\xi_2)\varphi_{\gamma}(x'(\xi_2))}{r(\xi_2)} (t-\tau_i)^+ \leq \frac{r(\tau_i^+)\varphi_{\gamma}(x'(\tau_i^+))}{r(\xi_2)} (t-\tau_i)^+ = \frac{r(\tau_i)\varphi_{\gamma}(b_1 x'(\tau_i)(t-\tau_i))}{r(\xi_2)}.
$$

(2.16)

Since $x(\tau_i) > 0$, we have

$$
\varphi_{\gamma}\left(\frac{x(t)}{x(\tau_i)} - a_i\right) \leq \frac{r(\tau_i)}{r(\xi_2)} \varphi_{\gamma}\left(\frac{b_1}{x(\tau_i)} x'(\tau_i)(t-\tau_i)\right).
$$

(2.17)
In addition,

$$x(t_i) > x(t_i) - x(t_i - \sigma) = x'(\xi_3)\sigma, \quad \xi_3 \in (t_i - \sigma, t_i). \quad (2.18)$$

Similar to the analysis of (2.11)–(2.14), we have

$$\frac{x'(t_i)}{x(t_i)} < \frac{1}{\sigma}. \quad (2.19)$$

From (2.17) and (2.19) and note that the monotone properties of $\varphi_i(y)$ and $r(t)$, we get

$$\frac{x(t)}{x(t_i)} < a_i + \frac{b_i}{\sigma}(t - t_i). \quad (2.20)$$

In view of $(A_3)$, we have

$$\frac{x(t_i)}{x(t)} > \frac{\sigma}{\sigma a_i + b_i(t - t_i)} \geq \frac{\sigma}{b_i(t + \sigma - t_i)} > 0. \quad (2.21)$$

On the other hand, using similar analysis of (2.11)–(2.19), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s - t_i + \sigma}, \quad s \in (t_i - \sigma, t_i). \quad (2.22)$$

Integrating (2.22) from $t - \sigma$ to $t_i$, where $t \in (t_i, t_i + \sigma)$, we have

$$\frac{x(t - \sigma)}{x(t_i)} > \frac{t - t_i}{\sigma} \geq 0. \quad (2.23)$$

From (2.21) and (2.23), we obtain

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - t_i}{b_i(t + \sigma - t_i)}, \quad t \in (t_i, t_i + \sigma). \quad (2.24)$$

Case (c) $(t \in [c_1, \tau_{k(c)+1}])$. Since $\tau_{k(c)} + \sigma < c_1$, then $t - \sigma \in [c_1 - \sigma, \tau_{k(c)+1} - \sigma] \subset (\tau_{k(c)}, \tau_{k(c)+1} - \sigma)$. So, there is no impulsive moment in $(t - \sigma, t)$. Similar to (2.14) of Case (a), we have

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_{k(c)} - \sigma}{t - \tau_{k(c)}}, \quad t \in [c_1, \tau_{k(c)+1}]. \quad (2.25)$$

Case (d) $(t \in (\tau_{k(d)}, d_1])$. Since $\tau_{k(d)} + \sigma > d_1$, then $t - \sigma \in (\tau_{k(d)} - \sigma, d_1 - \sigma] \subset (\tau_{k(d)} - \sigma, \tau_{k(d)})$. Hence, there is an impulsive moment $\tau_{k(d)}$ in $(t - \sigma, t)$. Making a similar analysis of Case (b), we obtain

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_{k(d)}}{b_{k(d)}(t + \sigma - \tau_{k(d)})} \geq 0, \quad t \in (\tau_{k(d)}, d_1]. \quad (2.26)$$
When \( x(t) < 0 \), we can choose interval \([c_2, d_2]\) to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof.

**Theorem 2.4.** Assume that for any \( T \geq t_0 \), there exists \( c_j, d_j \notin \{\tau_k\}, \ j = 1, 2 \), such that \( T < c_j < d_1 \leq c_2 < d_2 \) and (2.7) holds. If there exists \( w_j(t) \in \Omega_j(c_j, d_j) \) \((j = 1, 2)\) such that, for \( k(c_j) < k(d_j) \),

\[
\int_{c_j}^{\tau_k(c_j)+1} W_j(t) \frac{(t - \tau_k(c_j) - \sigma)}{(t - \tau_k(c_j))^{\gamma}} \, dt
\]

\[
+ \sum_{i = k(c_j)+1}^{k(d_j)-1} \int_{c_i}^{\tau_k(c_j)+1} W_j(t) \frac{(t - \tau_i)}{(t - \tau_i)^{\gamma}} \, dt + \int_{\tau_k(c_j)}^{\tau_k(c_j)+1} W_j(t) \frac{(t - \tau_k(c_j) - \sigma)}{(t - \tau_k(c_j))^{\gamma}} \, dt
\]

\[
+ \int_{c_j}^{d_j} W_j(t) \frac{(t - \tau_k(d_j))^{\gamma}}{b_k^{(d_j)}} \frac{(t - \nu_k(d_j) - \sigma)}{(t - \nu_k(d_j))^{\gamma}} \, dt - \int_{c_j}^{d_j} \left( r(t) |w_j(t)|^{\gamma+1} \right) \, dt
\]

\[
\geq M_j Q_j^{(d_j)} \left[ |w_j|^{\gamma+1} \right],
\]

and for \( k(c_j) = k(d_j) \),

\[
\int_{c_j}^{d_j} W_j(t) \frac{(t - c_j)^{\gamma}}{(t - c_j + \sigma)^{\gamma}} - r(t) |w_j(t)|^{\gamma+1} \, dt \geq 0,
\]

where \( W_j(t) = (p(t) + \alpha y^{-\gamma/\alpha} (x - \gamma)^{\gamma/\alpha - 1} q^{\gamma/\alpha} (e(t))^{1 - \gamma/\alpha}) w_j^{\gamma+1}(t) \), then (1.1) is oscillatory.

**Proof.** Assume, to the contrary, that \( x(t) \) is a nonoscillatory solution of (1.1). Without loss of generality, we assume that \( x(t) > 0 \) and \( x(t - \sigma) > 0 \) for \( t \geq t_0 \). In this case the interval of \( t \) selected for the following discussion is \([c_1, d_1]\).

We define

\[
u(t) = r(t) \frac{\nu(t)}{x(t)}, \quad \text{for} \ t \in [c_1, d_1].
\]

Differentiating \( u(t) \) and in view of (1.1) we obtain, for \( t \neq t_k \),

\[
u'(t) = -p(t) \frac{x^\gamma (t - \sigma)}{x^\gamma (t)} - q(t) \frac{f(x(t - \sigma))}{x^\gamma (t)} - \frac{|e(t)|}{x^\gamma (t)} - \frac{\gamma}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma}
\]

\[
\leq - \left\{ (p(t) + \eta q(t)) x^\gamma (t - \sigma) + |e(t)| x^{-\gamma} (t - \sigma) \right\} \frac{x^\gamma (t - \sigma)}{x^\gamma (t)} - \frac{\gamma}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma}.
\]
Putting $A = \eta q(t)$, $B = |e(t)|$, $X = x(t - \sigma)$, $\lambda = \alpha$, and $\delta = \gamma$, by Lemma 2.1, we see that

$$u'(t) \leq -\left[ p(t) + \alpha \gamma^{-\gamma/\alpha} (a - \gamma)^{(\gamma/\alpha)-1} (\eta q(t))^{\gamma/\alpha} |e(t)|^{1-(\gamma/\alpha)} \right] \frac{x'(t - \sigma)}{x'(t)} \frac{1}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma}$$

$$= -\frac{x'(t - \sigma)}{x'(t)} R(t) - \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma},$$

(2.31)

where $R(t) = p(t) + \alpha \gamma^{-\gamma/\alpha} (a - \gamma)^{(\gamma/\alpha)-1} (\eta q(t))^{\gamma/\alpha} |e(t)|^{1-(\gamma/\alpha)}$.

First, we consider the case $k(c_1) < k(d_1)$.

In this case, we assume impulsive moments in $[c_1, d_1]$ are $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)}$. Choosing a $w_1(t) \in \Omega_1(c_1, d_1)$, multiplying both sides of (2.31) by $w_1^{\gamma+1}(t)$ and then integrating it from $c_1$ to $d_1$, we obtain

$$\int_{c_1}^{d_1} u'(t)w_1^{\gamma+1}(t)dt + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} u'(t)w_1^{\gamma+1}(t)dt + \cdots + \int_{\tau_{k(d_1)}}^{d_1} u'(t)w_1^{\gamma+1}(t)dt$$

$$\leq -\int_{c_1}^{d_1} \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t)dt - \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t)dt$$

$$- \cdots - \int_{\tau_{k(d_1)}}^{d_1} \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t)dt - \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+1+\gamma}} \frac{x'(t - \sigma)}{x'(t)} W_1(t)dt$$

$$- \int_{\tau_{k(c_1)+1+\gamma}}^{\tau_{k(c_1)+2}} \frac{x'(t - \sigma)}{x'(t)} W_1(t)dt - \int_{\tau_{k(d_1)}}^{d_1} \frac{x'(t - \sigma)}{x'(t)} W_1(t)dt,$$

(2.32)

where $W_1(t) = R(t)w_1^{\gamma+1}(t)$. Using the integration by parts on the left side of above inequality and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\sum_{i=k(c_1)+1}^{k(d_1)} w_1^{\gamma+1}(\tau_i) [u(\tau_i) - u(\tau_i^+)]$$

$$\leq \int_{c_1}^{d_1} \left[ (\gamma + 1) |w_1(t)w_1'(t)||u(t)| - \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t) \right] dt$$

$$+ \sum_{i=k(c_1)+1}^{k(d_1)-1} \int_{\tau_i}^{\tau_i+1} \left[ (\gamma + 1) |w_1(t)w_1'(t)||u(t)| - \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t) \right] dt$$

$$+ \int_{\tau_{k(d_1)}}^{d_1} \left[ (\gamma + 1) |w_1(t)w_1'(t)||u(t)| - \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t) \right] dt.$$
Hence
\begin{equation}
- \int_{c_i}^{c_{i+1}} \frac{x(t - \sigma)}{x(t)} W_1(t) dt
- \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[ \int_{\tau}^{\tau+i} \frac{x(t - \sigma)}{x(t)} W_1(t) dt + \int_{\tau+i}^{\tau+i+1} \frac{x(t - \sigma)}{x(t)} W_1(t) dt \right]
- \int_{\tau(d_1)}^{d_1} \frac{x(t - \sigma)}{x(t)} W_1(t) dt.
\end{equation}

(2.33)

Letting \( y = |w'_1(t)|/|u(t)|, B = (\gamma + 1)|w'_1(t)|, A = \gamma/r^{1/\gamma}(t) \) and using (2.6), for the integrand function in above inequality we have that
\begin{equation}
(y + 1) |w'_1(t)w'_i(t)|u(t)| - \frac{\gamma}{r^{1/\gamma}(t)} |u(t)|^{(1 + \gamma)/\gamma} w_1^{(1 + \gamma)}(t) \leq r(t) |w'_1(t)|^{\gamma + 1}.
\end{equation}

Meanwhile, for \( t = \tau_k, k = 1, 2, \ldots \), we have
\begin{equation}
u(\tau_k^+) = \left( \frac{b_k}{\alpha_k} \right)^Y u(\tau_k).
\end{equation}

Hence
\begin{equation}
\sum_{i=k(c_1)+1}^{k(d_1)} w_1^{(1 + \gamma)}(\tau_i) [u(\tau_i) - u(\tau_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^Y - b_i^Y}{a_i^Y} w_1^{(1 + \gamma)}(\tau_i) u(\tau_i).
\end{equation}

(2.36)

Therefore, we get
\begin{equation}
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^Y - b_i^Y}{a_i^Y} w_1^{(1 + \gamma)}(\tau_i) u(\tau_i)
\leq \int_{c_1}^{d_1} r(t) |w'_1(t)|^{\gamma + 1} dt - \int_{c_1}^{c_{i+1}} \frac{x(t - \sigma)}{x(t)} W_1(t) dt
- \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[ \int_{\tau}^{\tau+i} \frac{x(t - \sigma)}{x(t)} W_1(t) dt + \int_{\tau+i}^{\tau+i+1} \frac{x(t - \sigma)}{x(t)} W_1(t) dt \right]
- \int_{\tau(d_1)}^{d_1} \frac{x(t - \sigma)}{x(t)} W_1(t) dt.
\end{equation}

(2.37)

On the other hand, for \( t \in (\tau_{i-1}, \tau_i) \subset [c_1, d_1], i = k(c_1) + 2, \ldots, k(d_1), \) we have
\begin{equation}
x(t) - x(\tau_{i-1}) = x'(\xi)(t - \tau_{i-1}), \quad \xi \in (\tau_{i-1}, t).
\end{equation}

(2.38)
In view of \( x(\tau_{i-1}) > 0 \), and note that the monotone properties of \( \varphi_t(\cdot), r(t)\varphi_t(x'(t)) \), and \( r(t) \), we obtain

\[
\varphi_t(x(t)) > \varphi_t(x'(\xi)) \varphi_t(t - \tau_{i-1}) \geq \frac{r(t)}{r(\xi)} \varphi_t(x'(t)) \varphi_t(t - \tau_{i-1}).
\] (2.39)

This is

\[
\frac{r(t)\varphi_t(x'(t))}{\varphi_t(x(t))} < \frac{r(\xi)}{(t - \tau_{i-1})^\gamma}.
\] (2.40)

Let \( t \to \tau_i^- \), it follows

\[
u(\tau_i) = \frac{r(\tau_i)\varphi_t(x'(\tau_i))}{\varphi_t(x(\tau_i))} < \frac{M_1}{(\tau_i - \tau_{i-1})^\gamma}, \quad i = k(c_1) + 2, \ldots, k(d_1).
\] (2.41)

Similar analysis on \( (c_1, \tau_{k(c_1)+1}) \), we can get

\[
u(\tau_{k(c_1)+1}) < \frac{M_1}{(\tau_{k(c_1)+1} - c_1)^\gamma}.
\] (2.42)

Then from (2.41), (2.42), and (A_2), we have

\[
\sum_{i=k(c_1)+1}^{k(d_1)} b_i^Y - a_i^Y \sum_{i=k(c_1)+1}^{k(d_1)} w_i^{1+1}(\tau_i) u(\tau_i) < M_1 \left[ w_1^{1+1}(\tau_{k(c_1)+1}) \theta(c_1) + \sum_{i=k(c_1)+2}^{k(d_1)} w_i^{1+1}(\tau_i) \zeta(\tau_i) \right]
\] (2.43)

\[
= M_1 Q_{c_1}^{d_1} \left[ w_1^{1+1} \right],
\]

where \( \theta(c_1) = (b_i^Y - a_i^Y)/(b_i^Y - a_i^Y) \) and \( \zeta(\tau_i) = (b_i^Y - a_i^Y)/(a_i^Y(\tau_i - \tau_{i-1})^\gamma) \). From (2.37) and (2.43) and applying Lemma 2.3, we obtain

\[
\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t) \frac{(t - \tau_{k(c_1)} - \sigma)^\gamma}{(t - \tau_{k(c_1)})^\gamma} dt
\]

\[+ \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[ \int_{\tau_i}^{\tau_{i+\gamma}} W_1(t) \frac{(t - \tau_i)^\gamma}{b_i^Y (t + \sigma - \tau_i)^\gamma} dt + \int_{\tau_{i+\gamma}}^{\tau_{i+\gamma}} W_1(t) \frac{(t - \tau_i - \sigma)^\gamma}{(t - \tau_i)^\gamma} dt \right]
\]

\[+ \int_{\tau_{k(d_1)}}^{d_1} W_1(t) \frac{(t - \tau_{k(d_1)})^\gamma}{b_{k(d_1)}^Y (t + \sigma - \tau_{k(d_1)})^\gamma} dt
\]

\[ - \int_{c_1}^{d_1} r(t) \left| w_i'(t) \right|^{1+1} dt
\]

\[< M_1 Q_{c_1}^{d_1} \left[ w_1^{1+1} \right].
\]

This contradicts (2.27).
Next we consider the case \( k(c_1) = k(d_1) \). By the condition (5.1) we know there is no impulsive moment in \([c_1, d_1]\). Multiplying both sides of (2.31) by \( w^{r+1}_1(t) \) and integrating it from \( c_1 \) to \( d_1 \), we obtain

\[
\int_{c_1}^{d_1} u'(t)w^{r+1}_1(t)dt \leq -\int_{c_1}^{d_1} \frac{Y}{r^{1/(r+1)}(t)} |u(t)|^{(r+1)/r}w^{r+1}_1(t)dt - \int_{c_1}^{d_1} \frac{x(t) - \sigma}{x(t)} W_1(t)dt. \tag{2.45}
\]

Using the integration by parts on the left-hand side and noting the condition \( w_1(c_1) = w_1(d_1) = 0 \), we obtain

\[
\int_{c_1}^{d_1} \left[(\gamma + 1)w_1^r(t)w_1'(t)u(t) - \frac{Y}{r^{1/(r+1)}(t)} w^{r+1}_1(t)|u(t)|^{(r+1)/r}\right]dt - \int_{c_1}^{d_1} \frac{x(t) - \sigma}{x(t)} W_1(t)dt \geq 0. \tag{2.46}
\]

It follows that

\[
\int_{c_1}^{d_1} \left[(\gamma + 1)w_1^r(t)w_1'(t) |u(t)| - \frac{Y}{r^{1/(r+1)}(t)} w^{r+1}_1(t)|u(t)|^{(r+1)/r}\right]dt - \int_{c_1}^{d_1} \frac{x(t) - \sigma}{x(t)} W_1(t)dt \geq 0. \tag{2.47}
\]

Letting \( A = \frac{\gamma}{r^{1/(r+1)}(t)}w^{r+1}_1(t) \), \( B = (\gamma + 1)|w_1^r(t)w_1'(t)| \), and \( y = |u(t)| \) and applying the inequality (2.6), we get

\[
\int_{c_1}^{d_1} \left[r(t)|w_1'(t)|^{r+1} - \frac{x(t) - \sigma}{x(t)} W_1(t)\right]dt \geq 0. \tag{2.48}
\]

Using same way as Case (a) we get

\[
\frac{x(t) - \sigma}{x(t)} > \frac{t - c_1}{t - c_1 + \sigma}, \quad t \in [c_1, d_1]. \tag{2.49}
\]

From (2.48) and (2.49) we obtain

\[
\int_{c_1}^{d_1} \left(W_1(t) \frac{(t - c_1)^r}{(t - c_1 + \sigma)^r} - r(t)|w_1'(t)|^{r+1}\right)dt < 0. \tag{2.50}
\]

This contradicts our assumption (2.28).

When \( x(t) < 0 \), we can choose interval \([c_2, d_2]\) to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof. \(\square\)

Remark 2.5. When \( \gamma = 1, p(t) = 0, f(x) = |x|^{\alpha-1}x \) and the delay term \( \sigma = 0 \), (1.1) reduces to that studied by Liu and Xu [13]. Therefore our Theorem 2.4 generalizes Theorem 2.1 of [13].

Remark 2.6. When \( \gamma = \alpha = 1, r(t) = 1 \) and \( p(t) = 0 \), (1.1) reduces to the (1.3) studied by Huang and Feng [16]. Therefore our Theorem 2.4 extends Theorem 2.1 of [16].
Remark 2.7. When \( a_k = b_k = 1 \), for all \( k = 1, 2, \ldots \), the impulses in (1.1) disappear, Theorem 2.4 reduces to the main results of [17, 18].

In the following we will establish a Kemenev type interval oscillation criteria for (1.1) by the ideas of Philos [19] and Kong [22].

Let \( D = \{ (t, s) : t_0 \leq s \leq t \} \), \( H_1, H_2 \in C^1(D, \mathbb{R}) \), then a pair function \( H_1, H_2 \) is said to belong to a function set \( \mathcal{J} \), defined by \( (H_1, H_2) \in \mathcal{J} \), if there exists \( h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R}) \) satisfying the following conditions:

\[
\begin{align*}
(A_4) & \quad H_1(t, t) = H_2(t, t) = 0, H_1(t, s) > 0, H_2(t, s) > 0 \text{ for } t > s; \\
(A_5) & \quad \frac{\partial}{\partial t} H_1(t, s) = h_1(t, s) H_1(t, s), \frac{\partial}{\partial t} H_2(t, s) = h_2(t, s) H_2(t, s).
\end{align*}
\]

We assume there exist \( c_j, d_j, \delta_j \not\in \{ \tau_k, k = 1, 2, \ldots \} \) \( (j = 1, 2) \) which satisfy \( T < c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2 \) for any \( T \geq t_0 \). Noticing whether or not there are impulsive moments of \( x(t) \) in \( [c_j, \delta_j] \) and \( [\delta_j, d_j] \), we should consider the following four cases, namely,

\[
\begin{align*}
(S1) & \quad k(c_j) < k(\delta_j) < k(d_j); \\
(S2) & \quad k(c_j) = k(\delta_j) < k(d_j); \\
(S3) & \quad k(c_j) < k(\delta_j) = k(d_j); \\
(S4) & \quad k(c_j) = k(\delta_j) = k(d_j).
\end{align*}
\]

Moreover, in the discussion of the impulse moments of \( x(t - \sigma) \), it is necessary to consider the following two cases:

\[
\begin{align*}
(\bar{S}1) & \quad \tau_k(\delta_j) + \sigma > \delta_j; \\
(\bar{S}2) & \quad \tau_k(\delta_j) + \sigma \leq \delta_j.
\end{align*}
\]

In the following theorem, we only consider the case of combination of (S1) with (\( \bar{S}1 \)). For the other cases, similar conclusions can be given and their proofs will be omitted here.

For convenience in the expressing blow, we define

\[
\begin{align*}
\Pi_{1,j} = & \quad \frac{1}{H_1(\delta_j, c_j)} \left\{ \int_{c_j}^{\tau_k(c_j) - 1} \frac{H_1(t, c_j) (t - \tau_k(c_j) - \sigma)^Y}{(t - \tau_k(c_j))^Y} \, dt \\
& \quad + \sum_{i = k(c_j) + 1}^{k(\delta_j) - 1} \left[ \int_{\tau_i}^{\tau_i + \sigma} \frac{H_1(t, c_j) (t - \tau_i)^Y}{b_i (t + \sigma - \tau_i)^Y} \, dt + \int_{\tau_i + \sigma}^{\tau_i + \sigma} \frac{H_1(t, c_j) (t - \tau_i - \sigma)^Y}{(t - \tau_i)^Y} \, dt \right] \\
& \quad + \int_{\tau_k(\delta_j)}^{\delta_j} \frac{H_1(t, c_j) (t - \tau_k(\delta_j))^Y}{b_k(\delta_j) (t + \sigma - \tau_k(\delta_j))^Y} \, dt \\
& \quad - \frac{1}{(Y + 1)^{Y+1}} \int_{c_j}^{\delta_j} r(t) H_1(t, c_j) |h_1(t, c_j)|^{Y+1} \, dt \right\},
\end{align*}
\]
Assume that for any \( \Theta = 2.8 \).

Proof. Without loss of generality, we assume that

\[
\begin{align*}
\Pi_{2,j} &= \frac{1}{H_2(d_j, \delta_j)} \left\{ \int_{\delta_j}^{\tau_k(\delta_j)+\gamma} H_2(d_j, t) \frac{(t - \tau_k(\delta_j))^\gamma}{b_k(\delta_j)\left(t + \sigma - \tau_k(\delta_j)\right)^\gamma} dt \\
&+ \int_{\tau_k(\delta_j)+\gamma}^{\tau_k(\delta_j)+1} H_2(d_j, t) \frac{(t - \tau_k(\delta_j) - \sigma)^\gamma}{(t - \tau_k(\delta_j))^\gamma} dt \\
&+ \sum_{i=k(\delta_j)+1}^{k(d_j)-1} \left[ \int_{\tau_i}^{\tau_i+\gamma} H_2(d_j, t) \frac{(t-\tau_i)^\gamma}{b_i^\gamma(t+\sigma-\tau_i)^\gamma} dt + \int_{\tau_i+\gamma}^{\tau_i+1} H_2(d_j, t) \frac{(t-\tau_i-\sigma)^\gamma}{(t-\tau_i)^\gamma} dt \right] \\
&+ \int_{\tau_k(d_j)}^{d_j} H_2(d_j, t) \frac{(t - \tau_k(d_j))^\gamma}{b_k(\delta_j)^\gamma(t + \sigma - \tau_k(d_j))^\gamma} dt \\
&- \frac{1}{(\gamma + 1)^{\gamma+1}} \int_{\delta_j}^{d_j} r(t) H_2(d_j, t) |H_2(d_j, t)|^{\gamma+1} dt \right\},
\end{align*}
\]

(2.51)

where \( H_1(t, c_j) = H_1(t, c_j)R(t), \ H_2(d_j, t) = H_2(d_j, t)R(t) \) \((j = 1, 2) \) and \( R(t) = p(t) + a\gamma^{-\gamma/\alpha}(\alpha - \gamma)^{\gamma/\alpha-1}\eta^{1/\alpha}q^{1/\alpha}(t)|e(t)|^{1-\gamma/\alpha} \).

Theorem 2.8. Assume that for any \( T \geq t_0, \) there exists \( c_j, d_j \notin \{\tau_k\}, j = 1, 2, \) such that \( T < c_1 < d_1 \leq c_2 < d_2 \) and if there exists a pair of \( (H_1, H_2) \in \mathcal{U} \) such that

\[
\Pi_{1,j} + \Pi_{2,j} > \frac{M_j}{H_1(\delta_j, c_j)} Q^\delta_{c_j} \left[ H_1(\cdot, c_j) \right] + \frac{M_j}{H_2(\delta_j, c_j)} Q^\delta_{c_j} \left[ H_2(\cdot, \delta_j) \right], \quad j = 1, 2,
\]

(2.52)

then (1.1) is oscillatory.

Proof. Assume, to the contrary, that \( x(t) \) is a nonoscillatory solution of (1.1). Without loss of generality, we assume that \( x(t) > 0 \) and \( x(t - \sigma) > 0 \) for \( t \geq t_0. \) In this case the interval of \( t \) selected for the following discussion is \([c_1, d_1]\). Using the same proof as in Theorem 2.4, we can get (2.31). Multiplying both sides of (2.31) by \( H_1(t, c_1) \) and integrating it from \( c_1 \) to \( \delta_1 \), we have

\[
\begin{align*}
\int_{c_1}^{\delta_1} H_1(t, c_1) u'(t) dt &\leq -\int_{c_1}^{\delta_1} H_1(t, c_1) \frac{Y}{r^{1/\gamma}(t)} |u(t)|^{(1+\gamma)/\gamma} dt \\
&- \int_{c_1}^{\delta_1} H_1(t, c_1) \frac{x^\gamma(t - \sigma)}{x^\gamma(t)} dt,
\end{align*}
\]

(2.53)
where \( \overline{H}_1(t, c_1) = H_1(t, c_1)(p(t) + a\gamma r^\gamma \alpha \gamma(t\gamma - \gamma)^{\gamma - 1/\alpha} t^{1-\gamma/\alpha}). \)

Noticing impulsive moments \( \tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(\delta_1)} \) are in \([c_1, \delta_1]\) and using the integration by parts on the left-hand side of above inequality, we obtain

\[
\int_{c_1}^{\delta_1} H_1(t, c_1) u'(t) dt = \left( \int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1} \right) H_1(t, c_1) du(t)
\]

\[
= \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{a_i^\gamma - b_i^\gamma}{a_i^\gamma} H_1(\tau_i, c_1) u(\tau_i) + H_1(\delta_1, c_1) u(\delta_1)
\]

\[
- \left( \int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1} \right) H_1(t, c_1) h_1(t, c_1) u(t) dt.
\]

Substituting (2.54) into (2.53), we have

\[
\int_{c_1}^{\delta_1} \overline{H}_1(t, c_1) \frac{x^\gamma(t - \sigma)}{x^\gamma(t)} dt \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\gamma - a_i^\gamma}{a_i^\gamma} H_1(\tau_i, c_1) u(\tau_i) - H_1(\delta_1, c_1) u(\delta_1)
\]

\[
+ \left( \int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1} \right) H_1(t, c_1) \left[ |h_1(t, c_1)| u(t) - \frac{\gamma}{r^\gamma(t)} |u(t)|^{(\gamma+1)/\gamma} \right] dt.
\]

Letting \( A = \gamma/r^\gamma(t), B = |h_1(t, c_1)|, y = |u(t)| \) and using (2.6) to the right side of above inequality, we have

\[
\int_{c_1}^{\delta_1} \overline{H}_1(t, c_1) \frac{x^\gamma(t - \sigma)}{x^\gamma(t)} dt \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\gamma - a_i^\gamma}{a_i^\gamma} H_1(\tau_i, c_1) u(\tau_i) - H_1(\delta_1, c_1) u(\delta_1)
\]

\[
+ \frac{1}{(\gamma + 1)^{\gamma+1}} \int_{c_1}^{\delta_1} r(t) H_1(t, c_1) |h_1(t, c_1)|^{\gamma+1} dt.
\]

Because there are different integration intervals in (2.56), we need to divide the integration interval \([c_1, \delta_1]\) into several subintervals for estimating the function \( x(t - \sigma)/x(t) \). Using Lemma 2.2, (2.6), we get estimation for the left-hand side of above inequality as follows,
From (2.56) and (2.57), we have

\[
\int_{c_1}^{\tau_{k(c_1)}+1} \tilde{H}_1(t, c_1) \frac{(t - \tau_{k(c_1)} - \sigma)^{\gamma}}{(t - \tau_{k(c_1)})^{\gamma}} \, dt \\
+ \sum_{i=k(c_1)+1}^{k(\beta_1)+1} \left[ \int_{\tau_i}^{\tau_{i+\sigma}} \tilde{H}_1(t, c_1) \frac{(t - \tau_{i})^{\gamma}}{b_i^{\gamma} (t + \sigma - \tau_{i})^{\gamma}} \, dt + \int_{\tau_{i+\sigma}}^{\tau_{i+1}} \tilde{H}_1(t, c_1) \frac{(t - \tau_{i} - \sigma)^{\gamma}}{(t - \tau_{i})^{\gamma}} \, dt \right] \\
+ \int_{\tau_{k(c_1)}}^{\delta_1} \tilde{H}_1(t, c_1) \frac{(t - \tau_{k(c_1)})^{\gamma}}{b_{k(c_1)}^{\gamma} (t + \sigma - \tau_{k(c_1)})^{\gamma}} \, dt - \frac{1}{(\gamma + 1)^{\gamma+1}} \int_{c_1}^{\delta_1} r(t) H_1(t, c_1) |h_1(t, c_1)|^{\gamma+1} \, dt \\
< \sum_{i=k(c_1)+1}^{k(\beta_1)+1} \frac{b_i^{\gamma} - a_i^{\gamma}}{a_i^{\gamma}} H_1(\tau_i, c_1) u(\tau_i) - H_1(\delta_1, c_1) u(\delta_1).
\]

(2.58)

Multiplying both sides of (2.31) by \(H_2(d_1, t)\) and using similar analysis to the above, we can obtain

\[
\int_{\delta_1}^{\tau_{k(\beta_1)+\sigma}} \tilde{H}_2(d_1, t) \frac{(t - \tau_{k(\beta_1)})^{\gamma}}{b_{k(\beta_1)}^{\gamma} (t + \sigma - \tau_{k(\beta_1)})^{\gamma}} \, dt \\
+ \sum_{i=k(\beta_1)+1}^{k(d_1)+1} \left[ \int_{\tau_i}^{\tau_{i+\sigma}} \tilde{H}_2(d_1, t) \frac{(t - \tau_{i})^{\gamma}}{b_i^{\gamma} (t + \sigma - \tau_{i})^{\gamma}} \, dt + \int_{\tau_{i+\sigma}}^{\tau_{i+1}} \tilde{H}_2(d_1, t) \frac{(t - \tau_{i} - \sigma)^{\gamma}}{(t - \tau_{i})^{\gamma}} \, dt \right] \\
+ \int_{\tau_{k(\beta_1)}}^{d_1} \tilde{H}_2(d_1, t) \frac{(t - \tau_{k(d_1)})^{\gamma}}{b_{k(d_1)}^{\gamma} (t + \sigma - \tau_{k(d_1)})^{\gamma}} \, dt - \frac{1}{(\gamma + 1)^{\gamma+1}} \int_{\delta_1}^{d_1} r(t) H_2(d_1, t) |h_2(d_1, t)|^{\gamma+1} \, dt \\
< \sum_{i=k(\beta_1)+1}^{k(d_1)+1} \frac{b_i^{\gamma} - a_i^{\gamma}}{a_i^{\gamma}} H_2(d_1, \tau_i) u(\tau_i) + H_2(d_1, \delta_1) u(\delta_1).
\]

(2.59)

Dividing (2.58) and (2.59) by \(H_1(\delta_1, c_1)\) and \(H_2(d_1, \delta_1)\), respectively, and adding them, we get

\[
\Pi_{1,1} + \Pi_{2,1} < \frac{1}{H_1(\delta_1, c_1)} \sum_{i=k(c_1)+1}^{k(\beta_1)+1} \frac{b_i^{\gamma} - a_i^{\gamma}}{a_i^{\gamma}} H_1(\tau_i, c_1) u(\tau_i) \\
+ \frac{1}{H_2(d_1, \delta_1)} \sum_{i=k(\beta_1)+1}^{k(d_1)+1} \frac{b_i^{\gamma} - a_i^{\gamma}}{a_i^{\gamma}} H_2(d_1, \tau_i) u(\tau_i).
\]

(2.60)
On the other hand, similar to (2.43), we have

\[
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^T - a_i^T}{a_i^T} H_1(\tau_i, c_1)u(\tau_i) \leq M_1 Q^{d_i}_1 [H_1(\cdot, c_1)],
\]

\[
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^T - a_i^T}{a_i^T} H_2(d_1, \tau_i)u(\tau_i) \leq M_1 Q^{d_i}_1 [H_2(d_1, \cdot)].
\]

(2.61)

From (2.60) and (2.61), we can obtain a contradiction to the condition (2.52).

When \( x(t) < 0 \), we can choose interval \([c_2, d_2]\) to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof. \( \square \)

Remark 2.9. Let \( H_1(t, s) = H_2(t, s) = H(t, s), \ h_1(t, s) = 2h_1^*(t, s)/\sqrt{H(t, s)} \) and \( h_2(t, s) = -2h_2^*(t, s)/\sqrt{H(t, s)} \), the conditions \((A_4), \ (A_5)\) can be changed into

\[
(A_6) \quad H(t, t) = 0, \ H(t, s) > 0, \text{ for } t > s;
\]

\[
(A_7) \quad (\partial / \partial t)H(t, s) = 2h_1^*(t, s)\sqrt{H(t, s)}, \ (\partial / \partial s)H(t, s) = -2h_2^*(t, s)\sqrt{H(t, s)}.
\]

We know that \((A_6)\) and \((A_7)\) are the main assumptions used in [13, 16] to obtain Kemenev type oscillation criteria. Therefore, Theorem 2.8 is a generalization of Theorem 2.3 in [13] and Theorem 2.5 in [16].

3. Examples

In this section, we give two examples to illustrate the effectiveness and nonemptiness of our results.

Example 3.1. Consider the following equation

\[
[q_i(x'(t))']' + \nu_1 \sin t q_i \left(x \left(t - \frac{\pi}{12}\right)\right) + \nu_2^2 \cos t q_i \left(x \left(t - \frac{\pi}{12}\right)\right) = -\cos 2t, \quad t \neq \tau_{k,i},
\]

\[
x(t^+) = a_k x(t), \ x'(t^+) = b_k x'(t), \quad t = \tau_{k,i},
\]

where \( \tau_{k,i} = 2k\pi + (i - 1)(\pi/2) + (-1)^{(i-1)}(\pi/8), \ i = 1, 2, k = 1, 2, \ldots, \ t \geq 0, \gamma = 3, \alpha = 6, \nu_1 \) and \( \nu_2 \) are positive constants.

For any \( T > 0 \), we can choose large \( n_0 \in \mathbb{N} \) such that

\[
T < c_1 = 2n\pi + \frac{\pi}{12}, \quad d_1 = 2n\pi + \frac{\pi}{4}, \quad c_2 = 2n\pi + \frac{\pi}{3}, \quad d_2 = 2n\pi + \frac{\pi}{2}, \quad n = n_0, n_0 + 1, \ldots.
\]

(3.2)

There are impulsive moments \( \tau_{n,1} = 2n\pi + \pi/8 \) in \([c_1, d_1]\) and \( \tau_{n,2} = 2n\pi + 3\pi/8 \) in \([c_2, d_2]\). From \( \tau_{n,2} - \tau_{n,1} = \pi/4 > \pi/12 \) and \( \tau_{n+1,1} - \tau_{n,2} = 13\pi/8 > \pi/12 \) for all \( n > n_0 \), we know that condition \( \tau_{k+1} - \tau_k > \sigma \) is satisfied. Moreover, we also see the conditions \((S1)\) and \((2.7)\) are satisfied.
Let \( \omega(t) = \sin 12t \). It is easy to get that \( W_1(t) = (\nu_1 \sin t + 2\nu_2 \sqrt{\cos t \cos 2t}) \sin^4 12t \). In view of \( \sum_{i=k+1}^{k+1} = 0 \) as \( k(c_1) + 1 > k(d_1) - 1 \), by a simple calculation, the left side of (2.27) is the following

\[
\int_{t_1}^{t_2} W_1(t) \left( \frac{(t - \tau(c_1) - \sigma)^Y}{(t - \tau(c_1))^3} \right) dt
+ \sum_{i=k+1}^{k+1} \left[ \int_{\tau(c_1)}^{\tau(c_1)+\tau} W_1(t) \left( \frac{(t - \tau(c_1))^Y}{b_1(t + \sigma - \tau_1)^3} \right) dt + \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t - \tau_1)^Y}{(t - \tau_1)^3} \right) dt \right]
+ \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t - \tau(c_1))^Y}{b_1(t + \sigma - \tau_1)^3} \right) dt - \int_{c_1}^{d_1} r(t) \left| w_1(t) \right| + 1 dt
= \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t - 2(n - 1) \pi - 3\pi/8 - \pi/12)^3}{t - 2(n - 1) \pi - 3\pi/8} \right) dt
+ \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t - 2n \pi - \pi/8)}{b_1(t + \pi/12 - 2n \pi - \pi/8)} \right)^3 dt - \int_{\tau(c_1)+\tau}^{\tau+\tau_1} 12^4 \cos^4 12tdt
= \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t + 3\pi/24)}{t + 3\pi/24} \right)^3 dt + \int_{\tau(c_1)+\tau}^{\tau+\tau_1} W_1(t) \left( \frac{(t - \pi/8)}{b_1(t - \pi/24)} \right)^3 dt - 12^4 \int_{\tau(c_1)+\tau}^{\tau+\tau_1} \cos^4 12tdt
\approx 0.671\nu_1 + 1.326\nu_2 + b_{n_1}^{-3}(0.008\nu_1 + 0.002\nu_2) - 6^4\pi.
\]

(3.3)

On the other hand, we have

\[
Q_{c_1}^d [w^4] = \left( \frac{24}{\pi} \right)^3 \frac{b_{n_1}^3 - a_{n_1}^3}{a_{n_1}^3}.
\]

(3.4)

Thus condition (2.27) is satisfied for \( t \in [c_1, d_1] \) if

\[
0.671\nu_1 + 1.326\nu_2 + b_{n_1}^{-3}(0.008\nu_1 + 0.002\nu_2) - 6^4\pi \geq \left( \frac{24}{\pi} \right)^3 \frac{b_{n_1}^3 - a_{n_1}^3}{a_{n_1}^3}.
\]

(3.5)

Similarly, for \( t \in [c_2, d_2] \) we can get the following condition

\[
0.012\nu_1 + 0.003\nu_2 + b_{n_2}^{-3}(0.012\nu_1 + 0.001\nu_2) - 6^4\pi \geq \left( \frac{24}{\pi} \right)^3 \frac{b_{n_2}^3 - a_{n_2}^3}{a_{n_2}^3}
\]

(3.6)
which ensures (2.27). Hence, by Theorem 2.4, if (3.5) and (3.6) hold, (3.1) is oscillatory. It is easy to see that (3.5) and (3.6) may be satisfied when \( v_1 \) or \( v_2 \) is large enough. Particularly, let \( a_{n,i} = b_{n,i} \), for \( i = 1,2 \) and \( n = 1,2,\ldots \), condition (3.5) and (3.6) become a simple form

\[
0.671v_1 + 1.326v_2 + b_{n,3}^{-3}(0.008v_1 + 0.002v_2) \geq 6^4\pi, \\
0.012v_1 + 0.003v_2 + b_{n,3}^{-3}(0.012v_1 + 0.001v_2) \geq 6^4\pi.
\]  

(3.7)

Example 3.2. Consider the following equation

\[
x''(t) + \mu_1 p(t) x\left(t - \frac{2}{3}\right) + \mu_2 q(t) x^2 \left(t - \frac{2}{3}\right) = e(t), \quad t \neq \tau_k, \\
x(t^+) = a_k x(t), \quad x'(t^+) = b_k x'(t), \quad t = \tau_k,
\]

(3.8)

where \( \tau_{n,1} = 8n + 1/2, \tau_{n,2} = 8n + 3/2, \tau_{n,3} = 8n + 9/2, \tau_{n,4} = 8n + (11/2) \) \( n = 0,1,2,\ldots \), \( \gamma = 1, \alpha = 2, \sigma = 2/3, b_k \geq a_k > 0, \mu_1, \mu_2 > 0 \) is a constant. Clearly, \( \tau_{k+1} - \tau_k > \sigma \). In addition, let

\[
p(t) = q(t) = \begin{cases} (t - 8n)^3, & t \in [8n, 8n + 2], \\
(8n + 4 - t)^3, & t \in [8n + 2, 8n + 4], \\
(t - 8n - 4)^3, & t \in [8n + 4, 8n + 8], \\
(8n + 8 - t)^3, & t \in [8n + 8, 8n + 12] 
\end{cases}
\]

(3.9)

\[
e(t) = \begin{cases} (t - 8n - 2)^3, & t \in [8n, 8n + 4], \\
(8n + 6 - t)^3, & t \in [8n + 4, 8n + 8] 
\end{cases}
\]

For any \( t_0 > 0 \), we choose \( n \) large enough such that \( t_0 < 8n \) and let \([c_1, d_1] = [8n, 8n + 2], [c_2, d_2] = [8n + 4, 8n + 6], c_1 = 8n + 1 \) and \( d_2 = 8n + 5 \). Then \( p(t), q(t), \) and \( e(t) \) on \([c_1, d_1]\) and \([c_2, d_2]\) satisfy (2.7). Let \( H_1(t,s) = H_2(t,s) = (t - s)^3 \), then \( h_1(t,s) = -h_2(t,s) = 3/(t - s) \). By simple calculation, we get

\[
\Pi_{1,1} := \frac{1}{H_1(\delta_1, c_1)} \left\{ \int_{c_1}^{\tau_{k(\delta)}} \tilde{H}_1(t, c_1) \left( \frac{(t - \tau_{k(c_1)} - \sigma)^T}{(t - \tau_{k(c_1)})^T} \right) dt \right. \\
+ \sum_{j=0}^{k(\delta) - 1} \left[ \int_{\tau_j}^{\tau_{j+1}} \tilde{H}_1(t, c_1) \left( \frac{(t - \tau_j)^T}{b_j^T (t + \sigma - \tau_j)^T} \right) dt + \int_{\tau_{j+1}}^{\tau_{k(\delta)}} \tilde{H}_1(t, c_1) \left( \frac{(t - \tau_j - \sigma)^T}{(t - \tau_j)^T} \right) dt \right] \\
+ \sum_{i=k(\delta) + 1}^{k(\delta) + 1} \left[ \int_{\tau_{k(\delta)}}^{\tau_{i+1}} \tilde{H}_1(t, c_1) \left( \frac{(t - \tau_{k(\delta)})^T}{b_j^T (t + \sigma - \tau_{k(\delta)})^T} \right) dt \right]
\]

\[
\left. + \int_{\tau_{k(\delta)}}^{\tau_{k(\delta)}} \tilde{H}_1(t, c_1) \left( \frac{(t - \tau_{k(\delta)})^T}{b_j^T (t + \sigma - \tau_{k(\delta)})^T} \right) dt \right\}.
\]
\[
\Pi_{2,1} = \frac{1}{H_2(d_1, b_1)} \left\{ \int_{\tau_{k(d1)} + \sigma}^{\tau_{k(d1)} + \sigma} \frac{t-\tau_{k(d1)}}{b_1^T(t+\sigma-\tau_{k(d1)})^T} dt + \int_{\tau_{k(d1)} + \sigma}^{\tau_{k(d1)} + \sigma} \frac{t-\tau_{k(d1)}}{b_1^T(t+\sigma-\tau_{k(d1)})^T} dt \right\}
\]

\[= \int_{8n+7/6}^{8n+1} (8n+2-t)^3 \left[ \mu_1(t-8n)^3 + 2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2} \right] \frac{t-8n-1/2}{b_{n,1}(t-8n+1/6)} dt + \int_{8n+3/2}^{8n+1} (8n+2-t)^3 \left[ \mu_1(t-8n)^3 + 2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2} \right] \frac{t-8n-7/6}{t-8n-1/2} dt + \int_{8n+3/2}^{8n+1} (8n+2-t)^3 \left[ \mu_1(t-8n)^3 + 2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2} \right] \frac{t-8n-3/2}{b_{n,2}(t-8n+5/6)} dt - \frac{9}{8} \]

\[= \frac{1}{b_{n,1}} \int_{1/3}^{7/6} \frac{u^3(2-u)^3(u-1/2)}{u+1/6} \frac{2\mu_2^{1/2}(2-u)^{7/2}(u-1/2)}{u+1/6} du + \frac{1}{b_{n,2}} \int_{1/3}^{7/6} \frac{u^3(2-u)^3(u-1/2)}{u+1/6} \frac{2\mu_2^{1/2}(2-u)^{7/2}(u-1/2)}{u+1/6} du + \frac{1}{b_{n,2}} \int_{1/3}^{7/6} \frac{u^3(2-u)^3(u-3/2)}{u-5/6} \frac{2\mu_2^{1/2}(2-u)^{7/2}(u-3/2)}{u-5/6} du - \frac{9}{8} \]

\[\approx \left( \frac{0.0389}{b_{n,1}} + \frac{0.0754}{b_{n,2}} + \frac{0.0087}{b_{n,2}} \right) \mu_1 + \left( \frac{0.0394}{b_{n,1}} + \frac{0.1281}{b_{n,2}} + \frac{0.0046}{b_{n,2}} \right) \mu_2^{1/2} - \frac{9}{8} \cdot (3.10)\]
Then the left side of the inequality (2.52) is

\[
\Pi_{1,1} + \Pi_{2,1} \approx \left( 0.0497 + \frac{0.0759}{b_{n,1}} + \frac{0.0087}{b_{n,2}} \right) \mu_1 + \left( 0.0475 + \frac{0.2541}{b_{n,1}} + \frac{0.0046}{b_{n,2}} \right) \mu_2^{1/2} - \frac{9}{4}.
\]

(3.11)

Because \(M_1 = M_2 = 1\), \(\tau_k(c_1) = \tau_k(\delta_1) = \tau_{n,1} = 8n + 1/2 \in (c_1, \delta_1)\) and \(\tau_k(\delta_1) = \tau_{n,2} = 8n + 3/2 \in (\delta_1, d_1)\), it is easy to get that the right side of the inequality (2.52) for \(j = 1\) is

\[
\frac{M_1}{H_1(\delta_1, c_1)} Q_{\delta_1}^d [H_1(\cdot, c_1)] + \frac{M_1}{H_2(d_1, \delta_1)} Q_{\delta_1}^d [H_2(d_1, \cdot)] = \frac{b_{n,1} - a_{n,1}}{4a_{n,1}^2} + \frac{29(b_{n,2} - a_{n,2})}{8a_{n,2}}.
\]

(3.12)

Thus (2.52) is satisfied for \(j = 1\) if

\[
\left( 0.0497 + \frac{0.0759}{b_{n,1}} + \frac{0.0087}{b_{n,2}} \right) \mu_1 + \left( 0.0475 + \frac{0.2541}{b_{n,1}} + \frac{0.0046}{b_{n,2}} \right) \mu_2^{1/2} > \frac{9}{4} + \frac{b_{n,1} - a_{n,1}}{4a_{n,1}^2} + \frac{29(b_{n,2} - a_{n,2})}{8a_{n,2}}.
\]

(3.13)

When \(j = 2\), with the similar argument above we get that the left side of inequality (2.52) is

\[
\Pi_{1,2} + \Pi_{2,2} = \mu_1 \int_{0}^{1/2} \frac{u^6(u + 11/6)}{u + 5/2} \, du + 2 \mu_2^{1/2} \int_{0}^{1/2} \frac{u^6(2 - u)^{1/2}(u + 11/6)}{u + 5/2} \, du
\]

\[
+ \frac{\mu_1}{b_{n,3}} \int_{1/2}^{1} \frac{u^6(u - 1/2)}{u + 1/6} \, du + \frac{2 \mu_2^{1/2}}{b_{n,3}} \int_{1/2}^{1} \frac{u^6(2 - u)^{1/2}(u - 1/2)}{u + 1/6} \, du
\]

\[
+ \frac{\mu_1}{b_{n,4}} \int_{1}^{7/6} \frac{u^3(2 - u)^3(u - 1/2)}{u + 1/6} \, du + \frac{2 \mu_2^{1/2}}{b_{n,4}} \int_{1}^{7/6} \frac{u^3(2 - u)^{3/2}(u - 1/2)}{u + 1/6} \, du
\]

\[
+ \frac{\mu_1}{b_{n,4}} \int_{3/2}^{7/6} \frac{u^3(2 - u)^3(u - 7/6)}{u - 1/2} \, du + \frac{2 \mu_2^{1/2}}{b_{n,4}} \int_{3/2}^{7/6} \frac{u^3(2 - u)^{3/2}(u - 7/6)}{u - 1/2} \, du
\]

\[
+ \frac{\mu_1}{b_{n,4}} \int_{3/2}^{2} \frac{u^3(2 - u)^3(u - 3/2)}{u - 5/6} \, du + \frac{2 \mu_2^{1/2}}{b_{n,4}} \int_{3/2}^{2} \frac{u^3(2 - u)^{3/2}(u - 3/2)}{u - 5/6} \, du - \frac{9}{4}
\]

\[
\approx \left( 0.0497 + \frac{0.0759}{b_{n,3}} + \frac{0.0087}{b_{n,4}} \right) \mu_1 + \left( 0.0475 + \frac{0.2541}{b_{n,3}} + \frac{0.0046}{b_{n,4}} \right) \mu_2^{1/2} - \frac{9}{4}
\]

(3.14)

and the right side of the inequality (2.52) is

\[
\frac{M_2}{H_2(d_2, c_2)} Q_{\delta_2}^d [H_1(\cdot, c_2)] + \frac{M_2}{H_2(d_2, \delta_2)} Q_{\delta_2}^d [H_2(d_2, \cdot)] = \frac{b_{n,3} - a_{n,3}}{4a_{n,3}^2} + \frac{29(b_{n,4} - a_{n,4})}{8a_{n,4}}.
\]

(3.15)
Therefore (2.52) is satisfied for $j=2$ if

$$
\left(\frac{0.0497}{b_{n,3}} + \frac{0.0759}{b_{n,4}} + \frac{0.087}{b_{n,4}}\right)\mu_1 + \left(\frac{0.0475}{b_{n,4}} + \frac{0.2541}{b_{n,4}} + \frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2} > \frac{9}{4} + \frac{b_{n,3} - a_{n,3}}{4a_{n,3}^2} + \frac{29(b_{n,4} - a_{n,4})}{8a_{n,4}}.
$$

Hence, by Theorem 2.8, (3.8) is oscillatory if (3.13) and (3.16) hold. Particularly, let $a_{n,i} = b_{n,i}$ for $i = 1, 2, 3, 4$, condition (3.13) and (3.16) become a simple form

$$
\left(\frac{0.0497}{b_{n,1}} + \frac{0.0759}{b_{n,2}} + \frac{0.087}{b_{n,4}}\right)\mu_1 + \left(\frac{0.0475}{b_{n,1}} + \frac{0.2541}{b_{n,2}} + \frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2} > \frac{9}{4},
$$

$$
\left(\frac{0.0497}{b_{n,3}} + \frac{0.0759}{b_{n,4}} + \frac{0.087}{b_{n,4}}\right)\mu_1 + \left(\frac{0.0475}{b_{n,3}} + \frac{0.2541}{b_{n,4}} + \frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2} > \frac{9}{4}.
$$

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**References**


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