Research Article
Common Fixed Points for Asymptotic Pointwise Nonexpansive Mappings in Metric and Banach Spaces

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Let $C$ be a nonempty bounded closed convex subset of a complete CAT$(0)$ space $X$. We prove that the common fixed point set of any commuting family of asymptotic pointwise nonexpansive mappings on $C$ is nonempty closed and convex. We also show that, under some suitable conditions, the sequence $\{x_k\}_{k=1}^{\infty}$ defined by

$$x_{k+1} = (1 - t_{m_k})x_k \oplus t_{m_k}T_{n_k}y_{(m-1)_k},$$

where $x_k = (1 - t_{m_k})x_k \oplus t_{m_k}T_{m_k}y_{(m-1)_k}$, $y_{(m-1)_k} = (1 - t_{(m-1)_k})x_{k-1} \oplus t_{(m-1)_k}T_{n_{k-1}}y_{(m-2)_k}$, $y_{(m-2)_k} = (1 - t_{(m-2)_k})x_{k-2} \oplus t_{(m-2)_k}T_{m_{k-2}}y_{(m-3)_k}$, ..., $y_1 = (1 - t_1)x_0 \oplus t_1T_1y_0$, $y_0 = x_0$, $k \in \mathbb{N}$, converges to a common fixed point of $T_1, T_2, ..., T_m$ where they are asymptotic pointwise nonexpansive mappings on $C$, $\{t_{ik}\}_{i=1}^{\infty}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., m$, and $\{n_k\}$ is an increasing sequence of natural numbers. The related results for uniformly convex Banach spaces are also included.

1. Introduction

A mapping $T$ on a subset $C$ of a Banach space $X$ is said to be asymptotic pointwise nonexpansive if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|, \quad (1.1)$$

where $\limsup_{n \to \infty} \alpha_n(x) \leq 1$, for all $x, y \in C$. This class of mappings was introduced by Kirk and Xu [1], where it was shown that if $C$ is a bounded closed convex subset of a uniformly convex Banach space $X$, then every asymptotic pointwise nonexpansive mapping $T : C \to C$ always has a fixed point. In 2009, Hussain and Khamsi [2] extended Kirk-Xu’s result to the case of metric spaces, specifically to the so-called CAT(0) spaces. Recently, Kozlowski [3]...
defined an iterative sequence for an asymptotic pointwise nonexpansive mapping $T : C \rightarrow C$ by $x_1 \in C$ and

$$
x_{k+1} = (1 - t_k)x_k + t_k T^{n_k} y_k, \quad y_k = (1 - s_k)x_k + s_k T^{m_k} x_k, \quad k \in \mathbb{N},
$$

(1.2)

where $\{t_k\}$ and $\{s_k\}$ are sequences in $[0, 1]$ and $\{n_k\}$ is an increasing sequence of natural numbers. He proved, under some suitable assumptions, that the sequence $\{x_k\}$ defined by (1.2) converges weakly to a fixed point of $T$ where $X$ is a uniformly convex Banach space which satisfies the Opial condition and $\{x_k\}$ converges strongly to a fixed point of $T$ provided $T^r$ is a compact mapping for some $r \in \mathbb{N}$. On the other hand, Khan et al. [4] studied the iterative process defined by

$$
x_{n+1} = (1 - \alpha_m) x_n + \alpha_m T^n_{m} y_{m(1)n}, \\
y_{m(1)n} = (1 - \alpha_{m(1)n}) x_n + \alpha_{m(1)n} T^n_{m-1} y_{m(2)n}, \\
y_{m(2)n} = (1 - \alpha_{m(2)n}) x_n + \alpha_{m(2)n} T^n_{m-2} y_{m(3)n}, \\
\vdots \\
y_{2n} = (1 - \alpha_{2n}) x_n + \alpha_{2n} T^n_{2} y_{1n}, \\
y_{1n} = (1 - \alpha_{1n}) x_n + \alpha_{1n} T^n_{1} y_{0n}, \\
y_{0n} = x_n, \quad n \in \mathbb{N},
$$

(1.3)

where $T_1, \ldots, T_m$ are asymptotically quasi-nonexpansive mappings on $C$ and $\{\alpha_{m}\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ for all $i = 1, 2, \ldots, m$.

In this paper, motivated by the results mentioned above, we ensure the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings in a CAT(0) space. Furthermore, we obtain $\Delta$ and strong convergence theorems of a sequence defined by

$$
x_{k+1} = (1 - t_{mk}) x_k \oplus t_{mk} T_{m}^{n_k} y_{m(1)k}, \\
y_{m(1)k} = (1 - t_{m(1)k}) x_k \oplus t_{m(1)k} T_{m-1}^{n_k} y_{m(2)k}, \\
y_{m(2)k} = (1 - t_{m(2)k}) x_k \oplus t_{m(2)k} T_{m-2}^{n_k} y_{m(3)k}, \\
\vdots \\
y_{2k} = (1 - t_{2k}) x_k \oplus t_{2k} T_{2}^{n_k} y_{1k}, \\
y_{1k} = (1 - t_{1k}) x_k \oplus t_{1k} T_{1}^{n_k} y_{0k}, \\
y_{0k} = x_k, \quad k \in \mathbb{N},
$$

(1.4)

where $T_1, \ldots, T_m$ are asymptotic pointwise nonexpansive mappings on a subset $C$ of a complete CAT(0) space and $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0, 1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers. We also note that our method can be used to prove the analogous results for uniformly convex Banach spaces.
2. Preliminaries

A metric space $X$ is a CAT(0) space if it is geodesically connected and if every geodesic triangle in $X$ is at least as “thin” as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [5]), $\mathbb{R}$-trees (see [6]), Euclidean buildings (see [7]), and the complex Hilbert ball with a hyperbolic metric (see [8]). For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [9, 10]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2, 11–22] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology, and computer science (see, e.g., [6, 23–26]).

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space $(X, d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\Delta$ be a geodesic triangle in $X$, and let $\overline{\Delta}$ be a comparison triangle for $\Delta$. Then, $\Delta$ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, by Lemma 2.1(iv) of [14] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{2.2}$$

We will use the notation $(1 - t)x \oplus ty$ for the unique point $z$ satisfying (2.2). We now collect some elementary facts about CAT(0) spaces.
Lemma 2.1. Let $X$ be a complete CAT(0) space.

(i) [5, Proposition 2.4] If $C$ is a nonempty closed convex subset of $X$, then, for every $x \in X$, there exists a unique point $P(x) \in C$ such that $d(x, P(x)) = \inf \{d(x, y) : y \in C\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from $X$ onto $C$.

(ii) [14, Lemma 2.4] For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z). \quad (2.3)$$

(iii) [14, Lemma 2.5] For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2. \quad (2.4)$$

We now give the concept of $\Delta$-convergence and collect some of its basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \quad (2.5)$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}, \quad (2.6)$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{ x \in X : r(x, \{x_n\}) = r(\{x_n\}) \}. \quad (2.7)$$

It is known from Proposition 7 of [27] that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.2 (see [28, 29]). A sequence $\{x_n\}$ in a CAT(0) space $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n} x = x$ and call $x$ the $\Delta$-limit of $\{x_n\}$.

Lemma 2.3. Let $X$ be a complete CAT(0) space.

(i) [28, page 3690] Every bounded sequence in $X$ has a $\Delta$-convergent subsequence.

(ii) [30, Proposition 2.1] If $C$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.

(iii) [14, Lemma 2.8] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Recall that a mapping $T : X \to X$ is said to be nonexpansive [31] if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X, \quad (2.8)$$
where $T$ is called \textit{asymptotically nonexpansive} \cite{32} if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n \to \infty} k_n = 1$ and such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall n \geq 1, \; x, y \in X,$$

where $T$ is called an \textit{asymptotic pointwise nonexpansive mapping} \cite{1} if there exists a sequence of functions $\alpha_n : X \to [0, \infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x) d(x, y), \quad \forall n \geq 1, \; x, y \in X,$$

where $\limsup_{n \to \infty} \alpha_n(x) \leq 1$. The following implications hold.

$$T \text{ is nonexpansive} \Rightarrow T \text{ is asymptotically nonexpansive} \Rightarrow T \text{ is asymptotic pointwise nonexpansive}.$$  \hfill (2.11)

A point $x \in X$ is called a fixed point of $T$ if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of $T$. The existence of fixed points for asymptotic pointwise nonexpansive mappings in CAT(0) spaces was proved by Hussain and Khamsi \cite{2} as the following result.

\textbf{Theorem 2.4.} Let $C$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. Suppose that $T : C \to C$ is an asymptotic pointwise nonexpansive mapping. Then, $F(T)$ is nonempty closed and convex.

\section{Existence Theorems}

Let $M$ be a metric space and $\mathcal{F}$ a family of subsets of $M$. Then, we say that $\mathcal{F}$ defines a \textit{convexity structure} on $M$ if it contains the closed balls and is stable by intersection.

\textbf{Definition 3.1 (see \cite{2}).} Let $\mathcal{F}$ be a convexity structure on $M$. We will say that $\mathcal{F}$ is \textit{compact} if any family $\{A_\alpha\}_{\alpha \in \Gamma}$ of elements of $\mathcal{F}$ has a nonempty intersection provided $\bigcap_{\alpha \in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.

Let $X$ be a complete CAT(0) space. We denote by $C(X)$ the family of all closed convex subsets of $X$. Then, $C(X)$ is a compact convexity structure on $X$ (see, e.g., \cite{2}).

The following theorem is an extension of Theorem 4.3 in \cite{33}. For an analog of this result in uniformly convex Banach spaces, see \cite{34}.

\textbf{Theorem 3.2.} Let $C$ be a nonempty bounded closed and convex subset of a complete CAT(0) space $X$. Then, for any commuting family $S$ of asymptotic pointwise nonexpansive mappings on $C$, the set $\mathcal{F}(S)$ of common fixed points of $S$ is nonempty closed and convex.

\textbf{Proof.} Let $\mathcal{T}$ be the family of all finite intersections of the fixed point sets of mappings in the commutative family $S$. We first show that $\mathcal{T}$ has the finite intersection property. Let $T_1, T_2, \ldots, T_n \in S$. By Theorem 2.4, $F(T_i)$ is a nonempty closed and convex subset of $C$. We
assume that \(A := \bigcap_{j=1}^{k-1} F(T_j)\) is nonempty closed and convex for some \(k \in \mathbb{N}\) with \(1 < k \leq n\). For \(x \in A\) and \(j \in \mathbb{N}\) with \(1 \leq j < k\), we have
\[
T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x). \tag{3.1}
\]
Thus, \(T_k(x)\) is a fixed point of \(T_j\), which implies that \(T_k(x) \in A\); therefore, \(A\) is invariant under \(T_k\). Again, by Theorem 2.4, \(T_k\) has a fixed point in \(A\), that is,
\[
\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \neq \emptyset. \tag{3.2}
\]
By induction, \(\bigcap_{j=1}^{n} F(T_j) \neq \emptyset\). Hence, \(\mathcal{T}\) has the finite intersection property. Since \(C(X)\) is compact,
\[
\mathcal{F}(S) = \bigcap_{T \in \mathcal{C}} T \neq \emptyset. \tag{3.3}
\]
Obviously, the set is closed and convex.

As a consequence of Lemma 2.1(i) and Theorem 3.2, we obtain an analog of Bruck’s theorem [35].

**Corollary 3.3.** Let \(C\) be a nonempty bounded closed and convex subset of a complete CAT(0) space \(X\). Then, for any commuting family \(S\) of nonexpansive mappings on \(C\), the set \(\mathcal{F}(S)\) of common fixed points of \(S\) is a nonempty nonexpansive retract of \(C\).

### 4. Convergence Theorems

Throughout this section, \(X\) stands for a complete CAT(0) space. Let \(C\) be a closed convex subset of \(X\). We shall denote by \(\mathcal{T}(C)\) the class of all asymptotic pointwise nonexpansive mappings from \(C\) into \(C\). Let \(T_1, \ldots, T_m \in \mathcal{T}(C)\), without loss of generality, we can assume that there exists a sequence of mappings \(\alpha_n : C \rightarrow [0, \infty)\) such that for all \(x, y \in C\), \(i = 1, \ldots, m\), and \(n \in \mathbb{N}\), we have
\[
d(T^n_i x, T^n_i y) \leq \alpha_n(x) d(x, y), \quad \limsup_{n \to \infty} \alpha_n(x) \leq 1. \tag{4.1}
\]
Let \(a_n(x) = \max\{\alpha_n(x), 1\}\). Again, without loss of generality, we can assume that
\[
d(T^n_i x, T^n_i y) \leq a_n(x) d(x, y), \quad \lim_{n \to \infty} a_n(x) = 1, \quad a_n(x) \geq 1, \tag{4.2}
\]
for all \(x, y \in C\), \(i = 1, \ldots, m\), and \(n \in \mathbb{N}\). We define \(b_n(x) = a_n(x) - 1\), then, for each \(x \in C\), we have \(\lim_{n \to \infty} b_n(x) = 0\).
The following definition is a mild modification of [3, Definition 2.3].

**Definition 4.1.** Define $\mathcal{T}_r(C)$ as a class of all $T \in \mathcal{T}(C)$ such that

$$\sum_{n=1}^{\infty} \sup_{x \in C} b_n(x) < \infty,$$

(4.3)

$a_n$ is a bounded function for every $n \in \mathbb{N}$.

Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$, and let $\left\{ t_{ik} \right\}_{k=1}^{\infty} \subset (0, 1)$ be bounded away from 0 and 1 for all $i = 1, 2, \ldots, m$, and $\{ n_k \}$ an increasing sequence of natural numbers. Let $x_1 \in C$, and define a sequence $\{ x_k \}$ in $C$ as

$$x_{k+1} = (1 - t_{mk}) x_k \oplus t_{mk} T_m^{n_k} y_{m-1}k,$$

$$y_{m-1}k = (1 - t_{(m-1)k}) x_k \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{m-2}k,$$

$$y_{m-2}k = (1 - t_{(m-2)k}) x_k \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{m-3}k,$$

$$\vdots$$

$$y_{2k} = (1 - t_{2k}) x_k \oplus t_{2k} T_2^{n_k} y_{1k},$$

$$y_{1k} = (1 - t_{1k}) x_k \oplus t_{1k} T_1^{n_k} y_{0k},$$

$$y_{0k} = x_k, \quad k \in \mathbb{N}.$$

We say that the sequence $\{ x_k \}$ in (4.4) is well defined if $\limsup_{k \to \infty} a_{n_k}(x_k) = 1$. As in [3], we observe that $\lim_{k \to \infty} a_k(x) = 1$ for every $x \in C$. Hence, we can always choose a subsequence $\{ a_{n_k} \}$ which makes $\{ x_k \}$ well defined.

**Lemma 4.2** (see [36, Lemma 2.2]). Let $\{ a_n \}$ and $\{ u_n \}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + u_n) a_n, \quad \forall n \in \mathbb{N}, \quad \sum_{n=1}^{\infty} u_n < \infty.$$

(4.5)

Then, (i) $\lim_{n} a_n$ exists, (ii) if $\liminf_{n} a_n = 0$, then $\lim_{n} a_n = 0$.

**Lemma 4.3** (see [37, 38]). Suppose $\{ t_n \}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{ u_n \}, \{ v_n \}$ are sequences in $X$ such that $\limsup_{n} d(\{ u_n \}, \{ v_n \}) \leq r$, $\limsup_{n} d(\{ v_n \}, \{ w \}) \leq r$, and $\lim_{n} d((1 - t_n) u_n \oplus t_n v_n, \{ w \}) = r$ for some $r \geq 0$. Then,

$$\lim_{n \to \infty} d(\{ u_n \}, \{ v_n \}) = 0.$$

(4.6)

**Lemma 4.4.** Let $C$ be a nonempty closed convex subset of $X$ and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$. Let $\left\{ t_{ik} \right\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$ and $\{ n_k \} \subset \mathbb{N}$ be such that $\{ x_k \}$ in (4.4) is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then,

(a) there exists a sequence $\{ u_k \}$ in $[0, \infty)$ such that $\sum_{k=1}^{\infty} u_k < \infty$ and $d(\{ x_{k+1} \}, \{ p \}) \leq (1 + u_k)^m d(\{ x_k \}, \{ p \})$, for all $p \in F$ and all $k \in \mathbb{N}$,
(b) there exists a constant $M > 0$ such that $d(x_{k+l}, p) \leq Md(x_k, p)$, for all $p \in F$ and $k, l \in \mathbb{N}$.

Proof. (a) Let $p \in F$ and $v_k = \sup_{x \in C} b_n(x)$ for all $k \in \mathbb{N}$. Since \( \sum_{k=1}^{\infty} \sup_{x \in C} b_n(x) < \infty \), we have \( \sum_{k=1}^{\infty} v_k < \infty \). Now,

\[
d(y_{jk}, p) \leq (1 - t_k) d(x_k, p) + t_k d(T^n_{j, k} x_k, p) \\
\leq (1 - t_k) d(x_k, p) + t_k (1 + b_k(p)) d(x_k, p) \\
= (1 + t_k b_k(p)) d(x_k, p) \\
\leq (1 + v_k) d(x_k, p).
\]

Suppose that $d(y_{jk}, p) \leq (1 + v_k)^i d(x_k, p)$ holds for some $1 \leq j \leq m - 2$. Then,

\[
d(y_{(j+1)k}, p) \leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k} d(T^n_{j+1, k} y_{jk}, p) \\
\leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k} (1 + b_k(p)) d(y_{jk}, p) \\
\leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k} (1 + v_k)^{i+1} d(x_k, p) \\
= [1 - t_{(j+1)k} + t_{(j+1)k} \left( 1 + \sum_{r=1}^{i+1} \frac{(j+1) \cdots (j+2-r)}{r!} v_k^r \right)] d(x_k, p) \\
= [1 + t_{(j+1)k} \sum_{r=1}^{i+1} \frac{(j+1) \cdots (j+2-r)}{r!} v_k^r] d(x_k, p) \\
\leq (1 + v_k)^{i+1} d(x_k, p).
\]

By induction, we have

\[
d(y_{ik}, p) \leq (1 + v_k)^i d(x_k, p), \quad \forall i = 1, 2, \ldots, m - 1.
\]

This implies

\[
d(x_{k+1}, p) \leq (1 - t_{mk}) d(x_k, p) + t_{mk} d(T^n_{m, k} y_{(m-1)k}, p) \\
\leq (1 - t_{mk}) d(x_k, p) + t_{mk} (1 + b_k(p)) d(y_{(m-1)k}, p) \\
\leq (1 - t_{mk}) d(x_k, p) + t_{mk} (1 + v_k)^{m-1} d(x_k, p) \\
\leq (1 - t_{mk}) d(x_k, p) + t_{mk} (1 + v_k)^{m-1} d(x_k, p) \\
= \left[ 1 - t_{mk} + t_{mk} \left( 1 + \sum_{r=1}^{m} \frac{m(m-1) \cdots (m-r+1)}{r!} v_k^r \right) \right] d(x_k, p) \\
= \left[ 1 + t_{mk} \sum_{r=1}^{m} \frac{m(m-1) \cdots (m-r+1)}{r!} v_k^r \right] d(x_k, p) \\
\leq (1 + v_k)^m d(x_k, p).
\]

This completes the proof of (a).
(b) We observe that \( (1 + \alpha)^n \leq e^{n\alpha} \) holds for all \( n \in \mathbb{N} \) and \( \alpha \geq 0 \). Thus, by (a), for \( k,l \in \mathbb{N} \), we have

\[
d(x_{k+l}, p) \leq (1 + v_{k+l-1})^m d(x_{k+l-1}, p)
\leq \exp\left\{ m v_{k+l-1} \right\} d(x_{k+l-1}, p) \leq \cdots \leq \exp\left( m \sum_{i=1}^{\infty} v_i \right) d(x_k, p)
\leq \exp\left( m \sum_{i=1}^{\infty} v_i \right) d(x_k, p).
\]

(4.11)

The proof is complete by setting \( M = \exp\left( m \sum_{i=1}^{\infty} v_i \right) \). \( \square \)

**Theorem 4.5.** Let \( C \) be a nonempty closed convex subset of \( X \) and \( T_1, \ldots, T_m \in T_r(C) \). Let \( \{ t_{ik} \}_{k=1}^{\infty} \subset [a, b] \subset (0, 1) \) and \( \{ n_k \} \subset \mathbb{N} \) be such that \( \{ x_k \} \) in (4.4) is well defined. Assume that \( F \neq \emptyset \). Then, \( \{ x_k \} \) converges to some point in \( F \) if and only if \( \liminf_{k \to \infty} d(x_k, F) = 0 \), where \( d(x, F) = \inf_{p \in F} d(x, p) \).

**Proof.** The necessity is obvious. Now, we prove the sufficiency. From Lemma 4.4(a), we have

\[
d(x_{k+1}, p) \leq (1 + v_k)^m d(x_k, p), \quad \forall p \in F, \forall k \in \mathbb{N}.
\]

(4.12)

This implies

\[
d(x_{k+1}, F) \leq (1 + v_k)^m d(x_k, F) = \left( 1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r \right) d(x_k, F).
\]

(4.13)

Since \( \sum_{k=1}^{\infty} v_k < \infty \), then \( \sum_{k=1}^{\infty} \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r < \infty \). By Lemma 4.2(ii), we get \( \lim_{k \to \infty} d(x_k, F) = 0 \). Next, we show that \( \{ x_k \} \) is a Cauchy sequence. From Lemma 4.4(b), there exists \( M > 0 \) such that

\[
d(x_{k+l}, p) \leq Md(x_k, p), \quad \forall p \in F, \ k,l \in \mathbb{N}.
\]

(4.14)

Since \( \lim_{k \to \infty} d(x_k, F) = 0 \), for each \( \varepsilon > 0 \), there exists \( k_1 \in \mathbb{N} \) such that

\[
d(x_k, F) < \frac{\varepsilon}{2M}, \quad \forall k \geq k_1.
\]

(4.15)

Hence, there exists \( z_1 \in F \) such that

\[
d(x_k, z_1) < \frac{\varepsilon}{2M}.
\]

(4.16)
By (4.14) and (4.16), for \( k \geq k_1 \), we have
\[
d(x_{k,i}, x_k) \leq d(x_{k,i}, z_1) + d(x_k, z_1) \\
\leq Md(x_k, z_1) + Md(x_{k,i}, z_1) \\
< 2M\left(\frac{\varepsilon}{2M}\right) \\
= \varepsilon.
\] (4.17)

This shows that \( \{x_k\} \) is a Cauchy sequence and so converges to some \( q \in C \). We next show that \( q \in F \). Let \( L = \sup \{a_1(x) : x \in C\} \). Then, for each \( \varepsilon > 0 \), there exists \( k_2 \in \mathbb{N} \) such that
\[
d(x_k, q) < \frac{\varepsilon}{2(1 + L)}, \quad \forall k \geq k_2.
\] (4.18)

Since \( \lim_{k \to \infty} d(x_k, F) = 0 \), there exists \( k_3 \geq k_2 \) such that
\[
d(x_k, F) < \frac{\varepsilon}{2(1 + L)}, \quad \forall k \geq k_3.
\] (4.19)

Thus, there exists \( z_2 \in F \) such that
\[
d(x_k, z_2) < \frac{\varepsilon}{2(1 + L)}.
\] (4.20)

By (4.18) and (4.20), for each \( i = 1, 2, \ldots, m \), we have
\[
d(T_iq, q) \leq d(T_iq, T_ix_k) + d(T_ix_k, z_2) + d(z_2, x_k) + d(x_k, q) \\
\leq Ld(x_k, q) + Ld(x_k, z_2) + d(x_k, z_2) + d(x_k, q) \\
\leq (1 + L)d(x_k, q) + (1 + L)d(x_k, z_2) \\
< (1 + L)\frac{\varepsilon}{2(1 + L)} + (1 + L)\frac{\varepsilon}{2(1 + L)} \\
= \varepsilon.
\] (4.21)

Since \( \varepsilon \) is arbitrary, we have \( T_iq = q \) for all \( i = 1, 2, \ldots, m \). Hence, \( q \in F \). \( \square \)

As an immediate consequence of Theorem 4.5, we obtain the following.

**Corollary 4.6.** Let \( C \) be a nonempty closed convex subset of \( X \) and \( T_1, \ldots, T_m \in \mathcal{T}_r(C) \). Let \( \{t_k\}_{k=3}^{\infty} \subset [a, b] \subset (0,1) \) and \( \{n_k\} \subset \mathbb{N} \) be such that \( x_k \) in (4.4) is well defined. Assume that \( F \neq \emptyset \). Then, \( \{x_k\} \) converges to a point \( p \in F \) if and only if there exists a subsequence \( \{x_{k_i}\} \) of \( \{x_k\} \) which converges to \( p \).

**Definition 4.7.** A strictly increasing sequence \( \{n_k\} \subset \mathbb{N} \) is called quasiperiodic \([39]\) if the sequence \( \{n_{k+1} - n_k\} \) is bounded or equivalently if there exists a number \( p \in \mathbb{N} \) such that any block of \( p \) consecutive natural numbers must contain a term of the sequence \( \{n_k\} \). The smallest of such numbers \( p \) will be called a quasiperiod of \( \{n_k\} \).
Lemma 4.8. Let $C$ be a nonempty closed convex subset of $X$ and $T_1, \ldots, T_m \in \mathcal{T}(C)$. Let $\{t_{jk}\}_{k=1}^\infty \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Then,

(i) $\lim_{k \to \infty} d(x_k, p)$ exists for all $p \in F$;
(ii) $\lim_{k \to \infty} d(x_k, T_j^{n_k} y_{(j-1)k}) = 0$, for all $j = 1, 2, \ldots, m$,
(iii) if the set $\mathcal{J} = \{ k \in \mathbb{N} : n_k+1 = 1 + n_k \}$ is quasiperiodic, then $\lim_{k \to \infty} d(x_k, T_j x_k) = 0$, for all $j = 1, 2, \ldots, m$.

Proof. (i) Follows from Lemmas 4.2(i) and 4.4(a).
(ii) Let $p \in F$, then, by (i), we have $\lim_{k \to \infty} d(x_k, p)$ exists. Let

$$
\lim_{k \to \infty} d(x_k, p) = c.
$$

By (4.9) and (4.22), we get that

$$
\lim_{k \to \infty} \sup d(y_{jk}, p) \leq c, \quad \text{for } 1 \leq j \leq m - 1.
$$

Note that

$$
d(x_{k+1}, p) \leq (1 - t_{mk})d(x_k, p) + t_{mk}d(T_m^{n_k} y_{(m-1)k}, p)
\leq (1 - t_{mk})d(x_k, p) + t_{mk}(1 + v_k)d(y_{(m-1)k}, p)
\vdots
\leq (1 - t_{mk}t_{(m-1)k} \cdots t_{(j+1)k})(1 + v_k)^{m-j}d(x_k, p)
\quad + t_{mk}t_{(m-1)k} \cdots t_{(j+1)k}(1 + v_k)^{m-j}d(y_{jk}, p).
$$

Thus,

$$
d(x_k, p) \leq \frac{d(x_k, p)}{\delta^{m-j}} - \frac{d(x_{k+1}, p)}{\delta^{m-j}(1 + v_k)^{m-j}} + d(y_{jk}, p),
$$

so that

$$
c \leq \lim_{k \to \infty} \inf d(y_{jk}, p), \quad \text{for } 1 \leq j \leq m - 1.
$$

From (4.23) and (4.26), we have

$$
\lim_{k \to \infty} d(y_{jk}, p) = c, \quad \text{for each } j = 1, 2, \ldots, m - 1.
$$

That is

$$
\lim_{k \to \infty} d( (1 - t_{jk})x_k \oplus t_{jk} T_j^{n_k} y_{(j-1)k}, p) = c,
$$

for each $j = 1, 2, \ldots, m - 1$. 

We also obtain from (4.23) that
\[
\limsup_{k \to \infty} d\left(T_{j}^{m_{k}}y_{(j-1)k}, p\right) \leq c, \quad \text{for each } j = 1, 2, \ldots, m - 1. \tag{4.29}
\]

By Lemma 4.3, we get that
\[
\lim_{k \to \infty} d\left(T_{j}^{m_{k}}y_{(j-1)k}, x_{k}\right) = 0, \quad \text{for each } j = 1, 2, \ldots, m - 1. \tag{4.30}
\]

For the case \(j = m\), by (4.1), we have
\[
d(T_{m}^{m_{k}}y_{(m-1)k}, p) \leq (1 + b_{m_{k}}(p))d(y_{(m-1)k}, p) \leq (1 + b_{m_{k}}(p))(1 + v_{m})^{-1}d(x_{k}, p). \tag{4.31}
\]

But since \(\lim_{k \to \infty} d(x_{k}, p) = c\), then
\[
\limsup_{k \to \infty} d(T_{m}^{m_{k}}y_{(m-1)k}, p) \leq c. \tag{4.32}
\]

Moreover,
\[
\lim_{k \to \infty} d((1 - t_{m_{k}})x_{k} \oplus t_{m_{k}}T_{m}^{m_{k}}y_{(m-1)k}, p) = \lim_{k \to \infty} d(x_{k+1}, p) = c. \tag{4.33}
\]

Again, by Lemma 4.3, we get that
\[
\lim_{k \to \infty} d(T_{m}^{m_{k}}y_{(m-1)k}, x_{k}) = 0. \tag{4.34}
\]

Thus, (4.30) and (4.34) imply that
\[
\lim_{k \to \infty} d\left(T_{j}^{m_{k}}y_{(j-1)k}, x_{k}\right) = 0, \quad \text{for each } j = 1, 2, \ldots, m. \tag{4.35}
\]

(iii) For \(j = 1\), from (ii), we have
\[
\lim_{k \to \infty} d\left(T_{1}^{m_{k}}x_{k}, x_{k}\right) = 0. \tag{4.36}
\]

If \(j = 2, 3, \ldots, m\), then we have
\[
d\left(T_{j}^{m_{k}}x_{k}, x_{k}\right) \leq d\left(T_{j}^{m_{k}}x_{k}, T_{j}^{m_{k}}y_{(j-1)k}\right) + d\left(T_{j}^{m_{k}}y_{(j-1)k}, x_{k}\right)
\leq a_{m_{k}}(x_{k})d\left(x_{k}, y_{(j-1)k}\right) + d\left(T_{j}^{m_{k}}y_{(j-1)k}, x_{k}\right)
\leq a_{m_{k}}(x_{k})t_{(j-1)k}d\left(x_{k}, T_{j}^{m_{k}}y_{(j-2)k}\right) + d\left(T_{j}^{m_{k}}y_{(j-1)k}, x_{k}\right). \tag{4.37}
\]
By (ii) and \( \limsup_{k \to \infty} a_{n_k}(x_k) = 1 \), we get
\[
\limsup_{k \to \infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \text{for } j = 2, 3, \ldots, m. \tag{4.38}
\]

By (4.36) and (4.38), we have
\[
\lim_{k \to \infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.39}
\]

By the construction of the sequence \( \{x_k\} \), we have from (4.35) that
\[
\lim_{k \to \infty} d(x_{k+1}, x_k) = 0. \tag{4.40}
\]

Next, we show that
\[
\lim_{k \to \infty} d(T_j x_k, x_k) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.41}
\]

It is enough to prove that \( d(T_j x_k, x_k) \to 0 \) as \( k \to \infty \) though \( \mathcal{J} \). Indeed, let \( p \) be a quasiperiod of \( \mathcal{J} \), and let \( \varepsilon > 0 \) be given. Then, there exists \( N_1 \in \mathbb{N} \) such that
\[
\lim_{k \to \infty} d(T_j x_k, x_k) < \frac{\varepsilon}{3}, \quad \forall k \in \mathcal{J} \text{ such that } k \geq N_1. \tag{4.42}
\]

By the quasiperiodicity of \( \mathcal{J} \), for each \( l \in \mathbb{N} \), there exists \( i_l \in \mathcal{J} \) such that \( |l - i_l| \leq p \). Without loss of generality, we can assume that \( 1 \leq i_l \leq l + p \) (the proof for the other case is identical). Let \( M = \sup\{a_1(x) : x \in C\} \). Then, \( M \geq 1 \). Since \( \lim_{l \to \infty} d(x_{i_l+1}, x_{i_l}) = 0 \) by (4.40), there exists \( N_2 \in \mathbb{N} \) such that
\[
d(x_{i_l+1}, x_{i_l}) < \frac{\varepsilon}{3pM}, \quad \forall l \geq N_2. \tag{4.43}
\]

This implies that
\[
d(x_{i_l}, x_{i_l}) \leq d(x_{i_l}, x_{i_l-1}) + \cdots + d(x_{i_l+1}, x_{i_l}) \leq p\left(\frac{\varepsilon}{3pM}\right) = \frac{\varepsilon}{3M}. \tag{4.44}
\]

By the definition of \( T \), we have
\[
d(T_j x_{i_l}, T_j x_{i_l}) \leq M d(x_{i_l}, x_{i_l}) \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{4.45}
\]

Let \( N = \max\{N_1, N_2\} \). Then, for \( l \geq N \), we have from (4.42), (4.44), and (4.45) that
\[
d(x_l, T_j x_l) \leq d(x_l, x_{i_l}) + d(x_{i_l}, T_j x_{i_l}) + d(T_j x_{i_l}, T_j x_l) < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon. \tag{4.46}
\]
To prove that \( d(T_n x_k, x_k) \to 0 \) as \( k \to \infty \) though \( \mathcal{J} \). Since \( \mathcal{J} = \{ k \in \mathbb{N} : n_{k+1} = n_k + 1 \} \) is quasiperiodic, for each \( k \in \mathcal{J} \), we have

\[
d(x_k, T_j x_k) \leq d(x_k, x_{k+1}) + d(x_{k+1}, T_j^n x_{k+1}) + d(T_j^n x_{k+1}, x_{k+1}) + d(T_j^n x_{k+1}, x_k)
\]

\[
\leq d(x_k, x_{k+1}) + d(x_{k+1}, T_j^n x_{k+1}) + a_{n_k}(x_{k+1})d(x_{k+1}, x_k) + a_1(x_k)d(T_j^n x_{k+1}, x_k).
\]

(4.47)

From this, together with (4.39) and (4.40), we can obtain that \( d(T_j x_k, x_k) \to 0 \) as \( k \to \infty \) through \( \mathcal{J} \).

The following lemma can be found in [3] (see also [2]).

**Lemma 4.9.** Let \( C \) be a nonempty closed convex subset of \( X \), and let \( T \in \mathcal{T}_r(C) \). If \( \lim_{n \to \infty} d(x_n, T x_n) = 0 \), then \( \lim_{n \to \infty} d(x_n, T^l x_n) = 0 \) for every \( l \in \mathbb{N} \).

**Lemma 4.10.** Let \( C \) be a nonempty closed convex subset of \( X \), and let \( T \in \mathcal{T}_r(C) \). Suppose \( \{x_n\} \) is a bounded sequence in \( C \) such that \( \lim_n d(x_n, T x_n) = 0 \) and \( \Delta \)-lim \( x_n = \omega \). Then, \( T \omega = \omega \).

By using Lemmas 2.3 and 4.10, we can obtain the following result. We omit the proof because it is similar to the one given in [38].

**Lemma 4.11.** Let \( C \) be a closed convex subset of \( X \), and let \( T : C \to C \) be an asymptotic pointwise nonexpansive mapping. Suppose \( \{x_n\} \) is a bounded sequence in \( C \) such that \( \lim_n d(x_n, T x_n) = 0 \) and \( d(x_n, \nu) \) converges for each \( \nu \in F(T) \), then \( \omega_T(x_n) \subset F(T) \). Here, \( \omega_T(x_n) = \bigcup A(\{u_n\}) \), where the union is taken over all subsequences \( \{u_n\} \) of \( \{x_n\} \). Moreover, \( \omega_T(x_n) \) consists of exactly one point.

Now, we are ready to prove our \( \Delta \)-convergence theorem.

**Theorem 4.12.** Let \( C \) be a nonempty closed convex subset of \( X \) and \( T_1, \ldots, T_m \in \mathcal{T}_r(C) \). Let \( \{t_{ik}\} \in [0, 1 - \delta] \) for some \( \delta \in (0, 1/2) \) and \( \{n_k\} \subset \mathbb{N} \) be such that \( \{x_k\} \) in (4.4) is well defined. Suppose that \( F := \bigcap_{i=1}^m F(T_i) \neq \emptyset \) and the set \( \mathcal{J} = \{ k \in \mathbb{N} : n_{k+1} = 1 + n_k \} \) is quasiperiodic. Then, \( \{x_k\} \Delta \)-converges to a common fixed point of the family \( \{T_i : i = 1, 2, \ldots, m\} \).

**Proof.** Let \( p \in F \), by Lemma 4.8, \( \lim_{k \to \infty} d(x_k, p) \) exists and hence \( \{x_k\} \) is bounded. Since \( \lim_{k \to \infty} d(x_k, T_j x_k) = 0 \) for all \( j = 1, 2, \ldots, m \), then by Lemma 4.11 \( \omega_T(x_k) \subset F(T_i) \) for all \( j = 1, 2, \ldots, m \), and hence \( \omega_T(x_k) \subset \bigcap_{i=1}^m F(T_i) = F \). Since \( \omega_T(x_n) \) consists of exactly one point, then \( \{x_k\} \Delta \)-converges to an element of \( F \).

Before proving our strong convergence theorem, we recall that a mapping \( T : C \to C \) is said to be semicompact if \( C \) is closed and, for any bounded sequence \( \{x_n\} \) in \( C \) with \( \lim_{n \to \infty} d(x_n, T x_n) = 0 \), there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) and \( x \in C \) such that \( \lim_{k \to \infty} x_{n_k} = x \).

**Theorem 4.13.** Let \( C \) be a nonempty closed convex subset of \( X \) and \( T_1, \ldots, T_m \in \mathcal{T}_r(C) \) such that \( T_i \) is semicompact for some \( i \in \{1, \ldots, m\} \) and \( l \in \mathbb{N} \). Let \( \{t_{ik}\}_{k=1}^\infty \subset [0, 1 - \delta] \) for some \( \delta \in (0, 1/2) \) and \( \{n_k\} \subset \mathbb{N} \) be such that \( \{x_k\} \) in (4.4) is well defined. Suppose that \( F := \bigcap_{i=1}^m F(T_i) \neq \emptyset \) and the set \( \mathcal{J} = \{ k \in \mathbb{N} : n_{k+1} = 1 + n_k \} \) is quasiperiodic. Then, \( \{x_k\} \) converges to a common fixed point of the family \( \{T_i : i = 1, 2, \ldots, m\} \).
Concluding Remarks

By Lemma 4.8, we have

$$\lim_{k \to \infty} d(x_k, T_i x_k) = 0, \quad \text{for } i = 1, \ldots, m.$$  \hfill (4.48)

Let $i \in \{1, \ldots, m\}$ be such that $T^i_1$ is semicompact. Thus, by Lemma 4.9,

$$\lim_{k \to \infty} d(x_k, T^i_1 x_k) = 0.$$  \hfill (4.49)

We can also find a subsequence $\{x_{n_j}\}$ of $\{x_k\}$ such that $\lim_{j \to \infty} x_{n_j} = q \in C$. Hence, from (4.48), we have

$$d(q, T_i q) = \lim_{j \to \infty} d(x_{n_j}, T_i x_{n_j}) = 0, \quad \forall i = 1, \ldots, m.$$  \hfill (4.50)

Thus, $q \in F$, and, by Corollary 4.6, $\{x_k\}$ converges to $q$. This completes the proof. \hfill \Box

5. Concluding Remarks

One may observe that our method can be used to obtain the analogous results for uniformly convex Banach spaces. Let $C$ be a nonempty closed convex subset of a Banach space $X$ and fix $x_1 \in C$. Define a sequence $\{x_k\}$ in $C$ as

$$x_{k+1} = (1 - t_{mk}) x_k + t_{mk} T^{{n_k}_k} y_{(m-1)k},$$

$$y_{(m-1)k} = (1 - t_{(m-1)k}) x_k + t_{(m-1)k} T^{{n_k}_k} y_{(m-2)k},$$

$$y_{(m-2)k} = (1 - t_{(m-2)k}) x_k + t_{(m-2)k} T^{{n_k}_k} y_{(m-3)k},$$

$$\vdots$$

$$y_{2k} = (1 - t_{2k}) x_k + t_{2k} T^{{n_k}_k} y_{1k},$$

$$y_{1k} = (1 - t_{1k}) x_k + t_{1k} T^{{n_k}_k} y_{0k},$$

$$y_{0k} = x_k, \quad k \in \mathbb{N},$$

where $T_1, \ldots, T_m \in \mathcal{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers.

**Theorem 5.1.** Let $X$ be a uniformly convex Banach space with the Opial property, and let $C$ be a nonempty closed convex subset of $X$. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1 - \delta]$ for some $\delta \in (0,1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.

**Theorem 5.2.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ such that $T^i_i$ is semicompact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1 - \delta]$ for some $\delta \in (0,1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$. 
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