Research Article

An Iterative Algorithm for a Hierarchical Problem

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A general hierarchical problem has been considered, and an explicit algorithm has been presented for solving this hierarchical problem. Also, it is shown that the suggested algorithm converges strongly to a solution of the hierarchical problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H . The hierarchical problem is of finding $\tilde{x} \in \text{Fix}(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T), \quad (1.1)$$

where S, T are two nonexpansive mappings and $\text{Fix}(T)$ is the set of fixed points of T . Recently, this problem has been studied by many authors (see, e.g., [1–15]). The main reason is that this problem is closely associated with some monotone variational inequalities and convex programming problems (see [16–19]).

Now, we briefly recall some historic results which relate to the problem (1.1).

For solving the problem (1.1), in 2006, Moudafi and Mainge [1] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1-s)[tS(x_{t,s}) + (1-t)T(x_{t,s})] \quad (1.2)$$

and proved that the net $\{x_{t,s}\}$ defined by (1.2) strongly converges to x_t as $s \rightarrow 0$, where x_t satisfies $x_t = \text{proj}_{\text{Fix}(P_t)} Q(x_t)$, where $P_t : C \rightarrow C$ is a mapping defined by

$$P_t(x) = tS(x) + (1-t)T(x), \quad \forall x \in C, \quad t \in (0,1), \quad (1.3)$$

or, equivalently, x_t is the unique solution of the quasivariational inequality

$$0 \in (I - Q)x_t + N_{\text{Fix}(P_t)}(x_t), \quad (1.4)$$

where the normal cone to $\text{Fix}(P_t)$, $N_{\text{Fix}(P_t)}$, is defined as follows:

$$N_{\text{Fix}(P_t)} : x \longrightarrow \begin{cases} \{u \in H : \langle y - x, u \rangle \leq 0\}, & \text{if } x \in \text{Fix}(P_t), \\ \emptyset, & \text{otherwise.} \end{cases} \quad (1.5)$$

Moreover, as $t \rightarrow 0$, the net $\{x_t\}$ in turn weakly converges to the unique solution x_∞ of the fixed point equation $x_\infty = \text{proj}_\Omega Q(x_\infty)$ or, equivalently, x_∞ is the unique solution of the variational inequality

$$0 \in (I - Q)x_\infty + N_\Omega(x_\infty). \quad (1.6)$$

Recently, Moudafi [2] constructed an explicit iterative algorithm:

$$x_{n+1} = (1 - \delta_n)x_n + \delta_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \quad \forall n \geq 0, \quad (1.7)$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are two real numbers in $(0,1)$. By using this iterative algorithm, Moudafi [2] only proved a weak convergence theorem for solving the problem (1.1).

In order to obtain a strong convergence result, Mainge and Moudafi [3] further introduced the following iterative algorithm:

$$x_{n+1} = (1 - \delta_n)Qx_n + \delta_n[\sigma_n Sx_n + (1 - \sigma_n)Tx_n], \quad \forall n \geq 0, \quad (1.8)$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are two real numbers in $(0,1)$, and proved that, under appropriate conditions, the iterative sequence $\{x_n\}$ generated by (1.8) has strong convergence.

Subsequently, some authors have studied some algorithms on hierarchical fixed problems (see, e.g., [4–15]).

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding $\tilde{x} \in \text{Fix}(T)$ such that, for any $n \geq 1$,

$$\langle W_n \tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T), \quad (1.9)$$

where W_n is the W -mapping defined by (2.3) below and T is a nonexpansive mapping, and introduce an explicit iterative algorithm which converges strongly to a solution $\tilde{x}$ of the hierarchical problem (1.9).

2. Preliminaries

Let C a nonempty closed convex subset of a real Hilbert space H . Recall that a mapping $Q : C \rightarrow C$ is said to be contractive if there exists a constant $\gamma \in (0, 1)$ such that

$$\|Qx - Qy\| \leq \gamma \|x - y\|, \quad \forall x, y \in C. \quad (2.1)$$

A mapping $T : C \rightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2.2)$$

Forward, we use $\text{Fix}(T)$ to denote the fixed points set of T .

Let $\{T_i\}_{i=1}^{\infty} : C \rightarrow C$ be an infinite family of nonexpansive mappings and $\{\xi_i\}_{i=1}^{\infty}$ a real number sequence such that $0 \leq \xi_i \leq 1$ for each $i \geq 1$.

For each $n \geq 1$, define a mapping $W_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n T_n U_{n,n+1} + (1 - \xi_n)I, \\ U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\ &\dots \\ U_{n,k} &= \xi_k T_k U_{n,k+1} + (1 - \xi_k)I, \\ U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1 - \xi_{k-1})I, \\ &\dots \\ U_{n,2} &= \xi_2 T_2 U_{n,3} + (1 - \xi_2)I, \\ W_n = U_{n,1} &= \xi_1 T_1 U_{n,2} + (1 - \xi_1)I. \end{aligned} \quad (2.3)$$

Such W_n is called the W -mapping generated by $\{T_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$.

Lemma 2.1 (see [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$. Let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \leq b < 1$ for each $i \geq 1$. Then one has the following results:*

- (1) *for any $x \in C$ and $k \geq 1$, the limit $\lim_{n \rightarrow \infty} U_{n,k}x$ exists;*
- (2) $\text{Fix}(W) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$.

Using Lemma 3.1 in [21], we can define a mapping W of C into itself by $Wx = \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x$ for all $x \in C$. Thus we have the following.

Lemma 2.2 (see [21]). *If $\{x_n\}$ is a bounded sequence in C , then one has*

$$\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0. \quad (2.4)$$

Lemma 2.3 (see [22]). Let C be a nonempty closed convex of a real Hilbert space H and $T : C \rightarrow C$ be nonexpansive mapping. Then T is demiclosed on C , that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \rightarrow 0$, then $x = Tx$.

Lemma 2.4 (see [23]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n + \eta_n, \quad \forall n \geq 1, \quad (2.5)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}, \{\eta_n\}$ are two sequences such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$;
- (iii) $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

In this section, we introduce our algorithm and give its convergence analysis.

Algorithm 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}_{n=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself. Let $Q : C \rightarrow C$ be a contraction with coefficient $\gamma \in [0, 1)$. For any $x_0 \in C$, let $\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n W_n x_n + (1 - \alpha_n) T(\beta_n Q x_n + (1 - \beta_n) x_n), \quad \forall n \geq 0, \quad (3.1)$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$ and W_n is the W -mapping defined by (2.3).

Now, we give the convergence analysis of the algorithm.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself. Let $Q : C \rightarrow C$ be a contraction with coefficient $\gamma \in [0, 1)$. Assume that the set Ω of solutions of the hierarchical problem (1.9) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0, 1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$;
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} (1/\beta_n) |(1/\alpha_n) - (1/\alpha_{n-1})| = 0$ and $\lim_{n \rightarrow \infty} (\prod_{i=1}^{n-1} \xi_i / \alpha_n \beta_n) = \lim_{n \rightarrow \infty} (1/\alpha_n) |1 - (\beta_{n-1} / \beta_n)| = 0$.

Then $\lim_{n \rightarrow \infty} (\|x_{n+1} - x_n\| / \alpha_n) = 0$ and every weak cluster point of the sequence $\{x_n\}$ solves the following variational inequality

$$\begin{aligned} \tilde{x} &\in \Omega, \\ \langle (I - Q)\tilde{x}, x - \tilde{x} \rangle &\geq 0, \quad \forall x \in \Omega. \end{aligned} \quad (3.2)$$

Proof. Set $y_n = \beta_n Qx_n + (1 - \beta_n)x_n$ for each $n \geq 0$. Then we have

$$\begin{aligned} y_n - y_{n-1} &= \beta_n Qx_n + (1 - \beta_n)x_n - \beta_{n-1} Qx_{n-1} - (1 - \beta_{n-1})x_{n-1} \\ &= \beta_n(Qx_n - Qx_{n-1}) + (\beta_n - \beta_{n-1})Qx_{n-1} + (1 - \beta_n)(x_n - x_{n-1}) \\ &\quad + (\beta_{n-1} - \beta_n)x_{n-1}. \end{aligned} \quad (3.3)$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \gamma\beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|) \\ &= [1 - (1 - \gamma)\beta_n]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|). \end{aligned} \quad (3.4)$$

From (3.1), we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n W_n x_n + (1 - \alpha_n)Ty_n - \alpha_{n-1}W_{n-1}x_{n-1} - (1 - \alpha_{n-1})Ty_{n-1} \\ &= \alpha_n(W_n x_n - W_n x_{n-1}) + (\alpha_n - \alpha_{n-1})W_n x_{n-1} + \alpha_{n-1}(W_n x_{n-1} - W_{n-1}x_{n-1}) \\ &\quad + (1 - \alpha_n)(Ty_n - Ty_{n-1}) + (\alpha_{n-1} - \alpha_n)Ty_{n-1}. \end{aligned} \quad (3.5)$$

Then we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\|W_n x_n - W_n x_{n-1}\| + (1 - \alpha_n)\|Ty_n - Ty_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}\|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\leq \alpha_n\|x_n - x_{n-1}\| + (1 - \alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) \\ &\quad + \alpha_{n-1}\|W_n x_{n-1} - W_{n-1}x_{n-1}\|. \end{aligned} \quad (3.6)$$

From (2.3), since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned} \|W_n x_{n-1} - W_{n-1}x_{n-1}\| &= \|\xi_1 T_1 U_{n,2} x_{n-1} - \xi_1 T_1 U_{n-1,2} x_{n-1}\| \\ &\leq \xi_1 \|U_{n,2} x_{n-1} - U_{n-1,2} x_{n-1}\| \\ &= \xi_1 \|\xi_2 T_2 U_{n,3} x_{n-1} - \xi_2 T_2 U_{n-1,3} x_{n-1}\| \\ &\leq \xi_1 \xi_2 \|U_{n,3} x_{n-1} - U_{n-1,3} x_{n-1}\| \\ &\leq \dots \\ &\leq \xi_1 \xi_2 \dots \xi_{n-1} \|U_{n,n} x_{n-1} - U_{n-1,n} x_{n-1}\| \\ &\leq M_1 \prod_{i=1}^{n-1} \xi_i, \end{aligned} \quad (3.7)$$

where M_1 is a constant such that $\sup_{n \geq 1} \{\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\|\} \leq M_1$. Substituting (3.4) and (3.7) into (3.6), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n) [1 - (1 - \gamma)\beta_n] \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}| (\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1} M_1 \prod_{i=1}^{n-1} \xi_i \\
&= [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}| (\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1} M_1 \prod_{i=1}^{n-1} \xi_i.
\end{aligned} \tag{3.8}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\alpha_n} &\leq [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_n} \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} (\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1} M_1 \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \\
&= [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad + [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \left(\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \right) \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} (\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1} M_1 \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \\
&\leq [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad + \left(\left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \right) M
\end{aligned}$$

$$\begin{aligned}
&= [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + (1 - \gamma)\beta_n(1 - \alpha_n) \\
&\quad \times \left\{ \frac{M}{(1 - \gamma)(1 - \alpha_n)} \left(\frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} \right. \right. \\
&\quad \left. \left. + \frac{1}{\beta_n} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n \beta_n} \right) \right\}, \tag{3.9}
\end{aligned}$$

where M is a constant such that

$$\sup_{n \geq 1} \{M_1, \|x_n - x_{n-1}\|, (\|W_n x_{n-1}\| + \|T y_{n-1}\|), (\|Q x_{n-1}\| + \|x_{n-1}\|)\} \leq M. \tag{3.10}$$

From (iii), we note that $\lim_{n \rightarrow \infty} (1/\alpha_{n-1})|\alpha_n - \alpha_{n-1}/\beta_n \alpha_n| = 0$, which implies that

$$\lim_{n \rightarrow \infty} \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0. \tag{3.11}$$

Thus it follows from (iii) and (3.11) that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{1}{\beta_n} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n \beta_n} \right) = 0. \tag{3.12}$$

Hence, applying Lemma 2.4 to (3.9), we immediately conclude that

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n} = 0. \tag{3.13}$$

This implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Thus, from (3.1) and (3.14), we have

$$\lim_{n \rightarrow \infty} \|x_n - T y_n\| = 0. \tag{3.15}$$

At the same time, we note that

$$y_n - x_n = \beta_n(Q x_n - x_n) \longrightarrow 0. \tag{3.16}$$

Hence we get

$$\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0. \tag{3.17}$$

Since the sequence $\{x_n\}$ is bounded, $\{y_n\}$ is also bounded. Thus there exists a subsequence of $\{y_n\}$, which is still denoted by $\{y_n\}$ which converges weakly to a point $\tilde{x} \in H$. Therefore, $\tilde{x} \in \text{Fix}(T)$ by (3.17) and Lemma 2.3. By (3.1), we observe that

$$x_{n+1} - x_n = \alpha_n(W_n x_n - x_n) + (1 - \alpha_n)(T y_n - y_n) + (1 - \alpha_n)\beta_n(Q x_n - x_n), \quad (3.18)$$

that is,

$$\frac{x_n - x_{n+1}}{\alpha_n} = (I - W_n)x_n + \frac{1 - \alpha_n}{\alpha_n}(I - T)y_n + \frac{\beta_n(1 - \alpha_n)}{\alpha_n}(I - Q)x_n. \quad (3.19)$$

Set $z_n = (x_n - x_{n+1})/\alpha_n$ for each $n \geq 1$, that is,

$$z_n = (I - W_n)x_n + \frac{1 - \alpha_n}{\alpha_n}(I - T)y_n + \frac{\beta_n(1 - \alpha_n)}{\alpha_n}(I - Q)x_n. \quad (3.20)$$

Using monotonicity of $I - T$ and $I - W_n$, we derive that, for all $u \in \text{Fix}(T)$,

$$\begin{aligned} & \langle z_n, x_n - u \rangle \\ &= \langle (I - W_n)x_n, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle (I - T)y_n - (I - T)u, y_n - u \rangle \\ & \quad + \frac{1 - \alpha_n}{\alpha_n} \langle (I - T)y_n, x_n - y_n \rangle + \frac{\beta_n(1 - \alpha_n)}{\alpha_n} \langle (I - Q)x_n, x_n - u \rangle \\ & \geq \langle (I - W_n)u, x_n - u \rangle + \frac{\beta_n(1 - \alpha_n)}{\alpha_n} \langle (I - Q)x_n, x_n - u \rangle + \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle (I - T)y_n, x_n - Qx_n \rangle \\ &= \langle (I - W)u, x_n - u \rangle + \langle (W - W_n)u, x_n - u \rangle + \frac{\beta_n(1 - \alpha_n)}{\alpha_n} \langle (I - Q)x_n, x_n - u \rangle \\ & \quad + \frac{(1 - \alpha_n)\beta_n}{\alpha_n} \langle (I - T)y_n, x_n - Qx_n \rangle. \end{aligned} \quad (3.21)$$

But, since $z_n \rightarrow 0$, $\beta_n/\alpha_n \rightarrow 0$ and $\lim_{n \rightarrow \infty} \|W_n u - W u\| = 0$ (by Lemma 2.2), it follows from the above inequality that

$$\limsup_{n \rightarrow \infty} \langle (I - W)u, x_n - u \rangle \leq 0, \quad \forall u \in \text{Fix}(T). \quad (3.22)$$

This suffices to guarantee that $\omega_w(x_n) \subset \Omega$. As a matter of fact, if we take any $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Therefore, we have

$$\langle (I - W)u, x^* - u \rangle = \lim_{j \rightarrow \infty} \langle (I - W)u, x_{n_j} - u \rangle \leq 0, \quad \forall u \in \text{Fix}(T). \quad (3.23)$$

Note that $x^* \in \text{Fix}(T)$. Hence x^* solves the following problem:

$$\begin{aligned} x^* &\in \text{Fix}(T), \\ \langle (I - W)u, x^* - u \rangle &\leq 0, \quad \forall u \in \text{Fix}(T). \end{aligned} \quad (3.24)$$

It is obvious that this is equivalent to the problem (1.9) since $W_n \rightarrow W$ uniformly in any bounded set (by Lemma 2.2). Thus $x^* \in \Omega$.

Let $\tilde{x}$ be the unique solution of the variational inequality (3.2). Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle (I - Q)\tilde{x}, x_n - \tilde{x} \rangle = \lim_{i \rightarrow \infty} \langle (I - Q)\tilde{x}, x_{n_i} - \tilde{x} \rangle. \quad (3.25)$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$. Then $\bar{x} \in \Omega$. Therefore, we have

$$\limsup_{n \rightarrow \infty} \langle (I - Q)\tilde{x}, x_n - \tilde{x} \rangle = \langle (I - Q)\tilde{x}, \bar{x} - \tilde{x} \rangle \geq 0. \quad (3.26)$$

This completes the proof. $\square$

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}_{n=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself. Let $Q : C \rightarrow C$ be a contraction with coefficient $\gamma \in [0, 1)$. Assume that the set Ω of solutions of the hierarchical problem (1.9) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0, 1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \beta_n / \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n^2 / \beta_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} (1/\beta_n)|(1/\alpha_n) - (1/\alpha_{n-1})| = 0$ and $\lim_{n \rightarrow \infty} \prod_{i=1}^{n-1} \xi_i / \alpha_n \beta_n = \lim_{n \rightarrow \infty} (1/\alpha_n)|1 - (\beta_{n-1}/\beta_n)| = 0$;
- (iv) *there exists a constant $k > 0$ such that $\|x - Tx\| \geq k \text{Dist}(x, \text{Fix}(T))$, where*

$$\text{Dist}(x, \text{Fix}(T)) = \inf_{y \in \text{Fix}(T)} \|x - y\|. \quad (3.27)$$

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\tilde{x} \in \text{Fix}(T)$, which solves the variational inequality problem (3.2).

Proof. From (3.1), we have

$$x_{n+1} - \tilde{x} = \alpha_n(W_n x_n - W_n \tilde{x}) + \alpha_n(W_n \tilde{x} - \tilde{x}) + (1 - \alpha_n)(T y_n - \tilde{x}). \quad (3.28)$$

Thus we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \|\alpha_n(W_n x_n - W_n \tilde{x}) + (1 - \alpha_n)(T y_n - \tilde{x})\|^2 + 2\alpha_n \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\leq (1 - \alpha_n) \|T y_n - \tilde{x}\|^2 + \alpha_n \|W_n x_n - W_n \tilde{x}\|^2 + 2\alpha_n \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \quad (3.29) \\
&\leq (1 - \alpha_n) \|y_n - \tilde{x}\|^2 + \alpha_n \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle.
\end{aligned}$$

At the same time, we observe that

$$\begin{aligned}
\|y_n - \tilde{x}\|^2 &= \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n(Qx_n - Q\tilde{x}) + \beta_n(Q\tilde{x} - \tilde{x})\|^2 \\
&\leq \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n(Qx_n - Q\tilde{x})\|^2 + 2\beta_n \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&\leq (1 - \beta_n) \|x_n - \tilde{x}\|^2 + \beta_n \|Qx_n - Q\tilde{x}\|^2 + 2\beta_n \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \quad (3.30) \\
&\leq (1 - \beta_n) \|x_n - \tilde{x}\|^2 + \beta_n \gamma^2 \|x_n - \tilde{x}\|^2 + 2\beta_n \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&= \left[1 - (1 - \gamma^2)\beta_n\right] \|x_n - \tilde{x}\|^2 + 2\beta_n \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle.
\end{aligned}$$

Substituting (3.30) into (3.29), we get

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n \|x_n - \tilde{x}\|^2 + (1 - \alpha_n) \left[1 - (1 - \gamma^2)\beta_n\right] \|x_n - \tilde{x}\|^2 \\
&\quad + 2\beta_n(1 - \alpha_n) \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle + 2\alpha_n \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&= \left[1 - (1 - \gamma^2)\beta_n(1 - \alpha_n)\right] \|x_n - \tilde{x}\|^2 + 2\beta_n(1 - \alpha_n) \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&\quad + 2\alpha_n \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&= \left[1 - (1 - \gamma^2)\beta_n(1 - \alpha_n)\right] \|x_n - \tilde{x}\|^2 + (1 - \gamma^2)\beta_n(1 - \alpha_n) \\
&\quad \times \left\{ \frac{2}{1 - \gamma^2} \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle + \frac{2}{(1 - \gamma^2)(1 - \alpha_n)} \times \frac{\alpha_n}{\beta_n} \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \right\}. \quad (3.31)
\end{aligned}$$

By Theorem 3.2, we note that every weak cluster point of the sequence $\{x_n\}$ is in Ω . Since $y_n - x_n \rightarrow 0$, then every weak cluster point of $\{y_n\}$ is also in Ω . Consequently, since $\tilde{x} = \text{proj}_\Omega(Q\tilde{x})$, we easily have

$$\limsup_{n \rightarrow \infty} \langle Q\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \leq 0. \quad (3.32)$$

On the other hand, we observe that

$$\langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle = \langle W_n \tilde{x} - \tilde{x}, \text{proj}_{\text{Fix}(T)} x_{n+1} - \tilde{x} \rangle + \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1} \rangle. \quad (3.33)$$

Since $\tilde{x}$ is a solution of the problem (1.9) and $\text{proj}_{\text{Fix}(T)} x_{n+1} \in \text{Fix}(T)$, we have

$$\langle W_n \tilde{x} - \tilde{x}, \text{proj}_{\text{Fix}(T)} x_{n+1} - \tilde{x} \rangle \leq 0. \quad (3.34)$$

Thus it follows that

$$\begin{aligned} \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle &\leq \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1} \rangle \\ &\leq \|W_n \tilde{x} - \tilde{x}\| \|x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1}\| \\ &= \|W_n \tilde{x} - \tilde{x}\| \times \text{Dist}(x_{n+1}, \text{Fix}(T)) \\ &\leq \frac{1}{k} \|W_n \tilde{x} - \tilde{x}\| \|x_{n+1} - Tx_{n+1}\|. \end{aligned} \quad (3.35)$$

We note that

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - Tx_n\| + \|Tx_n - Tx_{n+1}\| \\ &\leq \alpha_n \|W_n x_n - Tx_n\| + (1 - \alpha_n) \|Ty_n - Tx_n\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n \|W_n x_n - Tx_n\| + \|y_n - x_n\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n \|W_n x_n - Tx_n\| + \beta_n \|Qx_n - x_n\| + \|x_{n+1} - x_n\|. \end{aligned} \quad (3.36)$$

Hence we have

$$\begin{aligned} &\frac{\alpha_n}{\beta_n} \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\ &\leq \frac{\alpha_n^2}{\beta_n} \left(\frac{1}{k} \|W_n \tilde{x} - \tilde{x}\| \|W_n x_n - Tx_n\| \right) + \alpha_n \left(\frac{1}{k} \|W_n \tilde{x} - \tilde{x}\| \|Qx_n - x_n\| \right) \\ &\quad + \frac{\alpha_n^2}{\beta_n} \frac{\|x_{n+1} - x_n\|}{\alpha_n} \left(\frac{1}{k} \|W_n \tilde{x} - \tilde{x}\| \right). \end{aligned} \quad (3.37)$$

From Theorem 3.2, we have $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\|/\alpha_n = 0$. At the same time, we note that $\{(1/k)\|W_n \tilde{x} - \tilde{x}\|\|W_n x_n - Tx_n\|\}$, $\{(1/k)\|W_n \tilde{x} - \tilde{x}\|\|Qx_n - x_n\|\}$, and $\{(1/k)\|W_n \tilde{x} - \tilde{x}\|\}$ are all bounded. Hence it follows from (i) and the above inequality that

$$\limsup_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} \langle W_n \tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \leq 0. \quad (3.38)$$

Finally, by (3.31)–(3.38) and Lemma 2.4, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in \text{Fix}(T)$. This completes the proof. $\square$

Remark 3.4. In the present paper, we consider the hierarchical problem (1.9) which includes the hierarchical problem (1.1) as a special case.

From the above discussion, we can easily deduce the following result.

Algorithm 3.5. Let C be a nonempty closed convex subset of a real Hilbert space H and S a nonexpansive mapping of C into itself. Let $Q : C \rightarrow C$ be a contraction with coefficient $\gamma \in [0, 1)$. For any $x_0 \in C$, let $\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n Sx_n + (1 - \alpha_n)T(\beta_n Qx_n + (1 - \beta_n)x_n), \quad \forall n \geq 0, \quad (3.39)$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$.

Corollary 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping. Let $Q : C \rightarrow C$ be a contraction with coefficient $\gamma \in [0, 1)$. Assume that the set Ω' of solutions of the hierarchical problem (1.1) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0, 1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \beta_n / \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n^2 / \beta_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} (1/\beta_n)|(1/\alpha_n) - (1/\alpha_{n-1})| = 0$ and $\lim_{n \rightarrow \infty} (1/\alpha_n)|1 - (\beta_{n-1}/\beta_n)| = 0$;
- (iv) there exists a constant $k > 0$ such that $\|x - Tx\| \geq k \text{Dist}(x, \text{Fix}(T))$, where

$$\text{Dist}(x, \text{Fix}(T)) = \inf_{y \in \text{Fix}(T)} \|x - y\|. \quad (3.40)$$

Then the sequence $\{x_n\}$ defined by (3.39) converges strongly to a point $\tilde{x} \in \text{Fix}(T)$, which solves the hierarchical problem (1.1).

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