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Research Article

Coupled Coincidence Points in Partially Ordered Cone Metric Spaces with a c-Distance

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Cho et al. (2012) proved some coupled fixed point theorems in partially ordered cone metric spaces by using the concept of a c-distance in cone metric spaces. In this paper, we prove some coincidence point theorems in partially ordered cone metric spaces by using the notion of a c-distance. Our results generalize several well-known comparable results in the literature. Also, we introduce an example to support the usability of our results.

1. Introduction and Preliminaries

Fixed point theory is an essential tool in functional nonlinear analysis. Consequently, fixed point theory has wide applications areas not only in the various branches of mathematics (see, e.g., [1, 2]) but also in many fields, such as, chemistry, biology, statistics, economics, computer science, and engineering (see, e.g., [3–11]). For example, fixed point results are incredibly useful when it comes to proving the existence of various types of Nash equilibria (see, e.g., [7]) in economics. On the other hand, fixed point theorems are vital for the existence and uniqueness of differential equations, matrix equations, integral equations (see, e.g., [1, 2]). Banach contraction mapping principle (Banach fixed point theorem) is one of the most powerful theorems of mathematics and hence fixed point theory. Huang and Zhang [12] generalized the Banach contraction principle by replacing the notion of usual metric spaces by the notion of cone metric spaces. Then many authors obtained many fixed and common fixed point theorems in cone metric spaces. For some works in cone metric spaces, we may refer the reader (as examples) to [13–24]. The concept of a coupled fixed point of a mapping $F : X \times X \to X$ was initiated by Bhaskar and Lakshmikantham [25], while Lakshmikantham

and Ćirić [26] initiated the notion of coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ and studied some coupled coincidence point theorems in partially ordered metric spaces. For some coupled fixed point and coupled coincidence point theorems, we refer the reader to [27–34].

In the present paper, $\mathbf{N}^*$ is the set of positive integers and E stands for a real Banach space. Let P be a subset of E. We will always assume that the cone P has a nonempty interior $\operatorname{Int}(P)$ (such cones are called solid). Then P is called a cone if the following conditions are satisfied:

(1) P is closed and $P \neq \{\theta\}$,

(2) $a, b \in \mathbf{R}^+$, $x, y \in P$ implies $ax + by \in P$,

(3) $x \in P \cap -P$ implies $x = \theta$.

For a cone P, define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We will write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{Int}(P)$. It can be easily shown that $\lambda \operatorname{Int}(P) \subseteq \operatorname{Int}(P)$ for all positive scalar λ.

Definition 1.1 (see [12]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Bhaskar and Lakshmikantham [25] introduced the notion of mixed monotone property of the mapping $F : X \times X \to X$.

Definition 1.2 (see [25]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then the mapping F is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} x_1 \leq x_2 \text{ implies } F(x_1, y) \leq F(x_2, y), \quad \forall y \in X, \\ y_1 \leq y_2 \text{ implies } F(x, y_2) \leq F(x, y_1), \quad \forall x \in X. \end{aligned} \tag{1.1}$$

Inspired by Definition 1.2, Lakshmikantham and Ćirić in [26] introduced the concept of a g-mixed monotone mapping.

Definition 1.3 (see [26]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. Then the mapping F is said to have mixed gg-monotone property if $F(x, y)$ is monotone g-nondecreasing in x and is monotone g-nonincreasing in y; that is:

$$\begin{aligned} gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y), \quad \forall y \in X, \\ gy_1 \leq gy_2 \text{ implies } F(x, y_2) \leq F(x, y_1), \quad \forall x \in X. \end{aligned} \tag{1.2}$$

Definition 1.4 (see [25]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if

$$F(x, y) = x, \quad F(y, x) = y. \tag{1.3}$$

Definition 1.5 (see [26]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx, \quad F(y, x) = gy. \tag{1.4}$$

Recently, Cho et al. [35] introduced the concept of c-distance on cone metric space (X, d) which is a generalization of w-distance of Kada et al. [36] (see also [37, 38]).

Definition 1.6 (see [35]). Let (X, d) be a cone metric space. Then a function $q : X \times X \to E$ is called a c-distance on X if the following is satisfied:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $0 \leq e$ such that $q(z, x) \ll e$, and $q(z, y) \ll e$ implies $d(x, y) \ll c$.

Cho et al. [35] noticed the following important remark in the concept of c-distance on cone metric spaces.

Remark 1.7 (see [35]). Let q be a c-distance on a cone metric space (X, d). Then

(1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$,

(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Very recently, Cho et al. [39] proved the following existence theorems.

Theorem 1.8 (see [39]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and let $F : X \times X \to X$ be a continuous function having the mixed monotone property such that*

$$q(F(x, y), F(x*, y*)) \preceq \frac{k}{2}(q(x, x*) + q(y, y*)), \tag{1.5}$$

for some $k \in [0, 1)$ and all $x, y, x*, y* \in X$ with $(x \sqsubseteq x*) \wedge (y \sqsupseteq y*)$ or $(x \sqsupseteq x*) \wedge (y \sqsubseteq y*)$. If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $y_0 \sqsupseteq F(y_0, x_0)$, then F has a coupled fixed point (u, v). Moreover, one has $q(v, v) = q(u, u) = \theta$.

Theorem 1.9 (see [39]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X, and let $F : X \times X \to X$ be a function having the mixed monotone property such that*

$$q(F(x, y), F(x*, y*)) \preceq \frac{k}{4}(q(x, x*) + q(y, y*)), \tag{1.6}$$

for some $k \in [0,1)$ *and all* $x, y, x*, y* \in X$ *with* $(x \sqsubseteq x*) \wedge (y \sqsupseteq y*)$ *or* $(x \sqsupseteq x*) \wedge (y \sqsubseteq y*)$. *Also, suppose that* X *has the following properties:*

(1) *if* (x_n) *is a nondecreasing sequence in* X *with* $x_n \to x$, *then* $x_n \sqsubseteq x$ *for all* $n \geq 1$,

(2) *if* (x_n) *is a nonincreasing sequence in* X *with* $x_n \to x$, *then* $x \sqsubseteq x_n$ *for all* $n \geq 1$.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \sqsubseteq F(x_0, y_0)$ *and* $y_0 \sqsupseteq F(y_0, x_0)$, *then* F *has a coupled fixed point* (u, v). *Moreover, one has* $q(v, v) = q(u, u) = \theta$.

For other fixed point results using a c-distance, see [40].

In this paper, we prove some coincidence point theorems in partially ordered cone metric spaces by using the notion of c-distance. Our results generalize Theorems 1.8 and 1.9. We consider an application to illustrate our result is useful (see Section 3).

2. Main Results

The following lemma is essential in proving our results.

Lemma 2.1 (see [35]). *Let* (X, d) *be a cone metric space, and let* q *be a cone distance on* X. *Let* (x_n) *and* (y_n) *be sequences in* X *and* $x, y, z \in X$. *Suppose that* (u_n) *is a sequence in* P *converging to* θ. *Then the following holds.*

(1) *If* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq u_n$, *then* $y = z$.

(2) *If* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq u_n$, *then* (y_n) *converges to* z.

(3) *If* $q(x_n, x_m) \preceq u_n$ *for* $m > n$, *then* (x_n) *is a Cauchy sequence in* X.

(4) *If* $q(y, x_n) \preceq u_n$, *then* (x_n) *is a Cauchy sequence in* X.

In this section, we prove some coupled fixed point theorems by using c-distance in partially partially ordered cone metric spaces.

Theorem 2.2. *Let* $(X, \leq)$ *be a partially ordered set and suppose that* (X, d) *is a cone metric space. Let* q *be a* c*-distance on* X. *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that*

$$q(F(x, y), F(x*, y*)) + q(F(y, x), F(y*, x*)) \preceq k(q(gx, gx*) + q(gy, gy*)), \tag{2.1}$$

for some $k \in [0,1)$ *and for all* $x, y, x*, y* \in X$ *with* $(gx \leq gx*) \wedge (gy \geq gy*)$ *or* $(gx \geq gx*) \wedge (gy \leq gy*)$. *Assume that* F *and* g *satisfy the following conditions:*

(1) F *is continuous,*

(2) g *is continuous and commutes with* F,

(3) $F(X \times X) \subseteq gX$,

(4) (X, d) *is complete,*

(5) F *has the mixed* g*-monotone property.*

If there exist $x_0, y_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq gy_0$, *then* F *and* g *have a coupled coincidence point* (u, v). *Moreover, one has* $q(gu, gu) = \theta$ *and* $q(gv, gv) = \theta$.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_2 \preceq gy_1 \preceq gy_0$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that

$$\begin{aligned} gx_n &= F(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = F(x_n, y_n), \\ gy_{n+1} &= F(y_n, x_n) \preceq gy_n = F(y_{n-1}, x_{n-1}). \end{aligned} \tag{2.2}$$

Let $n \in \mathbf{N}^*$. Then by (2.1), we have

$$\begin{aligned} &q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\ &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\preceq k(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)). \end{aligned} \tag{2.3}$$

Repeating (2.3) n-times, we get

$$q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \preceq k^n(q(gx_0, gx_1) + q(gy_0, gy_1)). \tag{2.4}$$

Thus, we have

$$q(gx_n, gx_{n+1}) \preceq k^n(q(gx_0, gx_1) + q(gy_0, gy_1)), \tag{2.5}$$

$$q(gy_n, gy_{n+1}) \preceq k^n(q(gx_0, gx_1) + q(gy_0, gy_1)). \tag{2.6}$$

Let $m, n \in \mathbf{N}^*$ with $m > n$. Then by (q_2) and (2.5), we have

$$\begin{aligned} q(gx_n, gx_m) &\preceq \sum_{i=n}^{m-1} q(gx_i, gx_{i+1}) \\ &\preceq \sum_{i=n}^{m-1} k^i(q(gx_0, gx_1) + q(gy_0, gy_1)) \\ &\preceq \frac{k^n}{1-k}(q(gx_0, gx_1) + q(gy_0, gy_1)). \end{aligned} \tag{2.7}$$

Similarly, we have

$$q(gy_n, gy_m) \preceq \frac{k^n}{1-k}(q(gx_0, gx_1) + q(gy_0, gy_1)). \tag{2.8}$$

From part (3) of Lemma 2.1, we conclude that (gx_n) and (gy_n) are Cauchy sequences in in (X, d). Since X is complete, there are $u, v \in X$ such that $gx_n \to u$ and $gy_n \to v$. Using

the continuity of g, we get $g(gx_n) \to gu$ and $g(gy_n) \to gv$. Also, by continuity of F and commutativity of F and g, we have

$$\begin{aligned} gu &= \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) = F(u, v), \\ gv &= \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n) = F(v, u). \end{aligned} \tag{2.9}$$

Hence, (u, v) is a coupled coincidence point of F and g. Moreover, by (2.1) we have

$$\begin{aligned} q(gu, gu) + q(gv, gv) &= q(F(u, v), F(u, v)) + q(F(v, u), F(v, u)) \\ &\preceq k(q(gu, gu) + q(gv, gv)). \end{aligned} \tag{2.10}$$

Since $k < 1$, we conclude that $q(gu, gu) + q(gv, gv) = \theta$, and hence $q(gu, gu) = \theta$ and $q(gv, gv) = \theta$. □

The continuity of F in Theorem 2.2 can be dropped. For this, we present the following useful lemma which is a variant of Lemma 2.1, (1).

Lemma 2.3. *Let (X, d) be a cone metric space, and let q be a c-distance on X. Let (x_n) be a sequence in X. Suppose that (α_n) and (β_n) are sequences in P converging to θ. If $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$, then $y = z$.*

Proof. Let $c \gg \theta$ be arbitrary. Since $\alpha_n \to \theta$, so there exists $N_1 \in \mathbb{N}$ such that $\alpha_n \ll c/2$ for all $n \geq N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $\beta_n \ll c/2$ for all $n \geq N_2$. Thus, for all $N \geq \max\{N_1, N_2\}$, we have

$$q(x_n, y) \ll \frac{c}{2}, \quad q(x_n, z) \ll \frac{c}{2}. \tag{2.11}$$

Take $e = c/2$, so by $(q4)$, we get that $d(y, z) \ll c$ for each $c \gg \theta$; hence $y = z$. □

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$ and let $g : X \to X$ be two mappings such that*

$$q(F(x, y), F(x*, y*)) + q(F(y, x), F(y*, x*)) \preceq k(q(gx, gx*) + q(gy, gy*)), \tag{2.12}$$

for some $k \in [0, 1)$ and for all $x, y, x*, y* \in X$ with $(gx \leq gx*) \wedge (gy \geq gy*)$ or $(gx \geq gx*) \wedge (gy \leq gy*)$. Assume that F and g satisfy the following conditions:

(1) $F(X \times X) \subseteq gX$,

(2) *gX is a complete subspace of X,*

(3) *F has the mixed g-monotone property.*

Suppose that X has the following properties:

(i) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,

(ii) if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.

Assume there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$. Then F and g have a coupled coincidence point, say $(u, v) \in X \times X$. Also, $q(gu, gu) = q(gv, gv) = \theta$.

Proof. As in the proof of Theorem 2.2, we can construct two Cauchy sequences (gx_n) and (gy_n) in the complete cone metric space (gX, d). Then, there exist $u, v \in X$ such that $gx_n \to gu$ and $gy_n \to gv$. Similarly we have for all $m > n \geq 1$

$$\begin{aligned} q(gx_n, gx_m) &\preceq \frac{k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right], \\ q(gy_n, gy_m) &\preceq \frac{k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right]. \end{aligned} \tag{2.13}$$

By $(q3)$, we get that

$$q(gx_n, gu) \preceq \frac{k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right], \tag{2.14}$$

$$q(gy_n, gv) \preceq \frac{k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right]. \tag{2.15}$$

By summation, we get that

$$q(gx_n, gu) + q(gy_n, gv) \preceq 2\frac{k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right]. \tag{2.16}$$

Since (gx_n) is nondecreasing and (gy_n) is nonincreasing, using the properties (i), (ii) of X, we have

$$gx_n \preceq gu, \quad gv \preceq gy_n, \quad \forall n \geq 0. \tag{2.17}$$

From this and (2.14), we have

$$\begin{aligned} q(gx_n, F(u,v)\ ) + q(gy_n, F(v,u)) &= q(F(x_{n-1}, y_{n-1}), F(u,v))\ + q(F(y_{n-1}, x_{n-1}), F(v,u)) \\ &\preceq k(q(gx_{n-1}, gu) + q(gy_{n-1}, gv)). \end{aligned} \tag{2.18}$$

Therefore

$$q(gx_n, F(u,v)) + q(gy_n, F(v,u)) \preceq k\left[q(gx_{n-1}, gu) + q(gy_{n-1}, gv)\right]. \tag{2.19}$$

By (2.16), we have

$$\begin{aligned} q(gx_n, F(u,v)) + q(gy_n, F(v,u)) &\preceq k\left[q(gx_{n-1}, gu) + q(gy_{n-1}, gv)\right] \\ &\preceq k\frac{2k^{n-1}}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right] \\ &= \frac{2k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right]. \end{aligned} \tag{2.20}$$

This implies that

$$q(gx_n, F(u,v)) \preceq \frac{2k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right], \tag{2.21}$$

$$q(gy_n, F(v,u)) \preceq \frac{2k^n}{1-k}\left[q(gx_0, gx_1) + q(gy_0, gy_1)\right]. \tag{2.22}$$

By (2.14), (2.21) and Lemma 2.3, we obtain $gu = F(u,v)$. Similarly, by (2.15), (2.22), and Lemma 2.3, we obtain $gv = F(v,u)$. Also, adjusting as the proof of Theorem 2.2, we get that

$$q(gu, gu) = q(gv, gv) = \theta. \tag{2.23}$$

□

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$, and let $g : X \to X$ be two mappings such that*

$$q(F(x,y), F(x*, y*)) \preceq aq(gx, gx*) + bq(gy, gy*), \tag{2.24}$$

for some $a, b \in [0,1)$ with $a + b < 1$ and for all $x, y, x*, y* \in X$ with $(gx \leq gx*) \wedge (gy \geq gy*)$ or $(gx \geq gx*) \wedge (gy \leq gy*)$. Assume that F and g satisfy the following conditions:

(1) *F is continuous,*

(2) *g is continuous and commutes with F,*

(3) *$F(X \times X) \subseteq gX$,*

(4) *(X, d) is complete,*

(5) *F has the mixed g-monotone property.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then F and g have a coupled coincidence point (u, v). Moreover, one has $q(gu, gu) = \theta$ and $q(gv, gv) = \theta$.

Proof. Given $x, x*, y, y* \in X$ such that $(gx \leq gx*) \wedge (gy \geq gy*)$. By (2.24), we have

$$\begin{aligned} q(F(x,y), F(x*, y*)) &\preceq aq(gx, gx*) + bq(gy, gy*), \\ q(F(y,x), F(y*, x*)) &\preceq aq(gy, gy*) + bq(gx, gx*). \end{aligned} \tag{2.25}$$

Thus

$$q(F(x,y), F(x*, y*)) + q(F(y,x), F(y*, x*)) \preceq (a+b)\left(q(gx, gx*) + q(gy, gy*)\right). \tag{2.26}$$

Since $a + b < 1$, the result follows from Theorem 2.2. □

Corollary 2.6. *Let $(X, \leq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property such that*

$$q(F(x,y), F(x*, y*)) \preceq aq(x, x*) + bq(y, y*), \tag{2.27}$$

for some $a, b \in [0,1)$ *with* $a + b < 1$ *and for all* $x, y, x*, y* \in X$ *with* $(x \leq x*) \wedge (y \geq y*)$ *or* $(x \geq x*) \wedge (y \leq y*)$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq y_0$, *then* F *has a coupled fixed point* (x, y). *Moreover, one has* $q(x, x) = \theta$ *and* $q(y, y) = \theta$.

Proof. It follows from Corollary 2.5 by taking $g = I_X$ (the identity map). □

Corollary 2.7. *Let* $(X, \leq)$ *be a partially ordered set and suppose that* (X, d) *is a cone metric space. Let* q *be a c-distance on* X. *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that*

$$q(F(x, y), F(x*, y*)) \preceq aq((gx, gx*) + bq(gy, gy*)), \tag{2.28}$$

for some $a, b \in [0,1)$ *with* $a + b < 1$ *and for all* $x, y, x*, y* \in X$ *with* $(gx \leq gx*) \wedge (gy \geq gy*)$ *or* $(gx \geq gx*) \wedge (gy \leq gy*)$. *Assume that* F *and* g *satisfy the following conditions:*

(1) $F(X \times X) \subseteq gX$,

(2) gX *is a complete subspace of* X,

(3) F *has the mixed* g*-monotone property.*

Suppose that X has the following properties:

(i) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,

(ii) if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.

Assume there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Then F and g have a coupled coincidence point.

Proof. It follows from Theorem 2.4 by similar arguments to those given in proof of Corollary 2.5. □

Corollary 2.8. *Let* $(X, \leq)$ *be a partially ordered set and suppose that* (X, d) *is a complete cone metric space. Let* q *be a c-distance on* X. *Let* $F : X \times X \to X$ *be a mapping having the mixed monotone property such that*

$$q(F(x, y), F(x*, y*)) \preceq aq((x, x*) + bq(y, y*)), \tag{2.29}$$

for some $a, b \in [0,1)$ *with* $a + b < 1$ *and for all* $x, y, x*, y* \in X$ *with* $(x \leq x*) \wedge (y \geq y*)$ *or* $(x \geq x*) \wedge (y \leq y*)$. *Suppose that* X *has the following properties:*

(i) *if a nondecreasing sequence* $x_n \to x$, *then* $x_n \leq x$ *for all* n,

(ii) *if a nonincreasing sequence* $x_n \to x$, *then* $x \leq x_n$ *for all* n.

Assume there exist $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq y_0$. *Then* F *has a coupled fixed point.*

Proof. It follows from Corollary 2.7 by taking $g = I_X$ (the identity map). □

Corollary 2.9. *Let* $(X, \leq)$ *be a partially ordered set and suppose that* (X, d) *is a complete cone metric space. Let* q *be a c-distance on* X, *and let* $F : X \times X \to X$ *be a continuous mapping having the mixed monotone property such that*

$$q(F(x, y), F(x*, y*)) + q(F(y, x), F(y*, x*)) \preceq k(q(x, x*) + q(y, y*)), \tag{2.30}$$

for some $k \in [0,1)$ *and for all* $x, y, x*, y* \in X$ *with* $(x \leq x*) \wedge (y \geq y*)$ *or* $(x \geq x*) \wedge (y \leq y*)$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point (x, y). Moreover, we have $q(x, x) = \theta$ and $q(y, y) = \theta$.

Proof. It follows from Theorem 2.2 by taking $g = I_X$. □

Corollary 2.10. *Let $(X, \leq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Let $F : X \times X \to X$ be a mapping having the mixed monotone property such that*

$$q(F(x,y), F(x*, y*)) + q(F(y,x), F(y*, x*)) \preceq k(q(x, x*) + q(y, y*)), \tag{2.31}$$

for some $k \in [0,1)$ and for all $x, y, x*, y* \in X$ with $(x \leq x*) \wedge (y \geq y*)$ or $(x \geq x*) \wedge (y \leq y*)$. Suppose that X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

Assume there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$. Then F has a coupled fixed point.

Proof. It follows from Theorem 2.4 by taking $g = I_X$. □

Example 2.11. Let $E = C_R^1([0,1])$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Let $X = [0,1]$ with usual order $\leq$. Define $d : X \times X \to X$ by $d(x,y)(t) = |x - y|e^t$ for all $x, y \in X$. Then (X, d) is a partially ordered cone metric space. Define $q : X \times X \to E$ by $q(x,y)(t) = ye^t$ for all $x, y \in X$. Then q is a c-distance. Define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \dfrac{x-y}{2}, & x \geq y, \\ 0, & x < y. \end{cases} \tag{2.32}$$

Then,

(1) $q(F(x,y), F(x*, y*)) + q(F(y,x), F(y*, x*)) \leq 1/2(q(x, x*) + q(y, y*))$, for all $x \leq x*$ and $y \geq y*$,

(2) there is no $k \in [0,1)$ such that $q(F(x,y), F(x*, y*)) \leq (k/2)(q(x, x*) + q(y, y*))$ for all $x \leq x*$ and $y \geq y*$,

(3) there is no $k \in [0,1)$ such that $q(F(x,y), F(x*, y*)) \leq (k/4)(q(x, x*) + q(y, y*))$ for all $x \leq x*$ and $y \geq y*$.

Note that $0 \leq F(0,0)$ and $0 \geq F(0,0)$. Thus by Corollary 2.10, we have F which has a coupled fixed point. Here $(0,0)$ is a coupled fixed point of F.

Proof. The proof of (2.1) is easy. To prove (2.3), suppose the contrary; that is, there is $k \in [0,1)$ such that $q(F(x,y), F(x*, y*)) \leq k/2(q(x, x*) + q(y, y*))$ for all $x \leq x*$ and $y \geq y*$. Take $x = 0, y = 1, x* = 1$ and $y* = 0$. Then

$$q(F(0,1), F(1,0))(t) \leq \frac{k}{2}(q(0,1) + q(1,0))(t). \tag{2.33}$$

Thus

$$q\left(0,\frac{1}{2}\right)(t)=\frac{1}{2}e^{t}\leq\frac{k}{2}e^{t}. \tag{2.34}$$

Hence $k\geq 1$ is a contradiction. The proof of (2.5) is similar to proof of (2.3). □

Remark 2.12. Note that Theorems 3.1 and 3.2 of [39] are not applicable to Example 2.11.

Remark 2.13. Theorem 3.1 of [39] is a special case of Corollary 2.6 and Corollary 2.9.

Remark 2.14. Theorem 3.3 of [39] is a special case of Corollary 2.8 and Corollary 2.10.

3. Application

Consider the integral equations

$$\begin{aligned} x(t)&=\int_0^T f\big(t,x(s),y(s)\big)ds,\quad t\in[0,T],\\ y(t)&=\int_0^T f\big(t,y(s),x(s)\big)ds,\quad t\in[0,T], \end{aligned} \tag{3.1}$$

where $T>0$ and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Let $X=C([0,T],\mathbb{R})$ denote the space of $\mathbb{R}$-valued continuous functions on $I=[0,T]$. The purpose of this section is to give an existence theorem for a solution (x,y) to (3.1) that belongs to X, by using the obtained result given by Corollary 2.10. Let $E=\mathbb{R}^2$, and let $P\subset E$ be the cone defined by

$$P=\left\{(x,y)\in\mathbb{R}^2\mid\ x\geq 0,y\geq 0\right\}. \tag{3.2}$$

We endow X with the cone metric $d:X\times X\to E$ defined by

$$d(u,v)=\left(\sup_{t\in I}|u(t)-v(t)|,\sup_{t\in I}|u(t)-v(t)|\right),\qquad \forall u,v\in X. \tag{3.3}$$

It is clear that (X,d) is a complete cone metric space. Let $q(x,y)=d(x,y)$ for all $x,y\in X$. Then, q is a c-distance.

Now, we endow X with the partial order $\leq$ given by

$$u,v\in X,\quad u\leq v\Longleftrightarrow u(x)\leq v(x),\qquad \forall x\in I. \tag{3.4}$$

Also, the product space $X\times X$ can be equipped with the partial order (still denoted $\leq$) given as follows:

$$(x,y)\leq(u,v)\Longleftrightarrow x\leq u,y\geq v. \tag{3.5}$$

It is easy that (i) and (ii) given in Corollary 2.10 are satisfied.

Now, we consider the following assumptions:

(a) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous,

(b) for all $t \in [0,T]$, the function $f(t,\cdot,\cdot) : \mathbb{R} \to \mathbb{R}$ has the mixed monotone property,

(c) for all $t \in [0,T]$, for all $p,q,p',q' \in \mathbb{R}$ with $p \le q$ and $p' \ge q'$, we have

$$f(t,q,q') - f(t,p,p') \le \frac{1}{T}\varphi\left(\frac{q-p+p'-q'}{2}\right), \tag{3.6}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous nondecreasing an satisfies the following condition: There exists $0 < k < 1$ such that

$$\varphi(r) \le kr \quad \forall r \ge 0, \tag{3.7}$$

(d) there exists $x_0, y_0 \in C([0,T],\mathbb{R})$ such that

$$x_0(t) \le \int_0^T f(t,x_0(s),y_0(s))ds, \quad \int_0^T f(t,y_0(s),x_0(s))ds \le y_0(t), \quad \forall t \in [0,T]. \tag{3.8}$$

We have the following result.

Theorem 3.1. *Suppose that* (*a*)–(*d*) *hold. Then,* (3.1) *has at least one solution* $(x^*,y^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$.

Proof. Define the mapping $A : C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$A(x,y)(t) = \int_0^T f(t,x(s),y(s))ds, \quad x,y \in C([0,T],\mathbb{R}), \quad t \in [0,T]. \tag{3.9}$$

We have to prove that A has at least one coupled fixed point $(x^*,y^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$. □

From (b), it is clear that A has the mixed monotone property.

Now, let $x,y,u,v \in C([0,T],\mathbb{R})$ such that ($x \le u$ and $y \ge v$) or ($x \ge u$ and $y \le v$). Using (c), for all $t \in [0,T]$, we have

$$\begin{aligned} |A(u,v)(t) - A(x,y)(t)| &\le \int_0^T [f(t,u(s),v(s)) - f(t,x(s),y(s))]ds \\ &\le \frac{1}{T}\int_0^T \varphi\left(\frac{u(s)-x(s)+y(s)-v(s)}{2}\right)ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{T}\int_0^T \varphi\left(\frac{\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-v(z)|}{2}\right)ds \\
&= \varphi\left(\frac{\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-v(z)|}{2}\right) \\
&\leq k\left[\frac{\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-v(z)|}{2}\right],
\end{aligned} \tag{3.10}$$

which implies that

$$\sup_{z\in[0,T]}|A(u,v)(t)-A(x,y)(t)| \leq k\left[\frac{\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-v(z)|}{2}\right]. \tag{3.11}$$

Similarly, one can get

$$\sup_{z\in[0,T]}|A(v,u)(t)-A(y,x)(t)| \leq k\left[\frac{\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-v(z)|}{2}\right]. \tag{3.12}$$

We deduce

$$\begin{aligned}
&\sup_{z\in[0,T]}|A(u,v)(t)-A(x,y)(t)|+\sup_{z\in[0,T]}|A(v,u)(t)-A(y,x)(t)| \\
&\quad\leq k\left[\sup_{z\in[0,T]}|u(z)-x(z)|+\sup_{z\in[0,T]}|y(z)-\mathrm{v}(z)|\right].
\end{aligned} \tag{3.13}$$

Thus

$$d(A(u,v),A(x,y))+d(A(v,u),A(y,x)) \preceq k[d(x,u)+d(y,v)]. \tag{3.14}$$

Thus, we proved that condition (2.31) of Corollary 2.10 is satisfied. Moreover, from (d), we have $x_0 \leq A(x_0,y_0)$ and $A(y_0,x_0) \leq y_0$. Finally, applying our Corollary 2.10, we get the desired result.

References

[1] A. Yantir and S. G. Topal, "Positive solutions of nonlinear m-point BVP on time scales," *International Journal of Difference Equations*, vol. 3, no. 1, pp. 179–194, 2008.

[2] Y. Guo, "A generalization of Banachs contraction principle for some non-obviously contractive operators in a cone metric space," *Turkish Journal of Mathematics*, vol. 36, pp. 297–304, 2012.

[3] A. Cataldo, E. A. Lee, X. Liu, E. D. Matsikoudis, and H. Zheng, "A constructive fixed point theorem and the feedback semantics of timed systems," Tech. Rep. UCB/EECS-2006-4, EECS Department, University of California, Berkeley, Calif, USA, 2006.

[4] A. Noumsi, S. Derrien, and P. Quinton, "Acceleration of a content based image retrieval application on the RDISK cluster," in *Proceedings of the IEEE International Parallel and Distributed Processing Symposium*, April 2006.

[5] A. Hyväarinen, "Fast and robust fixed-point algorithms for independent component analysis," *IEEE Transactions on Neural Networks*, vol. 10, no. 3, pp. 626–634, 1999.

[6] D. P. Mandic, J. A. Chambers, and M. M. Bozic, "On global asymptotic stability of fully connected recurrent neural networks," in *Proceedings of the International Conference on Acoustics, Speech and Signal Processing (ICASSP '00)*, vol. 6, pp. 3406–3409, 2000.

[7] K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, New York, NY, USA, 1985.

[8] M. H. Escardó, "PCF extended with real numbers," *Theoretical Computer Science*, vol. 162, no. 1, pp. 79–115, 1996.

[9] J. B. Gillespie and C. J. Houghton, "A metric space approach to the information channel capacity of spike trains," *Journal of Computational Neuroscience*, vol. 30, no. 1, pp. 201–209, 2011.

[10] K. Menger, "Statistical metrics," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 28, pp. 535–537, 1942.

[11] D. H. Fenwick and R. P. Batycky, "Using metric space methods to analyse reservoir uncertainty," in *Proceedings of the Gussow Conference*, Alberta, Canada, 2011.

[12] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[13] A. Amini-Harandi and M. Fakhar, "Fixed point theory in cone metric spaces obtained via the scalarization method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3529–3534, 2010.

[14] D. Turkoglu and M. Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings," *Acta Mathematica Sinica*, vol. 26, no. 3, pp. 489–496, 2010.

[15] H. Aydi, H. K. Nashine, B. Samet, and H. Yazidi, "Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations," *Nonlinear Analysis*, vol. 74, no. 17, pp. 6814–6825, 2011.

[16] E. Graily, S. M. Vaezpour, R. Saadati, and Y. J. Cho, "Generalization of fixed point theorems in ordered metric spaces concerning generalized distance," *Fixed Point Theory and Applications*, vol. 2011, article 30, 2011.

[17] S. Janković, Z. Kadelburg, and S. Radenović, "On cone metric spaces: a survey," *Nonlinear Analysis*, vol. 74, no. 7, pp. 2591–2601, 2011.

[18] Z. Kadelburg, M. Pavlović, and S. Radenović, "Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 59, no. 9, pp. 3148–3159, 2010.

[19] Z. Kadelburg, P. P. Murthy, and S. Radenović, "Common fixed points for expansive mappings in cone metric spaces," *International Journal of Mathematical Analysis*, vol. 5, no. 25–28, pp. 1309–1319, 2011.

[20] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Common fixed point theorems for c-distance in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1969–1978, 2011.

[21] W. Shatanawi and F. Awawdeh, "Some fixed and coincidence point theorems for expansive maps in cone metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 19, 2012.

[22] W. Shatanawi, "On w-compatible mappings and common coupled coincidence point in cone metric spaces," *Applied Mathematics Letters*, vol. 25, pp. 925–931, 2012.

[23] W. Shatanawi, "Some coincidence point results in cone metric spaces," *Mathematical and Computer Modelling*, vol. 55, pp. 2023–2028, 2012.

[24] D. Turkoglu, M. Abuloha, and T. Abdeljawad, "KKM mappings in cone metric spaces and some fixed point theorems," *Nonlinear Analysis*, vol. 72, no. 1, pp. 348–353, 2010.

[25] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis*, vol. 65, no. 7, pp. 1379–1393, 2006.

[26] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis*, vol. 70, no. 12, pp. 4341–4349, 2009.

[27] W. Shatanawi, "Some common coupled fixed point results in cone metric spaces," *International Journal of Mathematical Analysis*, vol. 4, no. 45–48, pp. 2381–2388, 2010.

[28] H. Aydi, M. Abbas, and M. Postolache, "Coupled coincidence points for hybrid pair of mappings via mixed monotone property," *Journal of Advanced Mathematical Studies*, vol. 5, no. 1, pp. 118–126, 2012.

[29] Y. J. Cho, G. He, and N. J. Huang, "The existence results of coupled quasi-solutions for a class of operator equations," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 455–465, 2010.

[30] Y. J. Cho, M. H. Shah, and N. Hussain, "Coupled fixed points of weakly F-contractive mappings in topological spaces," *Applied Mathematics Letters*, vol. 24, no. 7, pp. 1185–1190, 2011.
[31] M. E. Gordji, Y. J. Cho, and H. Baghani, "Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 1897–1906, 2011.
[32] E. Karapınar, "Couple fixed point theorems for nonlinear contractions in cone metric spaces," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3656–3668, 2010.
[33] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces," *Fixed Point Theory and Applications*, vol. 2011, article 81, 2011.
[34] W. Shatanawi, "Partially ordered cone metric spaces and coupled fixed point results," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2508–2515, 2010.
[35] Y. J. Cho, R. Saadati, and S. Wang, "Common fixed point theorems on generalized distance in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1254–1260, 2011.
[36] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.
[37] L. Cirić, H. Lakzian, and V. Rakočević, "Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 3, 2012.
[38] Z. Kadelburg and S. Radenovic, "Coupled fxed point results under tvs-cone metric and w-cone distance," *Advances in Fixed Point Theory*, vol. 2, no. 1, pp. 29–46, 2012.
[39] Y. J. Cho, Z. Kadelburg, R. Saadati, and W. Shatanawi, "Coupled fixed point theorems under weak contractions," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 184534, 9 pages, 2012.
[40] M. Dornevic, D. Doric, Z. Kadelburg, S. Radenovic, and D. Spaic, "Fixed point results under c-distance in tvs-cone metric spaces," *Fixed Point Theory and Its Applications*, vol. 2011, article 29, 2011.