Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 325473, 17 pages
doi:10.1155/2012/325473

Research Article

Formulation and Solution of *n*th-Order Derivative Fuzzy Integrodifferential Equation Using New Iterative Method with a Reliable Algorithm

A. A. Hemeda

Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

Correspondence should be addressed to A. A. Hemeda, aahemeda@yahoo.com

Received 11 July 2012; Revised 9 September 2012; Accepted 10 September 2012

Academic Editor: Fazlollah Soleymani

Copyright © 2012 A. A. Hemeda. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The *n*th-order derivative fuzzy integro-differential equation in parametric form is converted to its crisp form, and then the new iterative method with a reliable algorithm is used to obtain an approximate solution for this crisp form. The analysis is accompanied by numerical examples which confirm efficiency and power of this method in solving fuzzy integro-differential equations.

1. Introduction

In 1975 Zadeh and then Dubois and Prade [1] introduced fuzzy numbers and fuzzy arithmetic. This concept propagated widely to various problems, for example, fuzzy linear systems [2, 3], fuzzy differential equations [4, 5], fuzzy integral equations [6–8], and fuzzy integro-differential equations [9–12]. Additional topics can be found in [13, 14]. Recently, several numerical methods were suggested to solve integro-differential equations, for example, Sine-Cosine wavelet used by Tavassoli Kajani et al. to obtain a solution of linear integro-differential equations [15] and variational iteration method used by Abbasbandy and Hashemi to formulate and solve fuzzy integro-differential equation [12], by Saberi-Nadjafi and Tamamgar for solving system of integro-differential equations [16], and by Shang and Han for solving *n*th-order integro-differential equations [17]. Some other worthwhile works can be found in [18, 19]. Recently, Daftardar-Gejji and Jafari [20, 21] proposed the new iterative method. This method has proven useful for solving a variety of linear and nonlinear equations such as algebraic equations, integral equations, ordinary and partial differential equations of integer and fractional order, and system of equations as well. The new iterative method is simple to understand and easy to implement using computer packages and

yields better results [21] than the existing Adomian decomposition method [22], homotopy perturbation method [23], or variational iteration method [24]. For more details, see [25–40].

In the present work we apply the new iterative method with a reliable algorithm to solve the nth-order derivative fuzzy integro-differential equation in its crisp form.

2. Preliminaries

In this section we set up the basic definitions of fuzzy numbers and fuzzy functions.

Definition 2.1 (see [2, 3, 12]). A fuzzy number in parametric form is an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$ which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$,

(2) $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$,

(3) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Remark 2.2 (see [6, 12]). Let $u(r) = (\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$ be a fuzzy number, and we can take

$$u^c(r) = \frac{1}{2}\left(\underline{u}(r) + \overline{u}(r)\right), \qquad u^d(r) = \frac{1}{2}\left(\overline{u}(r) - \underline{u}(r)\right). \tag{2.1}$$

It is clear that $u^d(r) \ge 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, and also a fuzzy number $u \in E$ is said symmetric if $u^c(r)$ is independent of r for all $0 \le r \le 1$.

Let $f : [a,b] \to E$ for each partition $P = \{t_0, \ldots, t_n\}$ of $[a,b]$ and for arbitrary $\xi_i \in t_{i-1} - t_i]$, $1 \le i \le n$ suppose that $R_P = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$ and $\Delta = \max\{|t_i - t_{i-1}|, i = 1, \ldots, n\}$. The definite integral of $f(t)$ over $[a,b]$ is $\int_a^b f(t)dt = \lim_{\Delta \to 0} R_P$ provided that this limit exists in the metric D. If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists. Also we have $(\underline{\int_a^b f(t,r)dt}) = \int_a^b \underline{f}(t,r)dt$ and $(\overline{\int_a^b f(t,r)dt}) = \int_a^b \overline{f}(t,r)dt$.

Definition 2.3 (see [33]). Let $f : (a,b) \to R_F$ and $t_0 \in (a,b)$. We say that f' is Hukuhara differentiable at t_0 if there exists an element $f'(t_0) \in R_F$, such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus_H f(t_0)$, $f(t_0) \ominus_H f(t_0 - h)$ and

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus_H f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus_H f(t_0 - h)}{h} = f'(t_0), \tag{2.2}$$

where $\ominus_H$ is Hukuhara difference, R_F is the set of fuzzy numbers, and the limit is in the metric D.

In parametric form, if $f = (\underline{f}, \overline{f})$, then $f' = (\underline{f}', \overline{f}')$, where $\underline{f}', \overline{f}'$ are Hukuhara differentiable of $\underline{f}, \overline{f}$, respectively.

From (2.2), the nth-order Hukuhara differentiable, $f^{(n)}$, of f at t_0 can be defined as in the following definition.

Definition 2.4. Let $f^{(n-1)} : (a,b) \to R_F$ and $t_0 \in (a,b)$ where $f^{(n-1)}$ is $(n-1)$th-orderHukuhara differentiable of f at t_0 for all $n > 1$. We say that $f^{(n)}$ is nth-order Hukuhara differentiable at

t_0 if there exists an element $f^{(n)}(t_0) \in R_F$ such that for all $h > 0$ sufficiently small, $\exists f^{(n-1)}(t_0 + h) \ominus_H f^{(n-1)}(t_0), f^{(n-1)}(t_0) \ominus_H f^{(n-1)}(t_0 - h)$ and

$$\lim_{h \to 0} \frac{f^{(n-1)}(t_0 + h) \ominus_H f^{(n-1)}(t_0)}{h} = \lim_{h \to 0} \frac{f^{(n-1)}(t_0) \ominus_H f^{(n-1)}(t_0 - h)}{h} = f^{(n)}(t_0), \tag{2.3a}$$

$n = 1, 2, \ldots$. In parametric form, as above, if $= (\underline{f}, \overline{f})$, then $f^{(n)} = (\underline{f}^{(n)}, \overline{f}^{(n)})$, where $\underline{f}^{(n)}, \overline{f}^{(n)}$ are nth-order Hukuhara differentiable of $\underline{f}, \overline{f}$, respectively. From (2.3a), in case $n = 1$, we obtain (2.2).

3. Fuzzy Integro-Differential Equation

Consider the nth-order derivative integro-differential equation [12] as follows:

$$u^{(n)}(t) + u(t) + \lambda \int_0^T k(s,t) u^{(m)}(s) ds = g(t), \tag{3.1a}$$

where $m, n \in \mathbb{N}$, $m < n$, $t \in [0, T]$, $\lambda \in \mathbb{R}$, $g(t)$ is a known function and the kernel $k(s,t) \geq 0$ with the initial conditions as follows:

$$u^{(k)}(0) = h_k, \quad k = 0, 1, \ldots, n-1, \ h_k \in \mathbb{R}, \tag{3.1b}$$

in the fuzzy case; that is, u and g be fuzzy functions. Let

$$\begin{aligned} u(t,r) &= (\underline{u}(t,r), \overline{u}(t,r)), \qquad & g(t,r) &= \left(\underline{g}(t,r), \overline{g}(t,r)\right), \\ u^{(n)}(t,r) &= \left(\underline{u}^{(n)}(t,r), \overline{u}^{(n)}(t,r)\right), \qquad & g^{(n)}(t,r) &= \left(\underline{g}^{(n)}(t,r), \overline{g}^{(n)}(t,r)\right), \end{aligned} \tag{3.2}$$

where all derivatives are, with respect to t, fuzzy functions. Therefore, related fuzzy integro-differential equation of (3.1a) can be written as follows:

$$\underline{u}^{(n)}(t,r) + \underline{u}(t,r) + \lambda \int_0^T k(s,t) \underline{u}^{(m)}(s,r) ds = \underline{g}(t,r), \tag{3.3a}$$

$$\overline{u}^{(n)}(t,r) + \overline{u}(t,r) + \lambda \int_0^T k(s,t) \overline{u}^{(m)}(s,r) ds = \overline{g}(t,r), \tag{3.3b}$$

and its two crisp equations can be written as follows:

$$u^{c(n)}(t,r) + u^{c}(t,r) + \lambda \int_0^T k(s,t) u^{c(m)}(s,r) ds = g^{c}(t,r), \tag{3.4a}$$

$$u^{d(n)}(t,r) + u^{d}(t,r) + \lambda \int_0^T k(s,t) u^{d(m)}(s,r) ds = g^{d}(t,r). \tag{3.4b}$$

4. New Iterative Method

For simplicity, we present a review of the new iterative method [20, 21, 34–39, 41], and then we introduce a suitable algorithm of this method for solving the nth-order derivative fuzzy integro-differential equations.

Consider the following general functional equation:

$$u = f + N(u), \tag{4.1}$$

where N is a nonlinear operator from a Banach space $B \to B$, and f is a known function.

We are looking for a solution u of (4.1) having the series form:

$$u = \sum_{i=0}^{\infty} u_i. \tag{4.2}$$

The nonlinear operator N can be decomposed as follows:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{4.3}$$

From (4.2) and (4.3), (4.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{4.4}$$

We define the recurrence relation:

$$\begin{gathered} u_0 = f, \\ u_1 = N(u_0), \\ u_{n+1} = N(u_0 + u_1 + \cdots + u_n) - N(u_0 + u_1 + \cdots + u_{n-1}), \quad n = 1, 2, \ldots. \end{gathered} \tag{4.5}$$

Then

$$\begin{gathered} (u_1 + \cdots + u_{n+1}) = N(u_0 + \cdots + u_n), \quad n = 1, 2, \ldots, \\ \sum_{i=0}^{\infty} u_i = f + \sum_{i=1}^{\infty} u_i. \end{gathered} \tag{4.6}$$

The n-term approximate solution of (4.1) is given by $u = u_0 + u_1 + \cdots + u_{n-1}$.

4.1. Reliable Algorithm

After the above presentation of the new iterative method, we introduce a reliable algorithm of this method for solving any nth-order derivative fuzzy integro-differential equation. Consider the nth-order derivative integro-differential equation (3.1a) and (3.1b) defined as

$$u^{(n)}(t) + u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds = g(t), \tag{4.7a}$$

with the following initial conditions:

$$u^{(k)}(0) = h_k, \quad k = 0,1,\ldots,n-1,\ h_k \in \mathbb{R}. \tag{4.7b}$$

The initial value problem (4.7a) and (4.7b) is equivalent to the following integral equation:

$$u(t) = u_0(t) - I_t^n\left[u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds\right] = u_0(t) - N(u), \tag{4.8}$$

where $u_0(t)$ is the solution of the nth-order differential equation:

$$\frac{d^n u_0}{dt^n} = g(t), \qquad u_0^{(k)}(0) = h_k, \quad k = 0,1,2,\ldots,n-1,\ h_k \in \mathbb{R}, \tag{4.9a}$$

$$N(u) = I_t^n\left[u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds\right], \tag{4.9b}$$

and I_t^n is an integral operator of order n. Also, the two crisp equations (3.4a) and (3.4b) are equivalent to the two integral equations:

$$u^c(t,r) = u_0^c(t,r) - I_t^n\left[u^c(t,r) + \lambda \int_0^T k(s,t)u^{c(m)}(s,r)ds\right] = u_0^c(t,r) - N(u^c), \tag{4.10a}$$

$$u^d(t,r) = u_0^d(t,r) - I_t^n\left[u^d(t,r) + \lambda \int_0^T k(s,t)u^{d(m)}(s,r)ds\right] = u_0^d(t,r) - N\left(u^d\right), \tag{4.10b}$$

where $u_0^c(t,r)$ and $u_0^d(t,r)$ are the solutions of the nth-order differential equations:

$$\frac{d^n u_0^c}{dt^n} = g^c(t), \qquad u_0^{c(k)}(0) = h_k^c, \quad k = 0,1,2,\ldots,n-1,\ h_k^c \in \mathbb{R}, \tag{4.10c}$$

$$\frac{d^n u_0^d}{dt^n} = g^d(t), \qquad u_0^{d(k)}(0) = h_k^d, \quad k = 0,1,2,\ldots,n-1,\ h_k^d \in \mathbb{R}. \tag{4.10d}$$

$N(u^c)$ and $N(u^d)$ are the following two integral equations:

$$N(u^c) = I_t^n\left[u^c(t,r) + \lambda\int_0^T k(s,t)u^{c(m)}(s,r)ds\right], \tag{4.10e}$$

$$N(u^c) = I_t^n\left[u^d(t,r) + \lambda\int_0^T k(s,t)u^{c(m)}(s,r)ds\right]. \tag{4.10f}$$

We get the solution of (4.8) or the two (4.10a), (4.10b) by employing the recurrence relation (4.5).

4.2. Convergence of the New Iterative Method

Now, we introduce the condition of convergence of the new iterative method, which is proposed by Daftardar-Gejji and Jafari in (2006) [20], also called (DJM) [39]. From (4.3), the nonlinear operator N is decomposed as follows [39]:

$$N(u) = N(u_0) + [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \cdots. \tag{4.11}$$

Let $G_0 = N(u_0)$ and

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1,2,\ldots. \tag{4.12}$$

Then $N(u) = \sum_{i=0}^{\infty} G_i$.

Set

$$u_0 = f, \tag{4.13}$$

$$u_n = G_{n-1}, \quad n = 1,2,\ldots. \tag{4.14}$$

Then

$$u = \sum_{i=0}^{\infty} u_i \tag{4.15}$$

is a solution of the general functional equation (4.1). Also, the recurrence relation (4.5) becomes

$$\begin{aligned} u_0 &= f, \\ u_n &= G_{n-1}, \quad n = 1,2,\ldots. \end{aligned} \tag{4.16}$$

Using Taylor series expansion for G_i, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
G_1 &= N(u_0 + u_1) - N(u_0) = N(u_0) + N'(u_0)u_1 + N''(u_0)\frac{u_1^2}{2!} \\
&\quad + \cdots - N(u_0) = \sum_{k=1}^{\infty} N^{(k)}(u_0)\frac{u_1^k}{k!}, \\
G_2 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) = N'(u_0 + u_1)u_2 + N''(u_0 + u_1)\frac{u_2^2}{2!} \\
&\quad + \cdots = \sum_{j=1}^{\infty}\left[\sum_{i=0}^{\infty} N^{(i+j)}(u_0)\frac{u_1^i}{i!}\right]\frac{u_2^j}{j!}, \\
G_3 &= \sum_{i_3=1}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty} N^{(i_1+i_2+i_3)}(u_0)\frac{u_3^{i_3}}{i_3!}\frac{u_2^{i_2}}{i_2!}\frac{u_1^{i_1}}{i_1!}.
\end{aligned} \tag{4.17}$$

In general:

$$G_n = \sum_{i_n=1}^{\infty}\sum_{i_{n-1}=0}^{\infty}\cdots\sum_{i_1=0}^{\infty}\left[N^{(\sum_{k=1}^{n} i_k)}(u_0)\left(\prod_{j=1}^{n}\frac{u_j^{i_j}}{i_j!}\right)\right]. \tag{4.18}$$

In the following theorem we state and prove the condition of convergence of the method.

Theorem 4.1. *If N is $C^{(\infty)}$ in a neighborhood of u_0 and*

$$\left\|N^{(n)}(u_0)\right\| = \sup\left\{N^{(n)}(u_0)(h_1, \ldots, h_n) : \|h_i\| \le 1, 1 \le i \le n\right\} \le L, \tag{4.19a}$$

for any n and for some real $L > 0$ and $\|u_i\| \le M < 1/e$, $i = 1, 2, \ldots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and, moreover,

$$\|G_n\| \le LM^n e^{n-1}(e-1), \qquad n = 1, 2, \ldots. \tag{4.19b}$$

Proof. In view of (4.18),

$$\|G_n\| \le LM^n \sum_{i_n=1}^{\infty}\sum_{i_{n-1}=0}^{\infty}\cdots\sum_{i_1=0}^{\infty}\left(\prod_{j=1}^{n}\frac{u_j^{i_j}}{i_j!}\right) = LM^n e^{n-1}(e-1). \tag{4.19c}$$

Thus, the series $\sum_{n=1}^{\infty}\|G_n\|$ is dominated by the convergent series $LM(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$, where $M < 1/e$. Hence, $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, due to the comparison test.

For more details, see [39]. □

5. Numerical Examples

Example 5.1. Consider the following fuzzy integro-differential equation:

$$u'(t,r) + \lambda \int_0^1 k(s,t)u(s,r)ds = g(t,r), \quad 0 \le t, r \le 1, \tag{5.1}$$

where $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r)) = (-2rt, 4 + rt)$ is the exact solution for (5.1), $g(t,r) = (\underline{g}(t,r), \overline{g}(t,r)) = (-2r - 2rt/3,\ r + 2t + rt/3$, $k(s,t) = st$, and $\lambda = 1$, with the initial conditions $\underline{u}(0,r) = 0$, $\overline{u}(0,r) = 4$. Taking $g^c(t,r) = (1/2)(\underline{g}(t,r) + \overline{g}(t,r)) = -r/2 + t - rt/6$, $g^d(t,r) = (1/2)(\overline{g}(t,r) - \underline{g}(t,r)) = 3r/2 + t + rt/2$. The exact solution for related crisp equations is $u^c(t,r) = 2 - rt/2$, $u^d(t,r) = 2 + 3rt/2$. At first, we identify $u^c(t,r)$ and $u^d(t,r)$. From (3.4a) and (3.4b), (5.1), where $n = 1$, we have

$$u^{c'}(t,r) + \int_0^1 stu^c(s,r)ds = g^c(t,r), \qquad u^c(0,r) = 2, \tag{5.2a}$$

$$u^{d'}(t,r) + \int_0^1 stu^d(s,r)ds = g^d(t,r), \qquad u^d(0,r) = 2. \tag{5.2b}$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t,r) = 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12}, \qquad u_0^d(t,r) = 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4}. \tag{5.3}$$

Therefore, from (4.10a), (4.10b) the fuzzy integro-differential equations (5.2a) and (5.2b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t,r) &= 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12} - I_t\left[\int_0^1 stu^c(s,r)ds\right], \\ u^d(t,r) &= 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4} - I_t\left[\int_0^1 stu^d(s,r)ds\right]. \end{aligned} \tag{5.4}$$

Let $N(u^c) = -I_t[\int_0^1 stu^c(s,r)ds], N(u^d) = -I_t[\int_0^1 stu^d(s,r)ds]$. Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.1):

$$\begin{aligned}
&u_0^c(t,r) = 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12}, \qquad u_0^d(t,r) = 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4},\\
&u_1^c(t,r) = N\left(u_0^c\right) = -\frac{9t^2}{16} + \frac{9rt^2}{96}, \qquad u_1^d(t,r) = N\left(u_0^d\right) = -\frac{9t^2}{16} - \frac{9rt^2}{32},\\
&u_2^c(t,r) = \frac{9t^2}{128} - \frac{9rt^2}{768}, \qquad u_2^d(t,r) = \frac{9t^2}{128} + \frac{9rt^2}{256},\\
&\qquad \vdots\\
&u_5^c(t,r) = -\frac{9t^2}{65536} + \frac{9rt^2}{393216}, \qquad u_5^d(t,r) = -\frac{9t^2}{65536} - \frac{9rt^2}{131072},
\end{aligned} \tag{5.5}$$

and so on. The 6-term approximate solution is

$$\begin{aligned}
u^c(t,r) &= \sum_{n=0}^{5} u_n^c = 2 - \frac{rt}{2} - \frac{t^2}{65536} + \frac{rt^2}{393216},\\
u^d(t,r) &= \sum_{n=0}^{5} u_n^d = 2 + \frac{3rt}{2} - \frac{t^2}{65536} - \frac{rt^2}{131072}.
\end{aligned} \tag{5.6}$$

It is clear that the iterations converge to the exact solution of the two crisp equations u^c, u^d as the number of iteration converges to ∞, that is, $\lim_{n\to\infty} u_n^c(t,r) = 2 - rt/2 = u_{\text{Exact}}^c$, and $\lim_{n\to\infty} u_n^d(t,r) = 2 + 3rt/2 = u_{\text{Exact}}^d$. From the relations between $\underline{u}, \overline{u}, u^c, u^d$, and u in Section 2, it follows immediately that this solution is the solution of the fuzzy integro-differential equation (5.1).

Example 5.2. Consider the following fuzzy integro-differential equation:

$$u''(t,r) + \lambda \int_0^1 k(s,t)u(s,r)ds = g(t,r), \quad 0 \le t, r \le 1. \tag{5.7}$$

In this example $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r)) = (2rt, 4rt)$ is the exact solution for (5.7), $g(t,r) = (\underline{g}(t,r), \overline{g}(t,r)) = 2rt/3, 4rt/3$, $k(s,t) = st$, $\lambda = 1$, and the initial conditions are $\underline{u}(0,r) = 0$, $\underline{u}'(0,r) = 2r$, $\overline{u}(0,r) = 0$, $\overline{u}'(0,r) = 4r$. Also, the exact solution for related crisp equations is $u^c(t,r) = 3rt$, $u^d(t,r) = rt$ with $g^c(t,r) = rt$, $g^d(t,r) = rt/3$. At first, we identify $u^c(t,r)$, $u^d(t,r)$. From (3.4a) and (3.4b), (5.7), where $n = 2$, we have

$$u^{c''}(t,r) + \int_0^1 stu^c(s,r)ds = g^c(t,r), \qquad u^c(0,r) = 0, \qquad u^{c'}(0,r) = 3r, \tag{5.8a}$$

$$u^{d''}(t,r) + \int_0^1 stu^d(s,r)ds = g^d(t,r), \qquad u^d(0,r) = 0, \qquad u^{d'}(0,r) = r. \tag{5.8b}$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t,r) = 3rt + \frac{rt^3}{6}, \qquad u_0^d(t,r) = rt + \frac{rt^3}{18}. \tag{5.9}$$

Therefore, from (4.10a), (4.10b), the fuzzy integro-differential equations (5.8a) and (5.8b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t,r) &= 3rt + \frac{rt^3}{6} - I_t^2\left[\int_0^1 stu^c(s,r)ds\right], \\ u^d(t,r) &= rt + \frac{rt^3}{18} - I_t^2\left[\int_0^1 stu^d(s,r)ds\right]. \end{aligned} \tag{5.10}$$

Let $N(u^c) = -I_t^2[\int_0^1 stu^c(s,r)ds]$, $N(u^d) = -I_t^2[\int_0^1 stu^d(s,r)ds]$. Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.7):

$$\begin{aligned} &u_0^c(t,r) = 3rt + \frac{rt^3}{6}, \qquad u_0^d(t,r) = rt + \frac{rt^3}{18}, \\ &u_1^c(t,r) = N\left(u_0^c\right) = -\frac{31rt^3}{180}, \qquad u_1^d(t,r) = N\left(u_0^d\right) = -\frac{31rt^3}{540}, \\ &u_2^c(t,r) = \frac{31rt^3}{5400}, \qquad u_2^d(t,r) = \frac{31rt^2}{16200}, \\ &\qquad \vdots \\ &u_5^c(t,r) = -\frac{31rt^3}{145800000}, \qquad u_5^d(t,r) = -\frac{29rt^2}{437400000}, \end{aligned} \tag{5.11}$$

and so on. The 6-term approximate solution is

$$u^c(t,r) = \sum_{n=0}^{5} u_n^c = 3rt - \frac{rt^3}{145800000}, \qquad u^d(t,r) = \sum_{n=0}^{5} u_n^d = rt - \frac{rt^3}{437400000}. \tag{5.12}$$

It is clear that the iterations converge to the exact solution of u^c, u^d as $n \to \infty$, that is, $\lim_{n\to\infty} u_n^c(t,r) = 3rt = u_{\text{Exact}}^c$, $\lim_{n\to\infty} u_n^d(t,r) = rt = u_{\text{Exact}}^d$. As the above example the solution of (5.7) follows immediately.

Example 5.3. The third example is the fuzzy integro-differential equation:

$$u''(t,r) + \lambda \int_0^1 k(s,t)u'(s,r)ds = g(t,r), \quad 0 \le t, r \le 1, \tag{5.13}$$

where $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r)) = (rt, 4 - 3rt)$ is the exact solution for (5.13), $g(t,r) = (\underline{g}(t,r), \overline{g}(t,r)) = (-rt/2, 3rt/2)$, $k(s,t) = st$, $\lambda = -1$, and the initial conditions are $\underline{u}(0,r) = 0$,

$\underline{u}'(0,r) = r$, $\overline{u}(0,r) = 4$, $\overline{u}'(0,r) = -3r$. The exact solution for related crisp equations is $u^c(t,r) = 2 - rt$, $u^d(t,r) = 2 - 2rt$ with $g^c(t,r) = rt/2$, $g^d(t,r) = rt$. At first, we identify $u^c(t,r)$, $u^d(t,r)$, and from (3.4a) and (3.4b), (5.13), where $n = 2$, we have

$$u^{c''}(t,r) - \int_0^1 stu^{c'}(s,r)ds = g^c(t,r), \qquad u^c(0,r) = 2, \qquad u^{c'}(0,r) = -r, \tag{5.14a}$$

$$u^{d''}(t,r) - \int_0^1 stu^{d'}(s,r)ds = g^d(t,r), \qquad u^d(0,r) = 2, \qquad u^{d'}(0,r) = -2r. \tag{5.14b}$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t,r) = 2 - rt + \frac{rt^3}{12}, \qquad u_0^d(t,r) = 2 - 2rt + \frac{rt^3}{6}. \tag{5.15}$$

Therefore, from (4.10a) and (4.10b), the fuzzy integro-differential equations (5.14a) and (5.14b) are equivalent to the integral equations:

$$\begin{aligned} u^c(t,r) &= 2 - rt + \frac{rt^3}{12} + I_t^2\left[\int_0^1 stu^{c'}(s,r)ds\right], \\ u^d(t,r) &= 2 - 2rt + \frac{rt^3}{6} + I_t^2\left[\int_0^1 stu^{d'}(s,r)ds\right]. \end{aligned} \tag{5.16}$$

Let $N(u^c) = I_t^2[\int_0^1 stu^{c'}(s,r)ds]$, $N(u^d) = I_t^2[\int_0^1 stu^{d'}(s,r)ds]$. Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.13):

$$\begin{aligned} &u_0^c(t,r) = 2 - rt + \frac{rt^3}{12}, \qquad u_0^d(t,r) = 2 - 2rt + \frac{rt^3}{6}, \\ &u_1^c(t,r) = N\left(u_0^c\right) = -\frac{7rt^3}{96}, \qquad u_1^d(t,r) = N\left(u_0^d\right) = -\frac{7rt^3}{48}, \\ &u_2^c(t,r) = -\frac{7rt^3}{768}, \qquad u_2^d(t,r) = -\frac{7rt^3}{384}, \\ &\qquad \vdots \\ &u_5^c(t,r) = -\frac{7rt^3}{393216}, \qquad u_5^d(t,r) = -\frac{7rt^3}{196608}, \end{aligned} \tag{5.17}$$

and so on. The 6-term approximate solution is

$$u^c(t,r) = \sum_{n=0}^{5} u_n^c = 2 - rt + \frac{rt^3}{393216}, \qquad u^d(t,r) = \sum_{n=0}^{5} u_n^d = 2 - 2rt + \frac{rt^3}{196608}. \tag{5.18}$$

It is clear that the iterations converge to the exact solution of u^c, u^d as $n \to \infty$, that is, $\lim_{n\to\infty} u_n^c(t,r) = 2 - rt = u^c_{\text{Exact}}$, $\lim_{n\to\infty} u_n^d(t,r) = 2 - 2rt = u^d_{\text{Exact}}$. Therefore the solution of (5.13) follows immediately.

Example 5.4. Let us consider the following fuzzy integro-differential equation:

$$u''(t,r) + u(t,r) + \lambda \int_0^1 k(s,t)u(s,r)ds = g(t,r), \quad 0 \le t, r \le 1, \tag{5.19}$$

with the exact solution $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r)) = (3rt, 8 - rt)$, $g(t,r) = (\underline{g}, \overline{g}) = (2rt, 8 - 4t - (2rt/3))$, $k(s,t) = st$, $\lambda = -1$, and the initial conditions $\underline{u}(0,r) = 0$, $\underline{u}'(0,r) = 3r$, $\overline{u}(0,r) = 8$, $\overline{u}'(0,r) = -r$. The exact solution for related crisp equations is $u^c(t,r) = 4 + rt$, $u^d(t,r) = 4 - 2rt$ with $g^c(t,r) = 4 - 2t + 2rt/3$, $g^d(t,r) = 4 - 2t - 4rt/3$. From (3.4a), (3.4b), and (5.19), where $n = 2$, the two crisp equations are

$$u^{c''}(t,r) + u^c(t,r) - \int_0^1 stu^c(s,r)ds = g^c(t,r), \qquad u^c(0,r) = 4, \qquad u^{c'}(0,r) = r, \tag{5.20a}$$

$$u^{d''}(t,r) + u^d(t,r) - \int_0^1 stu^d(s,r)ds = g^d(t,r), \qquad u^d(0,r) = 4, \qquad u^{d'}(0,r) = -2r. \tag{5.20b}$$

From (4.10c), (4.10d) we obtain

$$u_0^c(t,r) = 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9}, \qquad u_0^d(t,r) = 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9}. \tag{5.21}$$

Therefore, from (4.10a), (4.10b), the fuzzy integro-differential equations (5.20a) and (5.20b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t,r) &= 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9} + I_t^2\left[-u^c(t,r) + \int_0^1 stu^c(s,r)ds\right], \\ u^d(t,r) &= 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9} + I_t^2\left[-u^d(t,r) + \int_0^1 stu^d(s,r)ds\right]. \end{aligned} \tag{5.22}$$

Let

$$\begin{aligned} N(u^c) &= I_t^2\left[-u^c(t,r) + \int_0^1 stu^c(s,r)ds\right], \\ N\left(u^d\right) &= I_t^2\left[-u^d(t,r) + \int_0^1 stu^d(s,r)ds\right]. \end{aligned} \tag{5.23}$$

Therefore, from (4.5), we can obtain the following first approximate solutions for (5.19):

$$
\begin{aligned}
&u_0^c(t,r) = 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9}, \qquad u_0^d(t,r) = 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9},\\
&u_1^c(t,r) = -2t^2 + \frac{73t^3}{180} - \frac{29rt^3}{270} - \frac{t^4}{6} + \frac{t^5}{60} - \frac{rt^5}{180},\\
&u_1^d(t,r) = -2t^2 + \frac{73t^3}{180} + \frac{29rt^3}{135} - \frac{t^4}{6} + \frac{t^5}{60} + \frac{rt^5}{90},\\
&u_2^c(t,r) = -\frac{311t^3}{4200} - \frac{421rt^3}{113400} + \frac{t^4}{6} - \frac{73t^5}{3600} + \frac{29rt^5}{5400} + \frac{t^6}{180} - \frac{t^7}{2520} + \frac{rt^7}{7560},\\
&u_2^d(t,r) = -\frac{311t^3}{4200} + \frac{421rt^3}{56700} + \frac{t^4}{6} - \frac{73t^5}{3600} - \frac{29rt^5}{2700} + \frac{t^6}{180} - \frac{t^7}{2520} - \frac{rt^7}{3780},\\
&\qquad \vdots
\end{aligned}
\tag{5.24}
$$

and so on. In the same manner the rest of components can be obtained. The 5-term approximate solution is

$$
\begin{aligned}
u^c(t,r) &= \sum_{n=0}^{4} u_n^c = 4 + rt + \frac{2.8739t^3}{3143448} - \frac{0.1019rt^3}{9430344} + \frac{0.0523t^5}{27216} + \frac{0.0023rt^5}{20412}\\
&\quad - \frac{0.023t^7}{10584} - \frac{0.001rt^7}{95256} - \frac{0.13t^9}{108864} - \frac{0.01rt^9}{163296} + \frac{0.01t^{10}}{9072} - \frac{0.01t^{11}}{199584} + \frac{0.01rt^{11}}{598752},\\
u^d(t,r) &= \sum_{n=0}^{4} u_n^d = 4 - 2rt + \frac{2.8739t^3}{3143448} + \frac{0.1019rt^3}{4715172} + \frac{0.0523t^5}{27216} - \frac{0.0023rt^5}{10206}\\
&\quad - \frac{0.023t^7}{10584} + \frac{0.001rt^7}{47628} - \frac{0.13t^9}{108864} + \frac{0.01rt^9}{81648} + \frac{0.01t^{10}}{9072} - \frac{0.01t^{11}}{199584} - \frac{0.01rt^{11}}{299376}.
\end{aligned}
\tag{5.25}
$$

Now, let us define the absolute value of the nth-term error by $|e_n(t,r)| = |u^*(t,r) - u_n(t,r)|$, where $u^*(t,r)$ is the exact solution and $u_n(t,r)$ is the nth-term approximate solution respectively. It is clear from previous above solutions that the smallest value of the absolute nth-term error is at $t = r = 0$, that is, $|e_n(0,0)| = (|e_n^c(0,0)|, |e_n^d(0,0)|) = (0,0)$, $n = 0,1,\ldots,4$ and the largest value is at $t = r = 1$ as shown in Table 1. Therefore the iterations converge to the exact solution of u^c, u^d, that is, $\lim_{n\to\infty} u_n^c(t,r) = 4 + rt = u_{\text{Exact}}^c$, $\lim_{n\to\infty} u_n^d(t,r) = 4 - 2rt = u_{\text{Exact}}^d$, and the solution of (5.19) follows immediately.

Example 5.5. The final example is the following fuzzy integro-differential equation:

$$
u''(t,r) + u(t,r) + \lambda \int_0^1 k(s,t)u'(s,r)ds = g(t,r), \quad 0 \le t,r \le 1, \tag{5.26}
$$

Table 1

n	0	1	2	3	4
$\lvert e_n^c \rvert$	1.777778	7.9629629^{-2}	3.395065^{-4}	4.588^{-5}	5.67^{-7}
$\lvert e_n^d \rvert$	1.444444	7.4074074^{-2}	1.543214^{-4}	4.3601^{-5}	3.96^{-7}

with the exact solution $u(t,r) = (\underline{u}(t,r), \overline{u}(t,r)) = (3rt, 4+rt)$, $g(t,r) = (\underline{g}, \overline{g}) = (2rt, 4+2rt/3$, $k(s,t) = s^2t$, $\lambda = -1$ and the initial values $\underline{u}(0,r) = 0$, $\underline{u}'(0,r) = 3r$, $\overline{u}(0,r) = 4$, $\overline{u}'(0,r) = r$. The exact solution for related crisp equations is $u^c(t,r) = 2+2rt$, $u^d(t,r) = 2-rt$ with $g^c(t,r) = 2+4rt/3$, $g^d(t,r) = 2-2rt/3$. As above, where $n = 2$, the two crisp equations are

$$u^{c''}(t,r) + u^c(t,r) - \int_0^1 s^2 t u^{c'}(s,r)ds = g^c(t,r), \qquad u^c(0,r) = 2, \qquad u^{c'}(0,r) = 2r, \tag{5.27a}$$

$$u^{d''}(t,r) + u^d(t,r) - \int_0^1 s^2 t u^{d'}(s,r)ds = g^d(t,r), \qquad u^d(0,r) = 2, \qquad u^{d'}(0,r) = -r. \tag{5.27b}$$

As the above examples, we obtain

$$u_0^c(t,r) = 2 + 2rt + t^2 + \frac{2rt^3}{9}, \qquad u_0^d(t,r) = 2 - rt + t^2 - \frac{rt^3}{9}. \tag{5.28}$$

Therefore, the fuzzy integro-differential equations (5.27a) and (5.27b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t,r) &= 2 + 2rt + t^2 + \frac{2rt^3}{9} + I_t^2\left[-u^c(t,r) + \int_0^1 s^2 t u^{c'}(s,r)ds\right], \\ u^d(t,r) &= 2 - rt + t^2 - \frac{rt^3}{9} + I_t^2\left[-u^d(t,r) + \int_0^1 s^2 t u^{d'}(s,r)ds\right]. \end{aligned} \tag{5.29}$$

Let

$$\begin{aligned} N(u^c) &= I_t^2\left[-u^c(t,r) + \int_0^1 s^2 t u^{c'}(s,r)ds\right], \\ N\left(u^d\right) &= I_t^2\left[-u^d(t,r) + \int_0^1 s^2 t u^{d'}(s,r)ds\right]. \end{aligned} \tag{5.30}$$

Table 2

n	0	1	2	3	4
$\lvert e_n^c \rvert$	1.2222222	1.1111111^{-2}	2.2222222^{-3}	1.757303^{-4}	4.3324^{-6}
$\lvert e_n^d \rvert$	8.888889^{-1}	5.5555556^{-3}	2.23412699^{-3}	1.3038902^{-4}	1.19612^{-6}

Therefore, we obtain

$$\begin{aligned}
u_0^c(t,r) &= 2 + 2rt + t^2 + \frac{2rt^3}{9}, \qquad u_0^d(t,r) = 2 - rt + t^2 - \frac{rt^3}{9},\\
u_1^c(t,r) &= -t^2 + \frac{t^3}{12} - \frac{rt^3}{5} - \frac{t^4}{12} - \frac{rt^5}{90},\\
u_1^d(t,r) &= -t^2 + \frac{t^3}{12} + \frac{rt^3}{10} - \frac{t^4}{12} + \frac{rt^5}{180},\\
u_2^c(t,r) &= -\frac{91t^3}{1080} - \frac{403rt^3}{18900} + \frac{t^4}{12} - \frac{t^5}{240} + \frac{rt^5}{100} + \frac{t^6}{360} + \frac{rt^7}{3780},\\
u_2^d(t,r) &= -\frac{91t^3}{1080} + \frac{403rt^3}{37800} + \frac{t^4}{12} - \frac{t^5}{240} - \frac{rt^5}{200} + \frac{t^6}{360} - \frac{rt^7}{7560},\\
&\vdots
\end{aligned} \tag{5.31}$$

and so on. The 5-term approximate solution is

$$\begin{aligned}
u^c(t,r) &= \sum_{n=0}^{4} u_n^c = 2 + 2rt - \frac{0.337t^3}{27216} - \frac{5.0753rt^3}{1571724} + \frac{0.073t^5}{6048} + \frac{0.0041rt^5}{10206}\\
&\quad - \frac{0.01t^7}{9072} + \frac{0.017rt^7}{15876} - \frac{0.1t^9}{72576} - \frac{0.01rt^9}{27216} + \frac{0.01t^{10}}{18144} + \frac{0.01t^{11}}{299376},\\
u^d(t,r) &= \sum_{n=0}^{4} u_n^d = 2 - rt - \frac{0.337t^3}{27216} + \frac{5.0753rt^3}{3143448} + \frac{0.073t^5}{6048} - \frac{0.0041rt^5}{20412}\\
&\quad - \frac{0.01t^7}{9072} - \frac{0.017rt^7}{31752} - \frac{0.1t^9}{72576} + \frac{0.01rt^9}{54432} + \frac{0.01t^{10}}{18144} - \frac{0.01t^{11}}{598752}.
\end{aligned} \tag{5.32}$$

As the previous example, it is clear from the obtained results that the smallest value of the absolute nth-term error is $|e_n(0,0)| = (|e_n^c(0,0)|, |e_n^d(0,0)|) = (0,0)$, $n = 0,1,\ldots,4$, and the largest value is at $t = r = 1$ as shown in Table 2. Therefore, the iterations converge to the exact solution of u^c, u^d, that is, $\lim_{n\to\infty} u_n^c(t,r) = 2 + 2rt = u_{\text{Exact}}^c$, $\lim_{n\to\infty} u_n^d(t,r) = 2 - rt = u_{\text{Exact}}^d$. Therefore, the solution of (5.26) follows immediately.

6. Conclusion

In this work, the new iterative method used with a reliable algorithm to solve the nth-order derivative fuzzy integro-differential equations in crisp form and the cases of *first- and second-order* derivatives are taken into account. The obtained results concluded that the approximate solutions are in high agreement with corresponding exact solutions, which means that this method is suitable and effective to solve fuzzy integro-differential equations. Moreover, the solutions of the higher order fuzzy integro-differential equations can be calculated, as a future prospects, in a similar manner.

References

[1] D. Dubois and H. Prade, "Operations on fuzzy numbers," *International Journal of Systems Science*, vol. 9, no. 6, pp. 613–626, 1978.

[2] B. Asady, S. Abbasbandy, and M. Alavi, "Fuzzy general linear systems," *Applied Mathematics and Computation*, vol. 169, no. 1, pp. 34–40, 2005.

[3] M. S. Hashemi, M. K. Mirnia, and S. Shahmorad, "Solving fuzzy linear systems by using the Schur complement when coefficient matrix is an M-matrix," *Iranian Journal of Fuzzy Systems*, vol. 5, no. 3, pp. 15–29, 2008.

[4] S. Abbasbandy, T. A. Viranloo, Ó. López-Pouso, and J. J. Nieto, "Numerical methods for fuzzy differential inclusions," *Computers & Mathematics with Applications*, vol. 48, no. 10-11, pp. 1633–1641, 2004.

[5] S. Abbasbandy, J. J. Nieto, and M. Alavi, "Tuning of reachable set in one dimensional fuzzy differential inclusions," *Chaos, Solitons and Fractals*, vol. 26, no. 5, pp. 1337–1341, 2005.

[6] S. Abbasbandy, E. Babolian, and M. Alavi, "Numerical method for solving linear Fredholm fuzzy integral equations of the second kind," *Chaos, Solitons and Fractals*, vol. 31, no. 1, pp. 138–146, 2007.

[7] E. Babolian, H. Sadeghi Goghary, and S. Abbasbandy, "Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method," *Applied Mathematics and Computation*, vol. 161, no. 3, pp. 733–744, 2005.

[8] M. Friedman, M. Ma, and A. Kandel, "Numerical solutions of fuzzy differential and integral equations," *Fuzzy Sets and Systems*, vol. 106, no. 1, pp. 35–48, 1999, Fuzzy modeling and dynamics.

[9] Y. C. Kwun, M. J. Kim, B. Y. Lee, and J. H. Park, "Existence of solu-tions for the semilinear fuzzy integro-differential equations using by successive iteration," *Journal of Korean Institute of Intelligent Systems*, vol. 18, pp. 543–548, 2008.

[10] J. H. Park, J. S. Park, and Y. C. Kwun, *Controllabillty for the Semilinear Fuzzy Integro-Differential Equations with Nonlocal conditions, Fuzzy Systems and Knowledge Discovery*, vol. 4223 of *Lecture Notes in Computer Science*, Springer, Berlin, Germany.

[11] A. V. Plotnikov and A. V. Tumbrukaki, "Integrodifferential inclusions with Hukuhara's derivative," *Nelmĭnĭ Kolivannya*, vol. 8, no. 1, pp. 80–88, 2005.

[12] S. Abbasbandy and M. S. Hashemi, "Fuzzy integro-differential equations: formulation and solution using the variational iteration method," *Nonlinear Science Letters A*, vol. 1, no. 4, pp. 413–418, 2010.

[13] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets*, World Scientific, River Edge, NJ, USA, 1994.

[14] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, vol. 144, Academic Press, New York, NY, USA, 1980.

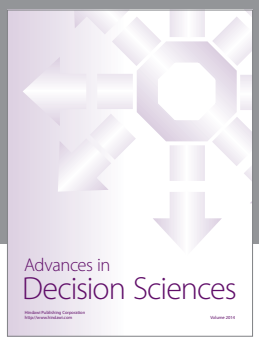
[15] C. M. Tavassoli Kajani, M. Ghasemi, and E. Babolian, "Numerical solution of linear integro-differential equation by using sine-cosine wavelets," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 569–574, 2006.

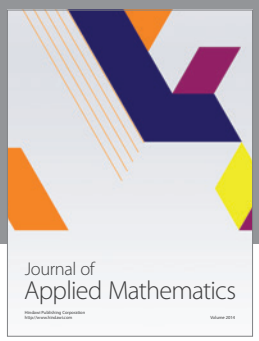
[16] J. Saberi-Nadjafi and M. Tamamgar, "The variational iteration method: a highly promising method for solving the system of integro-differential equations," *Computers & Mathematics with Applications*, vol. 56, no. 2, pp. 346–351, 2008.

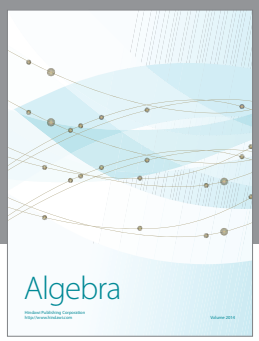
[17] X. Shang and D. Han, "Application of the variational iteration method for solving nth-order integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 234, no. 5, pp. 1442–1447, 2010.

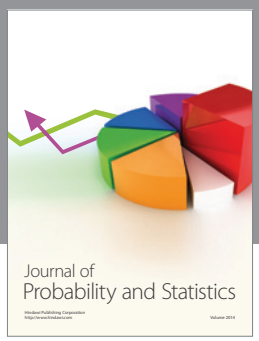
[18] A. A. Hemeda, "Variational iteration method for solving wave equation," *Computers & Mathematics with Applications*, vol. 56, no. 8, pp. 1948–1953, 2008.

[19] A. A. Hemeda, "Variational iteration method for solving non-linear partial differential equations," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1297–1303, 2009.
[20] V. Daftardar-Gejji and H. Jafari, "An iterative method for solving nonlinear functional equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 753–763, 2006.
[21] S. Bhalekar and V. Daftardar-Gejji, "New iterative method: application to partial differential equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 778–783, 2008.
[22] G. Adomain, *Solving Frontier Problems of Physics: the Decomposition Method*, Kluwer, 1994.
[23] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.
[24] A. F. Elsayed, "Comparison between variationaliteration method and homotopy perturbation method for thermal diffusion and diffusion thermo effects of thixotropic fluid through biological tissues with laser radiation existence," *Applied Mathematical Modelling*. In press.
[25] S. Haji Ghasemi, "Fuzzy Fredholm-Volterra integral equations and existance and uniqueness of solution of them," *Australian Journal of Basic and Applied Sciences*, vol. 5, no. 4, pp. 1–8, 2011.
[26] T. Allahviranloo and M. Ghanbari, "Solving fuzzy linear systems by homotopy perturbation method," *International Journal of Computational Cognition*, vol. 8, pp. 27–30, 2010.
[27] H.-K. Liu, "On the solution of fully fuzzy linear systems," *International Journal of Computational and Mathematical Sciences*, vol. 4, no. 1, pp. 29–33, 2010.
[28] B. Bede and S. G. Gal, "Solutions of fuzzy differential equations based on generalized differentiability," *Communications in Mathematical Analysis*, vol. 9, no. 2, pp. 22–41, 2010.
[29] Y. Chalco-Cano and H. Román-Flores, "Comparation between some approaches to solve fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1517–1527, 2009.
[30] Y. Chalco-Cano, H. Román-Flores, and M. D. Jiménez-Gamero, "Generalized derivative and π-derivative for set-valued functions," *Information Sciences. An International Journal*, vol. 181, no. 11, pp. 2177–2188, 2011.
[31] L. Stefanini, "A generalization of Hukuhara difference and division for interval and fuzzy arithmetic," *Fuzzy Sets and Systems*, vol. 161, no. 11, pp. 1564–1584, 2010.
[32] L. Stefanini and B. Bede, "Some notes on generalized Hukuhara differentiability of interval-valued functions and interval differential equations," Working Paper, University of Urbino, 2012.
[33] B. Bede and L. Stefanini, "Generalized differentiability of fuzzy-valued functions," Working Paper, University of Urbino, 2012.
[34] A. A. Hemeda, "New iterative method: application to nth-order integro-differential equations," *International Mathematical Forum*, vol. 7, no. 47, pp. 2317–2332, 2012.
[35] V. Daftardar-Gejji and S. Bhalekar, "Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method," *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1801–1809, 2010.
[36] S. Bhalekar and V. Daftardar-Gejji, "New iterative method: application to partial differential equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 778–783, 2008.
[37] S. Bhalekar and V. Daftardar-Gejji, "Solving evolution equations using a new iterative method," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 4, pp. 906–916, 2010.
[38] H. Jafari, S. Seifi, A. Alipoor, and M. Zabihi, "An iterative method for solving linear and nonlinear fractional diffusion-wave equation," *International e-Journal of Numerical Analysis and Related Topics*, vol. 3, pp. 20–32, 2009.
[39] S. Bhalekar and V. Daftardar-Gejji, "Convergence of the new iterative method," *International Journal of Differential Equations*, vol. 2011, Article ID 989065, 10 pages, 2011.
[40] O. S. Fard and M. Sanchooli, "Two successive schemes for numerical solution of linear fuzzy Fredholm integral equations of the second kind," *Australian Journal of Basic and Applied Sciences*, vol. 4, no. 5, pp. 817–825, 2010.
[41] V. Daftardar-Gejji and S. Bhalekar, "Solving fractional diffusion-wave equations using a new iterative method," *Fractional Calculus & Applied Analysis*, vol. 11, no. 2, pp. 193–202, 2008.









Hindawi

Submit your manuscripts at
http://www.hindawi.com