Research Article

Implicit Mann Type Iteration Method Involving Strictly Hemicontractive Mappings in Banach Spaces

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We proved that the modified implicit Mann iteration process can be applied to approximate the fixed point of strictly hemicontractive mappings in smooth Banach spaces.

1. Introduction

Let K be a nonempty subset of an arbitrary Banach space X and let X^* be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued mapping $T : X \rightarrow X$, $x \in X$ is called a *fixed point* of T iff $Tx = x$). We denote by J the *normalized duality mapping* from X to 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in X, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. In a smooth Banach space, J is singlevalued (we denote by j).

Remark 1.1. (1) X is called uniformly smooth if X^* is uniformly convex.

(2) In a uniformly smooth Banach space, J is uniformly continuous on bounded subsets of X .

Let $T : D(T) \subset X \rightarrow X$ be a mapping.

Definition 1.2. The mapping T is called *Lipshitz* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad (1.2)$$

for all $x, y \in D(T)$. If $L = 1$, then T is called *nonexpansive* and if $0 \leq L < 1$, then T is called *contractive*.

Definition 1.3 (see [1, 2]). (1) The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|, \quad (1.3)$$

for all $x, y \in D(T)$ and $r > 0$.

(2) The mapping T is said to be *strongly pseudocontractive* if there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \quad (1.4)$$

for all $x, y \in D(T)$ and $r > 0$.

(3) The mapping T is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \quad (1.5)$$

for all $y \in D(T)$ and $r > 0$.

(4) The mapping T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a constant $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|, \quad (1.6)$$

for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive mapping is local strongly pseudocontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Schu [3] generalized the result in [1] to both uniformly continuous strongly pseudocontractive mappings and real smooth Banach spaces. Park [4] extended the result in [1] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [5] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Afterwards, several generalizations have been made in various directions (see, e.g., [6–13]).

In 2001, Xu and Ori [14] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \dots, N\}$) with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= (1 - \alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 &= (1 - \alpha_2)x_1 + \alpha_2 T_2 x_2, \\ &\vdots \\ x_N &= (1 - \alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} &= (1 - \alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1}, \\ &\vdots \end{aligned} \tag{1.7}$$

which can be written in the following compact form:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T_n x_n, \quad n \geq 1, \tag{1.8}$$

where $T_n = T_{n \pmod N}$ (here the mod N function takes values in I). Xu and Ori [14] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [11], Osilike proved the following results.

Theorem 1.4. *Let X be a real Banach space and let K be a nonempty closed convex subset of X . Let $\{T_i : i \in I\}$ be N strictly pseudocontractive mappings from K to K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the following conditions:*

- (i) $0 < \alpha_n < 1$,
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$,
- (iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$.

Remark 1.5. One can easily see that for $\alpha_n = 1 - 1/n^{1/2}$, $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 = \infty$. Hence the results of Osilike [11] are needed to be improved.

Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X and let $T : K \rightarrow K$ be a continuous strictly hemiccontractive mapping. We proved that the implicit Mann type iteration method converges strongly to a unique fixed point of T .

The results presented in this paper extend and improve the corresponding results particularly in [1, 3, 4, 7, 8, 10, 11, 13, 15].

2. Preliminaries

We need the following results.

Lemma 2.1 (see [4]). Let X be a smooth Banach space. Suppose that one of the following holds:

- (a) J is uniformly continuous on any bounded subsets of X ,
- (b) $\langle x - y, j(x) - j(y) \rangle \leq \|x - y\|^2$ for all x, y in X ,
- (c) for any bounded subset D of X , there is a function $c : [0, \infty) \rightarrow [0, \infty)$ such that

$$\operatorname{Re} \langle x - y, j(x) - j(y) \rangle \leq c(\|x - y\|), \quad (2.1)$$

for all $x, y \in D$, where c satisfies $\lim_{t \rightarrow 0^+} (c(t)/t) = 0$.

Then for any $\epsilon > 0$ and any bounded subset K , there exists $\delta > 0$ such that

$$\|sx + (1 - s)y\|^2 \leq (1 - 2s)\|y\|^2 + 2s \operatorname{Re} \langle x, j(y) \rangle + 2s\epsilon, \quad (2.2)$$

for all $x, y \in K$ and $s \in [0, \delta]$.

Remark 2.2. (1) If X is uniformly smooth, then (a) in Lemma 2.1 holds.

(2) If X is a Hilbert space, then (b) in Lemma 2.1 holds.

Lemma 2.3 (see [8]). Let $T : D(T) \subset X \rightarrow X$ be a mapping with $F(T) \neq \emptyset$. Then T is strictly hemicontractive if and only if there exists a constant $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x - q) \in J(x - q)$ satisfying

$$\operatorname{Re} \langle x - Tx, j(x - q) \rangle \geq \left(1 - \frac{1}{t}\right) \|x - q\|^2. \quad (2.3)$$

Lemma 2.4 (see [10]). Let X be an arbitrary normed linear space and let $T : D(T) \subset X \rightarrow X$ be a mapping.

- (a) If T is a local strongly pseudocontractive mapping and $F(T) \neq \emptyset$, then $F(T)$ is a singleton and T is strictly hemicontractive.
- (b) If T is strictly hemicontractive, then $F(T)$ is a singleton.

Lemma 2.5 (see [10]). Let $\{\theta_n\}, \{\sigma_n\}$, and $\{\omega_n\}$ be nonnegative real sequences and let $\epsilon' > 0$ be a constant satisfying

$$\sigma_{n+1} \leq (1 - \theta_n)\sigma_n + \epsilon'\theta_n + \omega_n, \quad n \geq 1, \quad (2.4)$$

where $\sum_{n=1}^{\infty} \theta_n = \infty$, $\theta_n \leq 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \omega_n < \infty$. Then $\limsup_{n \rightarrow \infty} \sigma_n \leq \epsilon'$.

3. Main Results

We now prove our main results.

Lemma 3.1. Let X be a smooth Banach space. Suppose that one of the following holds:

- (a) J is uniformly continuous on any bounded subsets of X ,

- (b) $\langle x - y, j(x) - j(y) \rangle \leq \|x - y\|^2$ for all x, y in X ,
 (c) for any bounded subset D of X , there is a function $c : [0, \infty) \rightarrow [0, \infty)$ such that

$$\operatorname{Re} \langle x - y, j(x) - j(y) \rangle \leq c(\|x - y\|) \quad (3.1)$$

for all $x, y \in D$, where c satisfies $\lim_{t \rightarrow 0^+} c(t)/t = 0$.

Then for any $\epsilon > 0$ and any bounded subset K , there exists $\delta > 0$ such that

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &\leq (1 - 2\alpha)\|x\|^2 + 2\frac{\alpha\beta}{1-\alpha} \operatorname{Re} \langle y, j(x) \rangle \\ &\quad + 2\frac{\alpha\gamma}{1-\alpha} \operatorname{Re} \langle z, j(x) \rangle + 2\epsilon\alpha \end{aligned} \quad (3.2)$$

for all $x, y, z \in K$ and $\alpha, \beta, \gamma \in [0, \delta]; \alpha + \beta + \gamma = 1$.

Proof. For $\alpha, \beta, \gamma \in [0, \delta]; \alpha + \beta + \gamma = 1$, by using (2.2), consider

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \left\| \alpha x + (1 - \alpha) \left(\frac{\beta}{1-\alpha} y + \frac{\gamma}{1-\alpha} z \right) \right\|^2 \\ &\leq (1 - 2\alpha)\|x\|^2 + 2\epsilon\alpha + 2\alpha \operatorname{Re} \left\langle \frac{\beta}{1-\alpha} y + \frac{\gamma}{1-\alpha} z, j(x) \right\rangle \\ &= (1 - 2\alpha)\|x\|^2 + 2\epsilon\alpha + 2\frac{\alpha\beta}{1-\alpha} \operatorname{Re} \langle y, j(x) \rangle + 2\frac{\alpha\gamma}{1-\alpha} \operatorname{Re} \langle z, j(x) \rangle. \end{aligned} \quad (3.3)$$

This completes the proof. $\square$

Theorem 3.2. Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \rightarrow K$ be a continuous strictly hemicontractive mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying conditions

- (iv) $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$,
 (v) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
 (vi) $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$.

For a sequence $\{v_n\}$ in K , suppose that $\{x_n\}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T v_n, \quad n \geq 1, \quad (3.4)$$

satisfying $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Proof. By [2, Corollary 1], T has a unique fixed point q in K . It follows from Lemma 2.4 that $F(T)$ is a singleton. That is, $F(T) = \{q\}$ for some $q \in K$.

Set $M = 1 + \text{diam } K$. It is easy to verify that

$$M = \sup_{n \geq 1} \|x_n - q\| + \sup_{n \geq 1} \|Tx_n - q\| + \sup_{n \geq 1} \|Tv_n - q\|. \quad (3.5)$$

Also

$$\begin{aligned} \|v_n - q\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2\|v_n - x_n\|\|x_n - q\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2M\|v_n - x_n\|. \end{aligned} \quad (3.6)$$

Consider

$$\begin{aligned} \|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2 \\ &= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2 \\ &\leq \alpha_n \|x_{n-1} - q\|^2 \\ &\quad + \beta_n \|Tx_n - q\|^2 + \gamma_n \|Tv_n - q\|^2 \\ &\leq \|x_{n-1} - q\|^2 + M^2(\beta_n + \gamma_n), \end{aligned} \quad (3.7)$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Now we put $k = 1/t$, where t satisfies (2.3). Using (3.4) and Lemma 3.1, we infer that

$$\begin{aligned} \|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2 \\ &= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2 \\ &\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} \text{Re}\langle Tx_n - q, j(x_{n-1} - q) \rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} \text{Re}\langle Tv_n - q, j(x_{n-1} - q) \rangle + 2\epsilon\alpha_n \\ &= (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} \text{Re}\langle Tx_n - q, j(x_n - q) \rangle \\ &\quad + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} \text{Re}\langle Tx_n - q, j(x_{n-1} - q) - j(x_n - q) \rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} \text{Re}\langle Tv_n - q, j(v_n - q) \rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} \text{Re}\langle Tv_n - q, j(x_{n-1} - q) - j(v_n - q) \rangle + 2\epsilon\alpha_n \\ &\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} k\|x_n - q\|^2 \\ &\quad + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} \|Tx_n - q\| \|j(x_{n-1} - q) - j(x_n - q)\| \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} k\|v_n - q\|^2 \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} \|Tv_n - q\| \|j(x_{n-1} - q) - j(v_n - q)\| + 2\epsilon\alpha_n \end{aligned}$$

$$\begin{aligned}
&\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} k \|x_n - q\|^2 + 2M \frac{\alpha_n \beta_n}{1 - \alpha_n} \delta_n \\
&\quad + 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} k \|v_n - q\|^2 + 2M \frac{\alpha_n \gamma_n}{1 - \alpha_n} \eta_n + 2\epsilon \alpha_n \\
&\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} k \|x_n - q\|^2 \\
&\quad + 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} k \|v_n - q\|^2 + 2M \alpha_n \max\{\delta_n, \eta_n\} + 2\epsilon \alpha_n,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\delta_n &= \|j(x_{n-1} - q) - j(x_n - q)\|, \\
\eta_n &= \|j(x_{n-1} - q) - j(v_n - q)\|.
\end{aligned} \tag{3.9}$$

Also, we have

$$\begin{aligned}
\|x_{n-1} - x_n\| &= \|x_{n-1} - \alpha_n x_{n-1} - \beta_n T x_n - \gamma_n T v_n\| \\
&= \|\beta_n (x_{n-1} - T x_n) + \gamma_n (x_{n-1} - T v_n)\| \\
&\leq \beta_n \|x_{n-1} - T x_n\| + \gamma_n \|x_{n-1} - T v_n\| \\
&\leq 2M(\beta_n + \gamma_n) \\
&< \infty
\end{aligned} \tag{3.10}$$

implies

$$\|x_{n-1} - x_n\| \longrightarrow 0, \tag{3.11}$$

as $n \rightarrow \infty$, and consequently

$$\|x_{n-1} - v_n\| \leq \|x_{n-1} - x_n\| + \|x_n - v_n\| \longrightarrow 0 \tag{3.12}$$

as $n \rightarrow \infty$. Since J is uniformly continuous on any bounded subsets of X , we have

$$\delta_n, \eta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.13}$$

For any given $\epsilon > 0$ and the bounded subset K , there exists a $\delta > 0$ satisfying (2.2). Note that (3.13) and (vi) ensure that there exists an N such that

$$\beta_n, \gamma_n < \min\left\{\delta, \frac{\epsilon}{8M^2k}\right\}, \quad \delta_n, \eta_n \leq \frac{\epsilon}{4M}, \quad n \geq N. \tag{3.14}$$

Now substituting (3.6) in (3.8) to obtain

$$\begin{aligned}\|x_n - q\|^2 &\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2k\alpha_n\|x_n - q\|^2 \\ &\quad + 2M\alpha_n \max\{\delta_n, \eta_n\} + 2\epsilon\alpha_n \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right),\end{aligned}\tag{3.15}$$

by using (3.7), implies

$$\begin{aligned}\|x_n - q\|^2 &\leq (1 - 2(1 - k)\alpha_n)\|x_{n-1} - q\|^2 + 2\epsilon\alpha_n \\ &\quad + 2M^2k\alpha_n(\beta_n + \gamma_n) + 2M\alpha_n \max\{\delta_n, \eta_n\} \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right) \\ &\leq (1 - 2(1 - k)\alpha_n)\|x_{n-1} - q\|^2 + 3\epsilon\alpha_n \\ &\quad + 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right)\end{aligned}\tag{3.16}$$

for all $n \geq N$.

Put

$$\begin{aligned}\sigma_n &= \|x_{n-1} - q\|^2, \quad \theta_n = 2(1 - k)\alpha_n, \quad \epsilon' = \frac{3\epsilon}{2(1 - k)}, \\ \omega_n &= 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right),\end{aligned}\tag{3.17}$$

and we have from (3.16)

$$\sigma_{n+1} \leq (1 - \theta_n)\sigma_n + \epsilon'\theta_n + \omega_n, \quad n \geq 1.\tag{3.18}$$

For $k < 1/2$, set $\delta = 1/2(1 - k) < 1$. Because $\alpha_n \leq \delta$, we imply $1 - \alpha_n \geq 1 - \delta$ and $2(1 - k)\alpha_n \leq 1$. Now observe that $\sum_{n=1}^{\infty} \theta_n = \infty, \theta_n \leq 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \omega_n < \infty$. It follows from Lemma 2.5 that

$$\limsup_{n \rightarrow \infty} \|x_n - q\|^2 \leq \epsilon'.\tag{3.19}$$

Letting $\epsilon' \rightarrow 0^+$, we obtain that $\limsup_{n \rightarrow \infty} \|x_n - q\|^2 = 0$, which implies that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. $\square$

Corollary 3.3. *Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \rightarrow K$ be a Lipschitz strictly hemicontractive mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the conditions (iv)–(vi).*

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (3.4). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Corollary 3.4. Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \rightarrow K$ be a continuous strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the conditions (v) and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Corollary 3.5. Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \rightarrow K$ be a Lipschitz strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the conditions (v) and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Remark 3.6. Similar results can be found for the iteration processes involved error terms; we omit the details.

Remark 3.7. Theorem 3.2 and Corollary 3.3 extend and improve Theorem 1.4 in the following directions.

We do not need the assumption $\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$ as in Theorem 1.4.

4. Applications for Multistep Implicit Iterations

Let K be a nonempty closed convex subset of a smooth Banach space X and let $T, T_1, T_2, \dots, T_p : K \rightarrow K$ ($p \geq 2$) be a family of $p + 1$ mappings.

Algorithm 4.1. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the implicit iteration process of arbitrary fixed order $p \geq 2$:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^i &= \beta_n^i x_{n-1} + (1 - \beta_n^i) T_{i+1} y_n^{i+1}, \quad i = 1, 2, \dots, p-2, \\ y_n^{p-1} &= \beta_n^{p-1} x_{n-1} + (1 - \beta_n^{p-1}) T_p x_n, \quad n \geq 1, \end{aligned} \quad (4.1)$$

which is called the *multistep implicit iteration process*, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\beta_n^i\}$, $i = 1, 2, \dots, p-1$ are real sequences in $[0, 1]$ and $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$.

For $p = 3$, we obtain the following three-step implicit iteration process.

Algorithm 4.2. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + (1 - \beta_n^1) T_2 y_n^2, \\ y_n^2 &= \beta_n^2 x_{n-1} + (1 - \beta_n^2) T_3 x_n, \quad n \geq 1, \end{aligned} \quad (4.2)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are real sequences in $[0, 1]$ satisfying some certain conditions.

For $p = 2$, we obtain the following two-step implicit iteration process.

Algorithm 4.3. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + (1 - \beta_n^1) T_2 x_n, \quad n \geq 1, \end{aligned} \quad (4.3)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\beta_n^1\}$ are real sequences in $[0, 1]$ satisfying some certain conditions.

If $T_1 = T$, $T_2 = I$ and $\beta_n^1 = 0$ in (4.3), we obtain the following implicit Mann iteration process.

Algorithm 4.4. For any given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_n, \quad n \geq 1, \quad (4.4)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying some certain conditions.

Theorem 4.5. Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T, T_1, T_2, \dots, T_p : K \rightarrow K$ ($p \geq 2$) be $p + 1$ mappings. Let T, T_1 be continuous strictly hemiccontractive mappings. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\beta_n^i\}, i = 1, 2, \dots, p - 1$ be real sequences in $[0, 1]$ satisfying the conditions (iv)–(vi) and $\sum_{n=1}^{\infty} (1 - \beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^p F(T_i) \cap F(T) \neq \emptyset$.

Proof. By applying Theorem 3.2 under assumption that T and T_1 are continuous strictly hemiccontractive mappings, we obtain Theorem 4.5 which proves strong convergence of the iteration process defined by (4.1). Consider by taking $T_1 = T$ and $v_n = y_n^1$,

$$\begin{aligned} \|v_n - x_n\| &= \|y_n^1 - x_n\| \\ &= \|\beta_n^1 x_{n-1} + (1 - \beta_n^1) T_2 y_n^2 - x_n\| \\ &= \|\beta_n^1 (x_{n-1} - x_n) + (1 - \beta_n^1) (T_2 y_n^2 - x_n)\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + (1 - \beta_n^1) \|T_2 y_n^2 - x_n\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + M' (1 - \beta_n^1). \end{aligned} \quad (4.5)$$

From (4.5) and the condition $\sum_{n=1}^{\infty} (1 - \beta_n^1) < \infty$, we obtain

$$\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty. \quad (4.6)$$

This completes the proof. □

Corollary 4.6. Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T, T_1, T_2, \dots, T_p : K \rightarrow K$ ($p \geq 2$) be $p+1$ mappings. Let T, T_1 be Lipschitz strictly hemicontractive mappings. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\beta_n^i\}$, $i = 1, 2, \dots, p-1$ be real sequences in $[0, 1]$ satisfying the conditions (iv)–(vi) and $\sum_{n=1}^{\infty} (1 - \beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^p F(T_i) \cap F(T) \neq \emptyset$.

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