Research Article

An Iterative Algorithm on Approximating Fixed Points of Pseudocontractive Mappings

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Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E , and every nonempty closed convex bounded subset of K has the fixed point property for non-expansive self-mappings. Let $f : K \rightarrow K$ a contractive mapping and $T : K \rightarrow K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2)$ be a sequence satisfying the following conditions: (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. Define the sequence $\{x_n\}$ in K by $x_0 \in K$, $x_{n+1} = \lambda_n f(x_n) + (1 - 2\lambda_n)x_n + \lambda_n T x_n$, for all $n \geq 0$. Under some appropriate assumptions, we prove that the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$ which is the unique solution of the following variational inequality: $\langle f(p) - p, j(z - p) \rangle \leq 0$, for all $z \in F(T)$.

1. Introduction

Let E be a real Banach space with dual E^* . We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is well known that, if E is smooth, then J is single-valued. In the sequel, we will denote the single-valued normalized duality mapping by j . We use $D(T), R(T)$ to denote the domain and range of T , respectively.

An operator $T : D(T) \rightarrow R(T)$ is called pseudocontractive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \quad \forall x, y \in D(T). \quad (1.2)$$

A point $x \in K$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in K : Tx = x\}$.

Within the past 40 years or so, many authors have been devoted to the iterative construction of fixed points of pseudocontractive mappings (see [1–10]).

In 1974, Ishikawa [11] introduced an iterative scheme to approximate the fixed points of Lipschitzian pseudocontractive mappings and proved the following result.

Theorem 1.1 (see [11]). *If K is a compact convex subset of a Hilbert space H , $T : K \rightarrow K$ is a Lipschitzian pseudocontractive mapping. Define the sequence $\{x_n\}$ in K by*

$$\begin{aligned} x_0 &\in K, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad \forall n \geq 0, \end{aligned} \quad (1.3)$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

- (i) $0 \leq \alpha_n \leq \beta_n < 1$,
- (ii) $\lim_{n \rightarrow \infty} \beta_n = 0$,
- (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T .

In connection with the iterative approximation of fixed points of pseudo-contractions, in 2001, Chidume and Mutangadura [12] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iterative algorithm failed to converge. Chidume and Zegeye [13] introduced a new iterative scheme for approximating the fixed points of pseudocontractive mappings.

Theorem 1.2 (see [13]). *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E . Let $T : K \rightarrow K$ be a L -Lipschitzian pseudocontractive mapping such that $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0, 1]$ satisfying the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$,
- (ii) $\lambda_n(1 + \theta_n) \leq 1$, $\sum_{n=0}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n \rightarrow \infty} (\lambda_n / \theta_n) = 0$,
- (iii) $\lim_{n \rightarrow \infty} ((\theta_{n-1} / \theta_n - 1) / \lambda_n \theta_n) = 0$.

For given $x_1 \in K$ arbitrarily, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall n \geq 1. \quad (1.4)$$

Then, the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of T .

Prototypes for the iteration parameters are, for example, $\lambda_n = 1/(n+1)^a$ and $\theta_n = 1/(n+1)^b$ for $0 < b < a$ and $a + b < 1$. But we observe that the canonical choices of $\lambda_n = 1/n$ and $\theta_n = 1/n$ are impossible. This brings us a question.

Question 1. Under what conditions, $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$ are sufficient to guarantee the strong convergence of the iterative scheme (1.4) to a fixed point of T ?

In this paper, we explore an iterative scheme to approximate the fixed points of pseudocontractive mappings and prove that, under some appropriate assumptions, the proposed iterative scheme converges strongly to a fixed point of T , which solves some variational inequality. Our results improve and extend many results given in the literature.

2. Preliminaries

Let K be a nonempty closed convex subset of a real Banach space E . Recall that a mapping $f : K \rightarrow K$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in K. \quad (2.1)$$

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \dots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \dots) \in l^\infty$.

If μ is a Banach limit, then we have the following.

- (1) For all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$.
- (2) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r .
- (3) $\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$ for all $(a_0, a_1, \dots) \in l^\infty$.
- (4) If $s = (a_0, a_1, \dots) \in l^\infty$ with $a_n \rightarrow a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit μ .

For more details on Banach limits, we refer readers to [14]. We need the following lemmas for proving our main results.

Lemma 2.1 (see [15]). *Let E be a Banach space. Suppose that K is a nonempty closed convex subset of E and $T : K \rightarrow E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition: $T(x) \in \overline{I_K(x)}$ (where $\overline{I_K(x)}$ is the closure of $I_K(x)$) for each $x \in K$, where $I_K(x) = \{x + c(u - x) : u \in E \text{ and } c \geq 1\}$. Then, for each $z \in K$, there exists a unique continuous path $t \mapsto z_t \in K$ for all $t \in [0, 1)$, satisfying the following equation*

$$z_t = tTz_t + (1 - t)z. \quad (2.2)$$

Furthermore, if E is a reflexive Banach space with a uniformly Gâteaux differentiable norm and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings, then, as $t \rightarrow 1$, z_t converges strongly to a fixed point of T .

Lemma 2.2 (see [16]). (1) If E is smooth Banach space, then the duality mapping J is single valued and strong-weak* continuous.

(2) If E is a Banach space with a uniformly Gâteaux differentiable norm, then the duality mapping $J : E \rightarrow E^*$ is single valued and norm to weak star uniformly continuous on bounded sets of E .

Lemma 2.3 (see [17]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$ for all $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_n\}$ two sequences of real numbers such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. Then $\{a_n\}$ converges to zero as $n \rightarrow \infty$.

Lemma 2.4 (see [18]). Let E be a real Banach space, and let J be the normalized duality mapping. Then, for any given $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y). \quad (2.3)$$

Lemma 2.5 (see [14]). Let a be a real number, and let $(x_0, x_1, \dots) \in l^\infty$ such that $\mu_n x_n \leq a$ for all Banach limits. If $\limsup_{n \rightarrow \infty} (x_{n+1} - x_n) \leq 0$, then $\limsup_{n \rightarrow \infty} x_n \leq a$.

3. Main Results

Now, we are ready to give our main results in this paper.

Theorem 3.1. Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E , and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $f : K \rightarrow K$ a contractive mapping and $T : K \rightarrow K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:

$$(i) \lim_{n \rightarrow \infty} \lambda_n = 0,$$

$$(ii) \sum_{n=0}^{\infty} \lambda_n = \infty.$$

Define the sequence $\{x_n\}$ in K by

$$\begin{aligned} x_0 &\in K, \\ x_{n+1} &= \lambda_n f(x_n) + (1 - 2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{aligned} \quad (3.1)$$

If $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle f(p) - p, j(z - p) \rangle \leq 0, \quad \forall z \in F(T). \quad (3.2)$$

Proof. Take $p \in F(T)$, and let $S = I - T$. Then, we have

$$\langle Sx - Sy, j(x - y) \rangle \geq 0. \quad (3.3)$$

From (3.1), we obtain

$$\begin{aligned} x_n &= x_{n+1} + \lambda_n x_n - \lambda_n T x_n + \lambda_n x_n - \lambda_n f(x_n) \\ &= x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n f(x_n) \\ &= x_{n+1} + \lambda_n [x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n f(x_n)] + \lambda_n S x_n - \lambda_n f(x_n) \\ &= (1 + \lambda_n) x_{n+1} + \lambda_n^2 (x_n + S x_n) - \lambda_n^2 f(x_n) + \lambda_n S x_n - \lambda_n f(x_n) \\ &= (1 + \lambda_n) x_{n+1} + \lambda_n S x_{n+1} + \lambda_n^2 (x_n + S x_n) - \lambda_n^2 f(x_n) \\ &\quad + \lambda_n (S x_n - S x_{n+1}) - \lambda_n f(x_n). \end{aligned} \quad (3.4)$$

By (3.4), we have

$$\begin{aligned} x_n - p &= (1 + \lambda_n)(x_{n+1} - p) + \lambda_n (S x_{n+1} - S p) + \lambda_n^2 (x_n + S x_n) \\ &\quad - \lambda_n^2 f(x_n) + \lambda_n (S x_n - S x_{n+1}) + \lambda_n (p - f(x_n)). \end{aligned} \quad (3.5)$$

Combining (3.3) and (3.5), we have

$$\begin{aligned} &\langle x_n - p - \lambda_n^2 (x_n + S x_n) + \lambda_n^2 f(x_n) - \lambda_n (S x_n - S x_{n+1}) + \lambda_n (f(x_n) - p), j(x_{n+1} - p) \rangle \\ &= (1 + \lambda_n) \|x_{n+1} - p\|^2 + \lambda_n \langle S x_{n+1} - S p, j(x_{n+1} - p) \rangle \\ &\geq (1 + \lambda_n) \|x_{n+1} - p\|^2. \end{aligned} \quad (3.6)$$

Next, we prove that $\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0$. Indeed, taking $z = f(p)$ in Lemma 2.1, we have

$$z_t - x_n = (1 - t)(T z_t - x_n) + t(f(p) - x_n), \quad (3.7)$$

and, hence,

$$\begin{aligned} \|z_t - x_n\|^2 &= (1 - t) \langle T z_t - x_n, j(z_t - x_n) \rangle + t \langle f(p) - x_n, j(z_t - x_n) \rangle \\ &= (1 - t) \langle T z_t - T x_n, j(z_t - x_n) \rangle + (1 - t) \langle T x_n - x_n, j(z_t - x_n) \rangle \\ &\quad + t \langle f(p) - z_t, j(z_t - x_n) \rangle + t \|z_t - x_n\|^2 \\ &\leq \|z_t - x_n\|^2 + (1 - t) \|T x_n - x_n\| \|z_t - x_n\| \\ &\quad + t \langle f(p) - z_t, j(z_t - x_n) \rangle. \end{aligned} \quad (3.8)$$

Therefore, we have

$$\begin{aligned}\langle z_t - f(p), j(z_t - x_n) \rangle &\leq \frac{1-t}{t} \|Tx_n - x_n\| \|z_t - x_n\| \\ &\leq M_1 \frac{1-t}{t} \|Tx_n - x_n\|,\end{aligned}\tag{3.9}$$

where $M_1 > 0$ is some constant such that $\|z_t - x_n\| \leq M_1$ for all $t \in (0, 1]$ and $n \geq 1$. Letting $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \langle z_t - f(p), j(z_t - x_n) \rangle \leq 0.\tag{3.10}$$

From Lemma 2.1, we know $z_t \rightarrow p$ as $t \rightarrow 0$. Since the duality mapping $J : E \rightarrow E^*$ is norm to weak star uniformly continuous from Lemma 2.2, we have

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0.\tag{3.11}$$

From (3.6), we have

$$\begin{aligned}(1 + \lambda_n) \|x_{n+1} - p\|^2 &\leq \left\langle x_n - p - \lambda_n^2(x_n + Sx_n) + \lambda_n^2 f(x_n) - \lambda_n(Sx_n - Sx_{n+1}), j(x_{n+1} - p) \right\rangle \\ &\leq \|x_n - p\| \|x_{n+1} - p\| + M_2 \lambda_n^2 + M_2 \lambda_n \|Sx_{n+1} - Sx_n\| \\ &\quad + \lambda_n \|f(x_n) - f(p)\| \|x_{n+1} - p\| + \lambda_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\ &\leq \|x_n - p\| \|x_{n+1} - p\| + M_2 \lambda_n^2 + M_2 \lambda_n \|Sx_{n+1} - Sx_n\| \\ &\quad + \lambda_n \alpha \|x_n - p\| \|x_{n+1} - p\| + \lambda_n \langle f(p) - p, j(x_{n+1} - p) \rangle \\ &\leq \frac{1 + \lambda_n \alpha}{2} (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + M_2 \lambda_n^2 \\ &\quad + M_2 \lambda_n \|Sx_{n+1} - Sx_n\| + \lambda_n \langle f(p) - p, j(x_{n+1} - p) \rangle,\end{aligned}\tag{3.12}$$

where M_2 is a constant such that

$$\sup \{ \|x_n + Sx_n\| \|x_{n+1} - p\| + \|f(x_n)\| \|x_{n+1} - p\| + \|x_{n+1} - p\|, n \geq 0 \} \leq M_2.\tag{3.13}$$

It follows that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \frac{1 + \lambda_n \alpha}{1 + (2 - \alpha)\lambda_n} \|x_n - p\|^2 + M_2 \lambda_n^2 + M_2 \lambda_n \|Sx_{n+1} - Sx_n\| \\
 &\quad + \frac{\lambda_n}{1 + (2 - \alpha)\lambda_n} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
 &= \left[1 - \frac{2(1 - \alpha)}{1 + (2 - \alpha)\lambda_n} \lambda_n \right] \|x_n - p\|^2 + \frac{2(1 - \alpha)\lambda_n}{1 + (2 - \alpha)\lambda_n} \\
 &\quad \times \left\{ \frac{1 + (2 - \alpha)\lambda_n}{2(1 - \alpha)} M_2 \lambda_n + \frac{1 + (2 - \alpha)\lambda_n}{2(1 - \alpha)} M_2 \|Sx_{n+1} - Sx_n\| \right. \\
 &\quad \left. + \frac{1}{2(1 - \alpha)} \langle f(p) - p, j(x_{n+1} - p) \rangle \right\} \\
 &= (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n \beta_n,
 \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
 \alpha_n &= \frac{2(1 - \alpha)}{1 + (2 - \alpha)\lambda_n} \lambda_n, \\
 \beta_n &= \frac{1 + (2 - \alpha)\lambda_n}{2(1 - \alpha)} M_2 \lambda_n + \frac{1 + (2 - \alpha)\lambda_n}{2(1 - \alpha)} M_2 \|Sx_{n+1} - Sx_n\| \\
 &\quad + \frac{1}{2(1 - \alpha)} \langle f(p) - p, j(x_{n+1} - p) \rangle.
 \end{aligned} \tag{3.15}$$

Note that

$$\|x_{n+1} - x_n\| \leq \lambda_n \|x_n\| + \lambda_n \|Tx_n\| + \lambda_n \|x_n - f(x_n)\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.16}$$

By the uniform continuity of T , we have

$$\|Sx_{n+1} - Sx_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.17}$$

Hence, it is clear that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} \beta_n \leq 0$.

Finally, applying Lemma 2.3 to (3.14), we can conclude that $x_n \rightarrow p$. This completes the proof. $\square$

From Theorem 3.1, we can prove the following corollary.

Corollary 3.2. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E , and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $T : K \rightarrow K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:*

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$,
- (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Define the sequence $\{x_n\}$ in K by

$$\begin{aligned} u, x_0 &\in K, \\ x_{n+1} &= \lambda_n u + (1 - 2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{aligned} \quad (3.18)$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$.

Theorem 3.3. Let E be a uniformly smooth Banach space and K a nonempty bounded closed convex subset of E . Let $f : K \rightarrow K$ be a contractive mapping and $T : K \rightarrow K$ a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$,
- (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point $p \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle f(p) - p, j(z - p) \rangle \leq 0, \quad \forall z \in F(T). \quad (3.19)$$

Proof. Since every uniformly smooth Banach space E is reflexive and whose norm is uniformly Gâteaux differentiable, at the same time, every closed convex and bounded subset of K has the fixed point property for nonexpansive mappings. Hence, from Theorem 3.1, we can obtain the result. This completes the proof. $\square$

From Theorem 3.3, we can prove the following corollary.

Corollary 3.4. Let E be a uniformly smooth Banach space and K a nonempty bounded closed convex subset of E . Let $T : K \rightarrow K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$,
- (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Define the sequence $\{x_n\}$ in K by

$$\begin{aligned} u, x_0 &\in K, \\ x_{n+1} &= \lambda_n u + (1 - 2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{aligned} \quad (3.20)$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$.

Theorem 3.5. Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $f : K \rightarrow K$ be a contractive mapping and $T : K \rightarrow K$ be a uniformly continuous pseudocontractive mapping. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$,
- (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $D \cap F(T) \neq \emptyset$, where D is defined as (3.22) below, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point $p \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle f(p) - p, j(z - p) \rangle \leq 0, \quad \forall z \in F(T). \quad (3.21)$$

Proof. First, we note that the sequence $\{x_n\}$ is bounded. Now, if we define $g(x) = \mu_n \|x_n - x\|^2$, then $g(x)$ is convex and continuous. Also, we can easily prove that $g(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Since E is reflexive, there exists $y \in K$ such that $g(y) = \inf_{x \in K} g(x)$. So the set

$$D = \left\{ y \in K : g(y) = \inf_{x \in K} g(x) \right\} \neq \emptyset. \quad (3.22)$$

Clearly, D is closed convex subset of K .

Now, we can take $p \in D \cap F(T)$ and $t \in (0, 1)$. By the convexity of K , we have that $(1-t)p + tf(p) \in K$. It follows that

$$g(p) \leq g((1-t)p + tf(p)). \quad (3.23)$$

By Lemma 2.4, we have

$$\|x_n - p - t(f(p) - p)\|^2 \leq \|x_n - p\|^2 - 2t \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle. \quad (3.24)$$

Taking the Banach limit in (3.24), we have

$$\mu_n \|x_n - p - t(f(p) - p)\|^2 \leq \mu_n \|x_n - p\|^2 - 2t \mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle. \quad (3.25)$$

This implies

$$2t \mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle \leq g(p) - g((1-t)p + tf(p)). \quad (3.26)$$

Therefore, it follows from (3.23) and (3.26) that

$$\mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle \leq 0. \quad (3.27)$$

Since the normalized duality mapping j is single valued and norm-weak* uniformly continuous on bounded subset of E , we have

$$\langle f(p) - p, j(x_n - p) \rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle \rightarrow 0 \quad (t \rightarrow 0). \quad (3.28)$$

This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $t \in (0, \delta)$ and $n \geq 1$,

$$\langle f(p) - p, j(x_n - p) \rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle < \epsilon. \quad (3.29)$$

Taking the Banach limit and noting that (3.27), we have

$$\mu_n \langle f(p) - p, j(x_n - p) \rangle \leq \mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p)) \rangle + \epsilon \leq \epsilon. \quad (3.30)$$

By the arbitrariness of ϵ , we obtain

$$\mu_n \langle f(p) - p, j(x_n - p) \rangle \leq 0. \quad (3.31)$$

At the same time, we note that

$$\|x_{n+1} - x_n\| \leq \lambda_n (\|f(x_n)\| + 2\|x_n\| + \|Tx_n\|) \longrightarrow 0 \quad (n \longrightarrow \infty). \quad (3.32)$$

Since $\{x_n - p\}$, $\{f(p) - p\}$ are bounded and the duality mapping j is single valued and norm topology to weak star topology uniformly continuous on bounded sets in Banach space E with a uniformly Gâteaux differentiable norm, it follows that

$$\lim_{n \rightarrow \infty} \{ \langle f(p) - p, j(x_{n+1} - p) \rangle - \langle f(p) - p, j(x_n - p) \rangle \} = 0. \quad (3.33)$$

From (3.31), (3.33), and Lemma 2.5, we conclude that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_{n+1} - p) \rangle \leq 0. \quad (3.34)$$

Finally, by the similar arguments as that the proof in Theorem 3.1, it is easy to prove that the sequence $\{x_n\}$ converges to a fixed point of T . This completes the proof. $\square$

From Theorem 3.5, we can easily to prove the following result.

Corollary 3.6. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $f : K \rightarrow K$ be a contractive mapping and $T : K \rightarrow K$ a uniformly continuous pseudocontractive mapping. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:*

$$(i) \lim_{n \rightarrow \infty} \lambda_n = 0,$$

$$(ii) \sum_{n=0}^{\infty} \lambda_n = \infty.$$

Define the sequence $\{x_n\}$ in K by

$$\begin{aligned} u, x_0 &\in K, \\ x_{n+1} &= \lambda_n u + (1 - 2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{aligned} \quad (3.35)$$

If $D \cap F(T) \neq \emptyset$, where D is defined as (3.22), then the sequence $\{x_n\}$ defined by (3.35) converges strongly to a fixed point $p \in F(T)$.

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