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Research Article

Analytic Approximation of the Solutions of Stochastic Differential Delay Equations with Poisson Jump and Markovian Switching

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We are concerned with the stochastic differential delay equations with Poisson jump and Markovian switching (SDDEsPJMSs). Most SDDEsPJMSs cannot be solved explicitly as stochastic differential equations. Therefore, numerical solutions have become an important issue in the study of SDDEsPJMSs. The key contribution of this paper is to investigate the strong convergence between the true solutions and the numerical solutions to SDDEsPJMSs when the drift and diffusion coefficients are Taylor approximations.

1. Introduction

Recently there has been an increasing interest in the study of stochastic differential delay equations with Poisson jump and Markovian switching (SDDEsPJMSs). Svīshchuk and Kazmerchuk [1] investigated stability of stochastic differential delay equations with linear Poisson jumps and Markovian switchings. While Luo [2] discussed the comparison principle and several kinds of stability of Itô stochastic differential delay equations with Poisson jump and Markovian switching. Besides, Li and Chang [3] discussed the convergence of the numerical solutions of stochastic differential delay equations with Poisson jump and Markovian switching. In the present paper we will further research this topic and our focus is on the convergence of numerical solution to stochastic differential delay equation with Poisson jump and Markovian switching when the coefficients are Taylor approximations.

Stochastic differential delay equation with Poisson jump and Markovian switching may be considered as extension of stochastic differential delay equation with Poisson jump. Of course, it may also be regarded as an generalization of stochastic differential delay equation with Markovian switching. Similar to stochastic differential delay equations

with Poisson jump, explicit solutions can hardly be obtained for the stochastic differential delay equations with Poisson jump and Markovian switching. Thus appropriate numerical approximation schemes such as the Euler (or Euler-Maruyama) are needed if we apply them in practice or to study their properties. There is an extensive literature concerning the approximate schemes for either stochastic differential delay equations with Poisson jump or stochastic differential delay equations with Markovian switching [4–12].

However, the rate of convergence to the true solution by the numerical solution is different for different numerical schemes [13]. Recently, Janković and Ilić [14] have investigated the rate of convergence between the true solution and numerical solution of the stochastic differential equations in the sense of the L^p-norm when the drift and diffusion coefficients are Taylor approximations, up to arbitrary fixed derivatives. Moreover, Jiang et al. [15] generalized the Taylor method to stochastic differential delay equations with Poisson jump. Since the rate of convergence for such a numerical method is faster than the result obtained in [13], in this paper, we intend to generalize the above method to the SDDEsPJMSs case and consider the strong convergence between the true solution and numerical; solution to SDDEsPJMSs if the drift and diffusion coefficients are Taylor approximations, up to arbitrary fixed derivatives. To the best of our knowledge, so far there seem to be no existing results. Therefore, the aim of this paper is to close this gap.

In Section 2, we introduce necessary notations and approximation scheme. Then comes our main result that the Taylor approximate solutions will converge to the true solutions of SDDEsPJMSs. The proof of this main result is rather technical so we present several lemmas in Section 3 and then complete the proof in Section 4.

2. Approximation Scheme and Hypotheses

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous while $\mathcal{F}_0$ contains all P-null sets). Let $C([-\tau, 0]; R)$ be the family of continuous function ξ from $[-\tau, 0]$ to R with the norm $\|\xi\| = \sup_{-\tau\leq t\leq 0}|\xi(t)|$. We also denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R)$-valued random variables and denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; R)$ the family of all $\mathcal{F}_t$-measurable, $C([-\tau, 0]; R)$-valued random variables $\xi = \{\xi(t) : -\tau \leq t \leq 0\}$ satisfying $\sup_{-\tau\leq t\leq 0} E|\xi(t)|^p < \kappa < \infty$, where κ is a positive constant.

Let $w(t)$, $t \geq 0$, be a one-dimensional Brownian motion defined on the probability space and let $N(t)$, $t \geq 0$, be a scalar Poisson process with intensity λ which is independent of $w(t)$. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta), & \text{if } i = j, \end{cases} \tag{2.1}$$

where $\delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$ and Poisson process $N(t)$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\Re_+ = [0, \infty)$.

Consider stochastic differential delay equations with Poisson jump and Markovian switching of the form

$$\begin{aligned} dx(t) &= f(t, x(t), x(t-\tau), r(t))dt + g(t, x(t), x(t-\tau), r(t))dw(t) \\ &\quad + h(t, x(t), x(t-\tau), r(t))dN(t), \end{aligned} \tag{2.2}$$

on the time interval $[0, T]$ with initial data $\xi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0]; R)$ being independent of $w(t)$ and $N(t)$, and $r(0) = i_0 \in S$, where

$$f : [0, \infty) \times R \times R \times S \longrightarrow R, \qquad g : [0, \infty) \times R \times R \times S \longrightarrow R, \qquad h : [0, \infty) \times R \times R \times S \to R. \tag{2.3}$$

In the paper, $f_x^{(j)}$, $g_x^{(j)}$, and $h_x^{(j)}$ denote, respectively, jth-order partial derivatives of f, g and h with respect to x. Further, to ensure the existence and uniqueness of the solution to (2.2), we impose the following hypotheses.

(H_1) f, g, and h satisfy Lipschitz and linear growth condition; that is, there exists a positive constant $L > 0$ such that

$$\begin{aligned} &\left|f(t, x_1, x_2, i) - f(t, y_1, y_2, i)\right| \vee \left|g(t, x_1, x_2, i) - g(t, y_1, y_2, i)\right| \\ &\quad \vee \left|h(t, x_1, x_2, i) - h(t, y_1, y_2, i)\right| \le L\left(\left|x_1 - y_1\right|^2 + \left|x_2 - y_2\right|^2\right), \\ &\left|f(t, x_1, x_2, i)\right|^2 \vee \left|g(t, x_1, x_2, i)\right|^2 \vee |h(t, x_1, x_2, i)|^2 \le L^2\left(1 + |x_1|^2 + |x_2|^2\right), \end{aligned} \tag{2.4}$$

for $x_1, x_2, y_1, y_2 \in R$ and $i \in S$.

(H_2) There exist constants $K_1 > 0$ and $\gamma \in (0, 1]$ such that, for all $-\tau \le s < t \le 0$ and $\rho \ge 2$,

$$E|\xi(t) - \xi(s)|^\rho \le K_1(t-s)^\gamma. \tag{2.5}$$

(H_3) f, g, and h have Taylor approximations in the second argument, up to m_1th, m_2th, and m_3th derivatives, respectively.

(H_4) Partial derivatives of the order $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$ of the functions f, g, and h, $f_x^{(m_1+1)}(t, x, y, i)$, $g_x^{(m_2+1)}(t, x, y, i)$, and $h_x^{(m_3+1)}(t, x, y, i)$, are uniformly bounded; that is, there exist positive constants L_1, L_2, and L_3 obeying

$$\begin{aligned} &\sup_{[0,T]\times R\times R\times S} \left|f_x^{(m_1+1)}(t, x, y, i)\right| \le L_1, \\ &\sup_{[0,T]\times R\times R\times S} \left|g_x^{(m_2+1)}(t, x, y, i)\right| \le L_2, \\ &\sup_{[0,T]\times R\times R\times S} \left|h_x^{(m_3+1)}(t, x, y, i)\right| \le L_3. \end{aligned} \tag{2.6}$$

For some sufficiently large integer M, we define the time step by $h = \tau/M$, where $0 < h \ll 1$. Then, the approximation solution to (2.2) is computed by $y(t) = \xi(t)$ on $-\tau \le t \le 0$, and for any $t \ge 0$,

$$\begin{aligned} y(t) = \xi(0) &+ \int_0^t \sum_{j=0}^{m_1} \frac{f_x^{(j)}(s, z_1(s), z_2(s), \overline{r}(s))}{j!} \left(y(s) - z_1(s)\right)^j ds \\ &+ \int_0^t \sum_{j=0}^{m_2} \frac{g_x^{(j)}(s, z_1(s), z_2(s), \overline{r}(s))}{j!} \left(y(s) - z_1(s)\right)^j dw(s) \\ &+ \int_0^t \sum_{j=0}^{m_3} \frac{h_x^{(j)}(s, z_1(s), z_2(s), \overline{r}(s))}{j!} \left(y(s) - z_1(s)\right)^j dN(s), \end{aligned} \tag{2.7}$$

where, for $t_k = kh$ with integer $k \ge 0$,

$$z_1(t) = \sum_{k=0}^{\infty} y(t_k) I_{[t_k, t_{k+1})}(t), \qquad z_2(t) = \sum_{k=0}^{\infty} y(t_k - \tau) I_{[t_k, t_{k+1})}(t), \qquad \overline{r}(t) = \sum_{k=0}^{\infty} r(t_k) I_{[t_k, t_{k+1})}(t). \tag{2.8}$$

In order to show the strong convergence of the numerical solutions and the exact solutions to (2.2), we will need the following assumption.

(H_5) There exists a positive constant K_2 such that

$$E\left[\sup_{0 \le t \le T} |x(t)|^p\right] \vee E\left[\sup_{0 \le t \le T} \left|y(t)\right|^{(m+1)^2 p}\right] \le K_2, \tag{2.9}$$

for any $p > 0$, where $m = \max\{m_1, m_2, m_3\}$.

Remark 2.1. (H_1) shows that the exact solutions to (2.2) admit finite moments; see [15]. If $m_1 = m_2 = m_3 = 0$, then (H_1) shows also that the numerical solutions and the exact solutions admit finite moments (see, [1, 3]).

We can now state our main result of this paper.

Theorem 2.2. *Under assumptions* (H_1)–(H_5), *for any* $p \ge 2$, *then*

$$\lim_{h \to 0} E\left[\sup_{0 \le t \le T} \left|x(t) - y(t)\right|^p\right] = 0. \tag{2.10}$$

The proof of this theorem is rather technical. We will present a number of useful lemmas in Section 3 and then complete the proof in Section 4.

3. Lemmas

Throughout our analysis, C_i, $i = 1, 2, \ldots$ denote generic constants, independent of h. In order to prove the main theorem, the following lemmas are useful.

Lemma 3.1. *If assumptions* (H_1), (H_3), (H_4), *and* (H_5) *hold, then, for* $2 \le \rho \le (m+1)p$,

$$E\left|y(t) - z_1(t)\right|^{\rho} \le Ch^{\rho/2}, \quad t \ge 0, \tag{3.1}$$

where C *is a positive constant independent of* h.

Proof. For notation simplicity reason, let us denote that

$$\begin{aligned}
A\big(t, y(t), z_1(t), z_2(t), \overline{r}(t)\big) &= \sum_{j=0}^{m_1} \frac{f_x^{(j)}(t, z_1(t), z_2(t), \overline{r}(t))}{j!}\big(y(t) - z_1(t)\big)^j, \\
B\big(t, y(t), z_1(t), z_2(t), \overline{r}(t)\big) &= \sum_{j=0}^{m_2} \frac{g_x^{(j)}(t, z_1(t), z_2(t), \overline{r}(t))}{j!}\big(y(t) - z_1(t)\big)^j, \\
C\big(t, y(t), z_1(t), z_2(t), \overline{r}(t)\big) &= \sum_{j=0}^{m_3} \frac{h_x^{(j)}(t, z_1(t), z_2(t), \overline{r}(t))}{j!}\big(y(t) - z_1(t)\big)^j.
\end{aligned} \tag{3.2}$$

Obviously, for any $t \ge 0$, there exists an integer $k \ge 0$ such that $t \in [t_k, t_{k+1}]$. Then, by (2.7) and (2.8) we obtain

$$\begin{aligned}
y(t) - z_1(t) &= y(t) - y(t_k) \\
&= \int_{t_k}^{t} A\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)ds + \int_{t_k}^{t} B\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)dw(s) \\
&\quad + \int_{t_k}^{t} C\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)dN(s).
\end{aligned} \tag{3.3}$$

Hence, we have

$$\begin{aligned}
E\left|y(t) - z_1(t)\right|^{\rho} \le 3^{\rho-1}\Bigg[&E\left|\int_{t_k}^{t} A\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)ds\right|^{\rho} \\
&+ E\left|\int_{t_k}^{t} B\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)dw(s)\right|^{\rho} \\
&+ E\left|\int_{t_k}^{t} C\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)dN(s)\right|^{\rho}\Bigg].
\end{aligned} \tag{3.4}$$

By the Hölder inequality, we have

$$E\left|\int_{t_k}^{t} A\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)ds\right|^{\rho} \le (t - t_k)^{\rho-1}\int_{t_k}^{t} E\left|A\big(s, y(s), z_1(s), z_2(s), \overline{r}(s)\big)\right|^{\rho}ds. \tag{3.5}$$

By the Burkholder-Davis-Gundy inequality and the Hölder inequality, we yield, for some positive constant C_ρ,

$$\begin{aligned} &E\left|\int_{t_k}^{t} B\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)dw(s)\right|^{\rho} \\ &\quad\le C_\rho (t-t_k)^{\rho/2-1}\int_{t_k}^{t} E\big|B\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)\big|^{\rho}ds. \end{aligned} \tag{3.6}$$

For the jump integral, we convert to the compensated Poisson process $\widetilde{N}(t) := N(t) - \lambda t$, which is a martingale with

$$E\left|\int_{t_1}^{t_2} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)d\widetilde{N}(s)\right|^{2} = \lambda\int_{t_1}^{t_2} E\big|C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)\big|^{2}ds; \tag{3.7}$$

see, for example, [1, 4, 12, 16]. By the Burkholder-Davis-Gundy inequality and the Hölder inequality, for some positive constant $C_{\lambda,\rho}$, we then obtain

$$\begin{aligned} &E\left|\int_{t_k}^{t} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)dN(s)\right|^{\rho} \\ &\quad= E\left|\int_{t_k}^{t} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)d\widetilde{N}(s) + \lambda\int_{t_k}^{t} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)ds\right|^{\rho} \\ &\quad\le 2^{\rho-1}E\left|\int_{t_k}^{t} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)d\widetilde{N}(s)\right|^{\rho} \\ &\qquad + 2^{\rho-1}E\left|\lambda\int_{t_k}^{t} C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)ds\right|^{\rho} \\ &\quad\le 2^{\rho-1}C_{\lambda,\rho}(t-t_k)^{\rho/2-1}\int_{t_k}^{t} E\big|C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)\big|^{\rho}ds \\ &\qquad + 2^{\rho-1}\lambda^{\rho}(t-t_k)^{\rho-1}\int_{t_k}^{t} E\big|C\big(s,y(s),z_1(s),z_2(s),\bar{r}(s)\big)\big|^{\rho}ds. \end{aligned} \tag{3.8}$$

Now, by the mean value theorem there is a $\theta \in (0,1)$ such that

$$\begin{aligned} J_1(t) &= \int_{t_k}^{t} E\left|f(s,y(s),z_2(s),\bar{r}(s)) - \left[f(s,y(s),z_2(s),\bar{r}(s)) - A(s,y(s),z_1(s),z_2(s),\bar{r}(s))\right]\right|^{\rho} ds \\ &= \int_{t_k}^{t} E\left| f(s,y(s),z_2(s),\bar{r}(s)) - \frac{f_x^{(m_1+1)}(s,z_1(s)+\theta(y(s)-z_1(s)),z_2(s),\bar{r}(s))}{(m_1+1)!} \right. \\ &\qquad \left. \times (y(s)-z_1(s))^{m_1+1} \right|^{\rho} ds. \end{aligned} \tag{3.9}$$

This, together with (H_1), (H_3)–(H_5), yields that

$$\begin{aligned} J_1(t) &\le 2^{\rho-1} \int_{t_k}^{t} \left[E\left(\left|f(s,y(s),z_2(s),\bar{r}(s))\right|^2\right)^{\rho/2} + \frac{L_1^{\rho}}{[(m_1+1)!]^{\rho}} E\left|y(s)-z_1(s)\right|^{(m_1+1)\rho} \right] ds \\ &\le 2^{\rho-1} \int_{t_k}^{t} \left[3^{\rho/2} L^{\rho}\left(1 + E\left|y(s)\right|^{\rho} + E|z_2(s)|^{\rho}\right) + \frac{L_1^{\rho} 2^{(m_1+1)\rho}}{[(m_1+1)!]^{\rho}} \right. \\ &\qquad \left. \times \left(E\left|y(s)\right|^{(m_1+1)\rho} + E|z_1(s)|^{(m_1+1)\rho}\right) \right] ds \\ &\le 2^{\rho-1} \int_{t_k}^{t} \left[3^{\rho/2} L^{\rho}(1+2K_2+\kappa) + \frac{L_1^{\rho} 2^{(m_1+1)\rho+1}}{[(m_1+1)!]^{\rho}} K_2 \right] ds \\ &= C_1(t-t_k). \end{aligned} \tag{3.10}$$

Similarly, we can also show that there are two positive constant C_2 and C_3 for which

$$\begin{aligned} J_2(t) &= \int_{t_k}^{t} E\left|B(s,y(s),z_1(s),z_2(s),\bar{r}(s))\right|^{\rho} ds \\ &\le C_2(t-t_k), \\ J_3(t) &= \int_{t_k}^{t} E\left|C(s,y(s),z_1(s),z_2(s),\bar{r}(s))\right|^{\rho} ds \\ &\le C_3(t-t_k). \end{aligned} \tag{3.11}$$

Next, since the boundedness of $J_1(t)$, $J_2(t)$, and $J_3(t)$, we then have some positive constant C, independent of h, such that

$$E\left|y(t)-z_1(t)\right|^{\rho} \le Ch^{\rho/2}. \tag{3.12}$$

The desired assertion is complete. □

Lemma 3.2. *If assumptions* (H_1)–(H_5) *hold, then, for* $2 \le \rho \le (m+1)p$,

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} \le \overline{C}h^{\gamma}, \quad t \ge 0, \tag{3.13}$$

where $\gamma \in (0,1]$ *and* $\overline{C}$ *is a positive constant independent of* h.

Proof. Clearly, for any $t \ge 0$, there exists an integer $k \ge 0$ such that $t \in [t_k, t_{k+1})$. In what follows, we split the following three cases to complete the proof.

Case 1. If $-\tau \le t_k - \tau \le t - \tau \le 0$, we then have, from (H_2),

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} = E\left|y(t-\tau) - y(t_k-\tau)\right|^{\rho} = E|\xi(t-\tau) - \xi(t_k-\tau)|^{\rho} \le K_1 h^{\gamma}. \tag{3.14}$$

Case 2. If $0 \le t_k - \tau \le t - \tau$, then, by Lemma 3.1, we have

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} \le C_4 h^{\rho/2}. \tag{3.15}$$

Case 3. If $-\tau \le t_k - \tau \le 0 \le t - \tau$, note that

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} \le 2^{\rho-1}E\left|y(t-\tau) - \xi(0)\right|^{\rho} + 2^{\rho-1}E\left|y(t_k-\tau) - \xi(0)\right|^{\rho}. \tag{3.16}$$

Then, by the above two cases, it follows easily that

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} \le C_5\left(h^{\rho/2} + h^{\gamma}\right). \tag{3.17}$$

Now, combining the three cases altogether, for $2 \le \rho \le (m+1)P$ and $\gamma \in (0,1]$,

$$E\left|y(t-\tau) - z_2(t)\right|^{\rho} \le \overline{C}h^{\gamma}; \tag{3.18}$$

where $\overline{C}$ is a positive constant independent of h. The proof is complete. □

Lemma 3.3. *If assumptions* (H_1) *and* (H_5) *hold, then, for* $p \ge 2$,

$$\int_0^T E\left|f\left(s, y(s), z_2(s), r(s)\right) - f\left(s, y(s), z_2(s), \overline{r}(s)\right)\right|^p ds \le \widetilde{C}_1 h, \tag{3.19}$$

$$\int_0^T E\left|g\left(s, y(s), z_2(s), r(s)\right) - g\left(s, y(s), z_2(s), \overline{r}(s)\right)\right|^p ds \le \widetilde{C}_2 h, \tag{3.20}$$

$$\int_0^T E\left|h\left(s, y(s), z_2(s), r(s)\right) - h\left(s, y(s), z_2(s), \overline{r}(s)\right)\right|^p ds \le \widetilde{C}_3 h, \tag{3.21}$$

where $\widetilde{C}_1, \widetilde{C}_2$, *and* $\widetilde{C}_3$ *are positive constants dependent on* $\max_{0\le i\le N}(-\gamma_{ii})$, *but independent of* h.

Proof. Let $n = [T/h]$ be the integer part of T/h. Then

$$\begin{aligned} &E\int_0^T \left|f(s,y(s),z_2(s),r(s)) - f(s,y(s),z_2(s),\overline{r}(s)\right|^p ds \\ &\quad = \sum_{k=0}^{n} E\int_{t_k}^{t_{k+1}} \left|f(s,y(s),z_2(s),r(s)) - f(s,y(s),z_2(s),r(t_k))\right|^p ds, \end{aligned} \tag{3.22}$$

with t_{n+1} being T. By (H_1) and (H_5), we derive

$$\begin{aligned} &E\int_{t_k}^{t_{k+1}} \left|f(f(s,y(s),z_2(s),r(s)) - f(f(s,y(s),z_2(s),r(t_k))\right|^p ds \\ &\quad \le 2^{p-1}E\int_{t_k}^{t_{k+1}} \left[\left|f(s,y(s),z_2(s),r(s))\right|^p + \left|f(s,y(s),z_2(s),r(t_k))\right|^p\right] I_{\{r(s)\neq r(t_k)\}} ds \\ &\quad \le 2^p L^p E\int_{t_k}^{t_{k+1}} \left(1 + \left|y(s)\right|^2 + |z_2(s)|^2\right)^{p/2} I_{\{r(s)\neq r(t_k)\}} ds \\ &\quad \le 2^p L^p 3^{p/2-1}\int_{t_k}^{t_{k+1}} E\left[E\left[\left(1 + \left|y(s)\right|^p + |z_2(s)|^p\right) I_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right]\right] ds \\ &\quad \le 2^p L^p 3^{p/2-1}\int_{t_k}^{t_{k+1}} E\left[E\left[(1+2K_2+\kappa) \mid r(t_k)\right] E\left[I_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right]\right] ds. \end{aligned} \tag{3.23}$$

Now, by the Markov property [6], we have

$$\begin{aligned} E\left[I_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right] &= \sum_{i\in S} I_{\{r(t_k)=i\}} P(r(s)\neq i \mid r(t_k) = i) \\ &= \sum_{i\in S} I_{\{r(t_k)=i\}} \sum_{j\neq i} \left(\gamma_{ij}(s-t_k) + o(s-t_k)\right) \\ &\le \sum_{i\in S} I_{\{r(t_k)=i\}} \left(\max_{1\le i\le N}(-\gamma_{ij})\ \triangle + o(\triangle)\right). \end{aligned} \tag{3.24}$$

So, (3.19) is complete. Similarly, we can show (3.20) and (3.21). □

4. Proof of Theorem 2.2

Let us now begin to prove our main result Theorem 2.2. Clearly,

$$
\begin{aligned}
&E\left[\sup_{0\le t\le T}|x(t)-y(t)|^p\right] \\
&\quad = E\left\{\sup_{0\le t\le T}\left|\int_0^t\left[f(s,x(s),x(s-\tau),r(s))-A\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]ds\right.\right. \\
&\qquad\qquad +\int_0^t\left[g(s,x(s),x(s-\tau),r(s))-B\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]dw(s) \\
&\qquad\qquad \left.\left.+\int_0^t\left[h(s,x(s),x(s-\tau),r(s))-C(s,y(s),z_1(s),z_2(s),\overline{r}(s))\right]dN(s)\right|^p\right\} \\
&\quad \le 3^{p-1}\left\{E\left(\sup_{0\le t\le T}\left|\int_0^t\left[f(s,x(s),x(s-\tau),r(s))-A\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]ds\right|^p\right)\right. \\
&\qquad + E\left(\sup_{0\le t\le T}\left|\int_0^t\left[g(s,x(s),x(s-\tau),r(s))-B\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]dw(s)\right|^p\right) \\
&\qquad \left.+E\left(\sup_{0\le t\le T}\left|\int_0^t\left[h(s,x(s),x(s-\tau),r(s))-C\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]dN(s)\right|^p\right)\right\}.
\end{aligned}
\tag{4.1}
$$

So, for any $t_1 \le T$, by the Hölder inequality,

$$
\begin{aligned}
&E\left(\sup_{0\le t\le t_1}\left|\int_0^t\left[f(s,x(s),x(s-\tau),r(s))-A\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]ds\right|^p\right) \\
&\quad \le T^{p-1}\int_0^{t_1}E\left|f(s,x(s),x(s-\tau),r(s))-A\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right|^p ds.
\end{aligned}
\tag{4.2}
$$

By the Burkholder-Davis-Gundy inequality, for some positive constant C_p, we have

$$
\begin{aligned}
&E\left(\sup_{0\le t\le t_1}\left|\int_0^t\left[g(s,x(s),x(s-\tau),r(s))-B\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right]dw(s)\right|^p\right) \\
&\quad \le C_pT^{p/2-1}\int_0^{t_1}E\left|g(s,x(s),x(s-\tau),r(s))-B\big(s,y(s),z_1(s),z_2(s),\overline{r}(s)\big)\right|^p ds.
\end{aligned}
\tag{4.3}
$$

By Doob's martingale inequality and the compensated Poisson integral, for some positive constant $C_{\lambda,p,T}$, we yield

$$\begin{aligned}
&E\left(\sup_{0\le t\le t_1}\left|\int_0^t \left[h(s,x(s),x(s-\tau),r(s)) - C\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right]dN(s)\right|^p\right) \\
&\quad\le \left(\frac{p}{p-1}\right)^p 2^{p-1}C_{\lambda,p,T}\int_0^{t_1} E\left|h(s,x(s),x(s-\tau),r(s)) - C\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&\qquad + 2^{p-1}\lambda^p T^{p-1}\int_0^{t_1} E\left|h(s,x(s),x(s-\tau),r(s)) - C\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds.
\end{aligned} \tag{4.4}$$

Now, by virtue of the assumption (H_1), we obtain that

$$\begin{aligned}
J_4(t) &= \int_0^{t_1} E\left|f(s,x(s),x(s-\tau),r(s)) - A\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&= \int_0^{t_1} E\Big|f(s,x(s),x(s-\tau),r(s)) - f\left(s,y(s),z_2(s),r(s)\right) \\
&\qquad + f\left(s,y(s),z_2(s),r(s)\right) - f\left(s,y(s),z_2(s),\overline{r}(s)\right) \\
&\qquad + f\left(s,y(s),z_2(s),\overline{r}(s)\right) - A\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\Big|^p ds \\
&\le 3^{p-1}\left[\int_0^{t_1} E\left|f(s,x(s),x(s-\tau),r(s)) - f\left(s,y(s),z_2(s),r(s)\right)\right|^p ds\right] \\
&\qquad + \int_0^{t_1} E\left|f\left(s,y(s),z_2(s),r(s)\right) - f\left(s,y(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&\qquad + \int_0^{t_1} E\left|f\left(s,y(s),z_2(s),\overline{r}(s)\right) - A\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds.
\end{aligned} \tag{4.5}$$

Moreover, by (H_1) and Lemma 3.2, we have

$$\begin{aligned}
&\int_0^{t_1} E\left|f(s,x(s),x(s-\tau),r(s)) - f\left(s,y(s),z_2(s),r(s)\right)\right|^p ds \\
&\quad\le L^p\int_0^{t_1} E\left(\left|x(s)-y(s)\right| + \left|x(s-\tau)-z_2(s)\right|\right)^p ds \\
&\quad\le L^p 2^{2p-2}\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p + E\left|x(s-\tau)-y(s-\tau)\right|^p + E\left|y(s-\tau)-z_2(s)\right|^p\right)ds \\
&\quad\le L^p 2^{2p-2}\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p + E\left|x(s-\tau)-y(s-\tau)\right|^p\right)ds + L^p 2^{2p-2}T\overline{C}h^{\gamma},
\end{aligned} \tag{4.6}$$

and by Lemma 3.1 and (H_4), there exists a $\theta_1 \in (0,1)$ such that

$$\begin{aligned}
&\int_0^{t_1} E\left|f\left(s,y(s),z_2(s),\overline{r}(s)\right)-A\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&\quad=\int_0^{t_1} E\frac{\left|f_x^{(m_1+1)}\left(s,z_1(s)+\theta_1\left(y(s)-z_1(s)\right),z_2(s),\overline{r}(s)\right)\right|^p}{[(m_1+1)!]^p}\left|y(s)-z_1(s)\right|^{(m_1+1)p}ds \\
&\quad\le\frac{L_1^pTC}{[(m_1+1)!]^p}h^{(m_1+1)p/2}.
\end{aligned} \tag{4.7}$$

Hence, by (4.6) and (4.7), together with Lemma 3.3, we yield

$$\begin{aligned}
J_4(t)\le 3^{p-1}\Bigg[&\widetilde{C}_1h+\frac{L_1^pTC}{[(m_1+1)!]^p}h^{(m_1+1)p/2}+L^p2^{2p-2}T\overline{C}h^{\gamma} \\
&+L^p2^{2p-2}\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p+E\left|x(s-\tau)-y(s-\tau)\right|^p\right)ds\Bigg].
\end{aligned} \tag{4.8}$$

Similarly, we have

$$\begin{aligned}
J_5(t)&=\int_0^{t_1} E\left|g(s,x(s),x(s-\tau),r(s))-B\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&\le 3^{p-1}\Bigg[\widetilde{C}_2h+\frac{L_2^pTC}{[(m_2+1)!]^p}h^{(m_2+1)p/2}+L^p2^{2p-2}T\overline{C}h^{\gamma} \\
&\qquad+L^p2^{2p-2}\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p+E\left|x(s-\tau)-y(s-\tau)\right|^p\right)ds\Bigg],
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
J_6(t)&=\int_0^{t_1} E\left|h(s,x(s),x(s-\tau),r(s))-C\left(s,y(s),z_1(s),z_2(s),\overline{r}(s)\right)\right|^p ds \\
&\le 3^{p-1}\Bigg[\widetilde{C}_3h+\frac{L_3^pTC}{[(m_3+1)!]^p}h^{(m_3+1)p/2}+L^p2^{2p-2}T\overline{C}h^{\gamma} \\
&\qquad+L^p2^{2p-2}\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p+E\left|x(s-\tau)-y(s-\tau)\right|^p\right)ds\Bigg].
\end{aligned} \tag{4.10}$$

Now we use K to denote a generic positive constant that may change between occurrences. Hence, by (4.2)–(4.4) and (4.8)–(4.10), we yield that

$$\begin{aligned} &E\left(\sup_{0\le t\le t_1}\left|x(t)-y(t)\right|^p\right) \\ &\quad\le Kh+Kh^{(m+1)p/2}+Kh^{\gamma}+K\int_0^{t_1}\left(E\left|x(s)-y(s)\right|^p+E\left|x(s-\tau)-y(s-\tau)\right|^p\right)ds \\ &\quad\le Kh+Kh^{(m+1)p/2}+Kh^{\gamma}+2K\int_0^{t_1}E\left(\sup_{0\le r\le s}\left|x(s)-y(s)\right|^p ds\right). \end{aligned} \tag{4.11}$$

The continuous Gronwall inequality then gives

$$E\left(\sup_{0\le t\le T}\left|x(t)-y(t)\right|^p\right)\le\left(Kh+Kh^{(m+1)p/2}+Kh^{\gamma}\right)e^{2KT}. \tag{4.12}$$

This proof is therefore complete.

Remark 4.1. If $m_1 = m_2 = m_3 = 0$, then the approximate scheme (2.7) is reduced to Euler-Maruyama method for stochastic differential delay equations with Poisson Jump and Markovian switching, which has been discussed in [3].

Remark 4.2. As is well known, Taylor approximation is effectively applicable in engineering if the equations can be solved explicitly. If not, since polynomials are very useful analytic functions, the approximation in the paper can be useful in other applications of stochastic Taylor expansion, especially in the construction of various time discrete approximations of Itô processes by using Itô-Taylor expansion such as Euler-Maruyama approximation and Milstein approximation, which has order 1/2 and 1, respectively [3, 8, 13]. All these show that numerical methods based on Taylor expansions of higher degrees could be improved by combining them with analytic approximations presented in this paper.

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