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Research Article

Stability and Bifurcation Analysis of a Three-Dimensional Recurrent Neural Network with Time Delay

Yingguo Li

School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, China

Correspondence should be addressed to Yingguo Li, yguoli@fjnu.edu.cn

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We consider the nonlinear dynamical behavior of a three-dimensional recurrent neural network with time delay. By choosing the time delay as a bifurcation parameter, we prove that Hopf bifurcation occurs when the delay passes through a sequence of critical values. Applying the normal form method and center manifold theory, we obtain some local bifurcation results and derive formulas for determining the bifurcation direction and the stability of the bifurcated periodic solution. Some numerical examples are also presented to verify the theoretical analysis.

1. Introduction

Starting with the work of Hopfield [1] on neural networks, recurrent neural networks including Hopfield neural networks, Cohen-Grossberg neural networks, and cellular neural networks have been used extensively in different areas such as signal processing, pattern recognition, optimization, and associative memories. Many researchers studied the dynamical behavior of Recurrent neural network systems, and most of papers are devoted to the stability of equilibrium, existence and stability of periodic solutions, bifurcation, and chaos [2–5]. In [5], Ruiz et al. considered a particular configuration of a recurrent neural network, illustrated in Figure 1. In Figure 1, $u(t)$ is the input and $y(t)$ is the output of the network. This recurrent neural network can be described by the following system:

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + f(x_2(t)),\\
&\vdots\\
\dot{x}_{n-1}(t) &= -x_{n-1}(t) + u(t),
\end{aligned}$$

$$\begin{aligned}\dot{x}_n(t) &= -x_n(t) + w_1 f(x_1(t)) + \cdots + w_{n-1} f(x_{n-1}(t)),\\ y(t) &= f(x_n(t)).\end{aligned} \tag{1.1}$$

Here, $x(t) \in R^n$ is the state, $w_i \in R$, $i = 1, \ldots, n-1$ are the network parameters or weights, $u(t)$ is a smooth input, and $y(t)$ is the output. The transfer function of the neurons is taken as $f(\cdot) = \tanh(\cdot)$. A three-node network of the form (1.1) in feedback configuration, with $u(t) = y(t)$, has been studied in [5]; that is,

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + \tanh(x_2(t)),\\ \dot{x}_2(t) &= -x_2(t) + \tanh(x_3(t)),\\ \dot{x}_3(t) &= -x_3(t) + w_1 \tanh(x_1(t)) + w_2 \tanh(x_2(t)).\end{aligned} \tag{1.2}$$

Ruiz et al. found and analyzed the Hopf bifurcation behavior in system (1.2). In [4], Maleki et al. considered system (1.1) with a transfer function $f(x) = \sum_{i=1}^{\infty} \alpha_{2i-1} x^{2i-1}$, where $\alpha_{2i-1} > 0$ for i odd and $\alpha_{2i-1} < 0$ for i even. The authors analyzed the Bogdanov-Takens bifurcation in the system.

It is well known that there exist time delays in the information processing of neurons. The delayed axonal signal transmissions in the neural network models make the dynamical behaviors become more complicated and may destabilize the stable equilibria and admit periodic oscillation, bifurcation, and chaos. Therefore, the delay is an important control parameter in living nervous system: different ranges of delays correspond to different patterns of neural activities (see, e.g., [6–11]).

In the present paper, we consider the following three-dimensional recurrent neural network model with time delay

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + f(x_2(t-\tau)),\\ \dot{x}_2(t) &= -x_2(t) + f(x_3(t-\tau)),\\ \dot{x}_3(t) &= -x_3(t) + w_1 f(x_1(t-\tau)) + w_2 f(x_2(t-\tau)).\end{aligned} \tag{1.3}$$

By choosing the time delay as a bifurcation parameter, we prove that Hopf bifurcation occurs in the neuron and study the properties of periodic solutions of this model.

The organization of this paper is as follows. In Section 2, by analyzing the characteristic equation of the linearized system of system (1.3) at the equilibrium, we discuss the stability of the equilibrium and the existence of the Hopf bifurcation occurring at the equilibrium. In Section 3, the formulae determining the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions on the center manifold are obtained by using the normal form theory and the center manifold theorem due to Hassard et al. [12]. We do some computer observations to validate our theoretical results in Section 4.

2. Stability and Existence of Hopf Bifurcation

For most of the models in the literature, including the ones in [5, 7, 8], the activation function f is $f(u) = \tanh(cu)$. However, we only make the following assumption on function f:

(H) $f \in C^3(R)$, $f(0) = 0$, and $f'(0) \neq 0$.

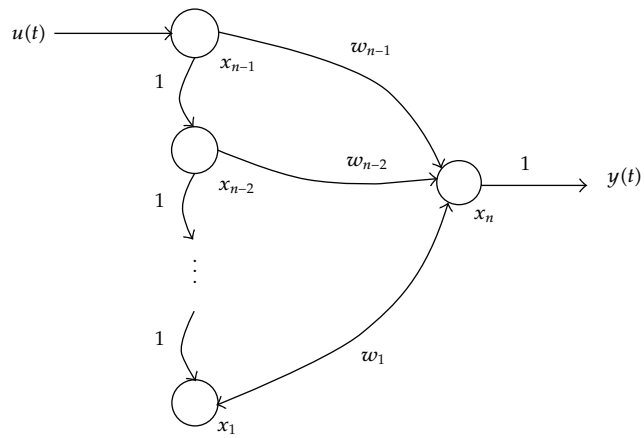

Figure 1: Class of recurrent neural networks.

Clearly, $(x_1, x_2, x_3)^T = (0, 0, 0)^T$ is equilibrium of system (1.3). Linearization of (1.3) at the zero equilibrium yields

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + f'(0)x_2(t-\tau), \\
\dot{x}_2(t) &= -x_2(t) + f'(0)x_3(t-\tau), \\
\dot{x}_3(t) &= -x_3(t) + w_1 f'(0)x_1(t-\tau) + w_2 f'(0)x_2(t-\tau),
\end{aligned} \tag{2.1}$$

whose characteristic equation is

$$\det\begin{pmatrix} \lambda+1 & -f'(0)e^{-\lambda\tau} & 0 \\ 0 & \lambda+1 & -f'(0)e^{-\lambda\tau} \\ -w_1 f'(0)e^{-\lambda\tau} & -w_2 f'(0)e^{-\lambda\tau} & \lambda+1 \end{pmatrix} = 0, \tag{2.2}$$

that is,

$$(\lambda+1)^3 - w_2 f'^2(0)(\lambda+1)e^{-2\lambda\tau} - w_1 f'^3(0)e^{-3\lambda\tau} = 0. \tag{2.3}$$

The zero equilibrium is stable if all roots of (2.3) have negative real parts and unstable if at least one root has positive real part. Therefore, in order to study the local stability of the zero equilibrium of system (2.3), we need to investigate the distribution of the roots of (2.3).

When $\tau = 0$, characteristic equation (2.3) yields

$$(\lambda+1)^3 - w_2 f'^2(0)(\lambda+1) - w_1 f'^3(0) = 0. \tag{2.4}$$

Let $y = \lambda + 1$, (2.4) reduces to

$$y^3 - w_2 f'^2(0)y - w_1 f'^3(0) = 0. \tag{2.5}$$

Denote

$$\Delta = \frac{w_1^2 f'^6(0)}{4} - \frac{w_2^3 f'^6(0)}{27}, \qquad \varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$
$$\alpha = \sqrt[3]{\frac{w_1 f'^3(0)}{2} + \sqrt{\Delta}}, \qquad \beta = \sqrt[3]{\frac{w_1 f'^3(0)}{2} - \sqrt{\Delta}}. \tag{2.6}$$

From Cardano formula for the third-degree algebra equation, we have the following lemma.

Lemma 2.1. *(1) If $\Delta > 0$, then (2.5) has a real root $\alpha + \beta$ and a pair of conjugate complex roots $-(\alpha+\beta)/2 \pm i(\sqrt{3}/2)(\alpha - \beta)$. Furthermore, the roots of (2.4) are given by $\lambda_1 = -1 + \alpha + \beta$ and $\lambda_{2,3} = -1 - (\alpha+\beta)/2 \pm i(\sqrt{3}/2)(\alpha - \beta)$.*

(2) If $\Delta = 0$, then (2.5) has a simple root 2α and a multiple root $-\alpha$ with the multiplicity of 2. Furthermore, the roots of (2.4) are given by $\lambda_1 = -1 + 2\alpha$ and $\lambda_{2,3} = -1 - \alpha$. Meanwhile, if $w_1 = w_2 = 0$, that is, $\alpha = 0$, then (2.4) has a multiple root -1 with the multiplicity of 3.

(3) If $\Delta < 0$, then (2.5) has three real roots $2\,\mathrm{Re}\{\alpha\}$, $2\,\mathrm{Re}\{\alpha\varepsilon\}$, and $2\,\mathrm{Re}\{\alpha\bar{\varepsilon}\}$. Furthermore, the roots of (2.4) are given by $\lambda_1 = -1 + 2\,\mathrm{Re}\{\alpha\}$, $\lambda_2 = -1 + 2\,\mathrm{Re}\{\alpha\varepsilon\}$, and $\lambda_3 = -1 + 2\,\mathrm{Re}\{\alpha\bar{\varepsilon}\}$.

Applying Lemma 2.1, we have the following lemma.

Lemma 2.2. *All roots of (2.4) have negative real parts if one of the following holds.*

(1) $\Delta > 0$ and $-2 < \alpha + \beta < 1$.

(2) $\Delta = 0$ and $-1 < \alpha < 1/2$.

(3) $\Delta < 0$ and $\max\{\mathrm{Re}\{\alpha\}, \mathrm{Re}\{\alpha\varepsilon\}, \mathrm{Re}\{\alpha\bar{\varepsilon}\}\} < 1/2$.

Let $z = (\lambda + 1)e^{\lambda\tau}$, then (2.3) becomes

$$z^3 - w_2 f'^2(0) z - w_1 f'^3(0) = 0. \tag{2.7}$$

We notice that (2.7) and (2.5) have the same coefficients.

Denote the three roots of (2.7) by $z_n = R_n + iI_n$ $(n = 1, 2, 3)$. Hence, (2.3) is equivalent to

$$(\lambda + 1)e^{\lambda\tau} = z_n \quad (n = 1, 2, 3). \tag{2.8}$$

Clearly, $i\omega$ $(\omega > 0)$ is a root of (2.3) if and only if ω satisfies

$$\cos\omega\tau - \omega\sin\omega\tau = R_n,$$
$$\sin\omega\tau + \omega\cos\omega\tau = I_n, \tag{2.9}$$

which implies that

$$\cos\omega\tau = \frac{R_n + \omega I_n}{1 + \omega^2}, \qquad \sin\omega\tau = \frac{I_n - \omega R_n}{1 + \omega^2} \quad (n = 1, 2, 3). \tag{2.10}$$

This yields

$$\omega^2 = R_n^2 + I_n^2 - 1 \quad (n = 1, 2, 3). \tag{2.11}$$

Obviously, we have the following lemma.

Lemma 2.3. *Equation* (2.11) *is meaningless when* $|z_n| = \sqrt{R_n^2 + I_n^2} \leq 1$ $(n = 1, 2, 3)$. *If* (2.7) *has a root, denoted by* z_j, *satisfying* $|z_n| > 1$, *then* (2.11) *has a positive root given by*

$$\omega_j = \sqrt{R_n^2 + I_n^2 - 1}. \tag{2.12}$$

Summarizing the discussion above, we have the following.

Lemma 2.4. *If* $\Delta \geq 0$, *then* (2.11) *has at most two positive roots. If* $\Delta < 0$, *then* (2.11) *has at most three positive roots.*

Without loss of generality, one assumes that (2.11) has three positive roots ω_n $(n = 1, 2, 3)$.

Define

$$\tau_j^{(n)} = \frac{1}{\omega_n}\left[\arccos\left(\frac{R_n + \omega_n I_n}{1 + \omega_n^2}\right) + 2j\pi\right], \quad n = 1, 2, 3, \ j = 0, 1, 2, \ldots. \tag{2.13}$$

Then, $\pm i\omega_n$ is a pair of purely imaginary roots of (2.3) with $\tau = \tau_j^{(n)}$.

Let $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ be the root of (2.3) near $\tau = \tau_j^n$ satisfying

$$\sigma\left(\tau_j^n\right) = 0, \quad \omega\left(\tau_j^n\right) = \omega_n \quad (n = 1, 2, 3, \ j = 0, 1, 2, \ldots). \tag{2.14}$$

We have the following.

Lemma 2.5. *One has*

$$\left.\frac{d\sigma(\tau)}{d\tau}\right|_{\tau=\tau_j^{(n)}} > 0, \quad n = 1, 2, 3, \ j = 0, 1, 2, 3, \ldots. \tag{2.15}$$

Proof. Differentiating both sides of (2.8) with respect to τ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{1}{\lambda(\lambda + 1)} - \frac{\tau}{\lambda}. \tag{2.16}$$

Note that $\lambda(\tau_j^{(n)}) = i\omega_n$, therefore

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\tau=\tau_j^{(n)}} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} = \operatorname{sign}\left\{\operatorname{Re}\left[\frac{-1}{i\omega_n(i\omega_n+1)}\right]\right\} = \operatorname{sign}\left\{\frac{1}{\omega^2+1}\right\}. \tag{2.17}$$

We have

$$\left.\frac{d\sigma(\tau)}{d\tau}\right|_{\tau=\tau_j^{(n)}} = \left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\tau=\tau_j^{(n)}} > 0. \tag{2.18}$$

This completes the proof. □

For convenience, we let $\cup_{n=1}^{3}\{\tau_j^{(n)}\}_{j=0}^{+\infty} = \{\tau_i\}_{i=0}^{+\infty}$, such that

$$\tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots, \tag{2.19}$$

where

$$\tau_0 = \min\left\{\tau_1^{(0)}, \tau_2^{(0)}, \tau_3^{(0)}\right\} \tag{2.20}$$

and $\tau_j^{(n)}$ is defined in (2.13).

Applying Lemmas 2.1–2.5 and Corollary 2.4 of Ruan and Wei [13], we have the following results.

Lemma 2.6. *All roots of* (2.3) *have negative real parts if one of the following holds:*

(H_1) $\Delta > 0, -1 < \alpha + \beta < 1$ *and* $(1/4)(\alpha+\beta)^2 + (3/4)(\alpha-\beta)^2 < 1$,

(H_2) $\Delta = 0$ *and* $-1 < \alpha < 1/2$,

(H_3) $\Delta < 0$ *and* $\max\{\operatorname{Re}\{\alpha\}, \operatorname{Re}\{\alpha\varepsilon\}, \operatorname{Re}\{\alpha\bar{\varepsilon}\}\} < 1/2$.

Lemma 2.7. *Suppose that one of the following hypothesis is satisfied:*

(H_4) $\Delta > 0, -2 < \alpha + \beta < -1$,

(H_5) $\Delta > 0, -1 < \alpha + \beta < 1$ *and* $(1/4)(\alpha+\beta)^2 + (3/4)(\alpha-\beta)^2 > 1$.

Then, there exists a sequence values of τ *defined by* (2.19) *such that all roots of* (2.3) *have negative real parts for all* $\tau \in [0, \tau_0)$, *and* (2.3) *has at least one root with positive real part when* $\tau > \tau_0$, *and* (2.3) *exactly has a pair of purely imaginary roots* $\pm i\omega_n$ $(n = 1, 2, 3)$ *when* $\tau = \tau_j^{(n)}$ $(n = 1, 2, 3;\ j = 0, 1, 2, 3, \ldots)$, *where* ω_n *and* $\tau = \tau_j^{(n)}$ *are defined by* (2.12) *and* (2.13), *respectively.*

From Lemmas 2.5–2.7 and the Hopf bifurcation theorem for functional differential equations in [14], we have the theorem.

Theorem 2.8. *(1) If one of the hypothesis* (H_1), (H_2), (H_3) *is satisfied, then the zero solution of system* (1.3) *is asymptotically stable for all* $\tau \geq 0$.

(2) If (H_4) *or* (H_5) *is satisfied, then the zero solution of system* (1.3) *is asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$, *and system* (1.3) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_j$ $(j = 0, 1, 2, 3, \ldots)$.

3. Direction of Hopf Bifurcations and Stability of the Bifurcating Periodic Orbits

In this section, we will study the direction of the Hopf bifurcation and stability of bifurcating periodic solutions by using the normal theory and the center manifold theorem due to Hassard et al. [12].

Let $u_1(t) = x_1(\tau t)$, $u_2(t) = x_2(\tau t)$, $u_3(t) = x_3(\tau t)$, then system (1.3) becomes functional differential equation in $\mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R}^3)$ as

$$\begin{aligned}
&\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \end{pmatrix} \\
&\quad = \tau B_1 \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} + \tau B_2 \begin{pmatrix} u_1(t-1) \\ u_2(t-1) \\ u_3(t-1) \end{pmatrix} \\
&\quad + \tau \begin{pmatrix} \dfrac{f''(0)}{2!} u_2^2(t-1) + \dfrac{f'''(0)}{3!} u_2^3(t-1) + \cdots \\ \dfrac{f''(0)}{2!} u_3^2(t-1) + \dfrac{f'''(0)}{3!} u_3^3(t-1) + \cdots \\ \dfrac{w_1 f''(0)}{2!} u_1^2(t-1) + \dfrac{w_1 f'''(0)}{3!} u_1^3(t-1) + \dfrac{w_2 f''(0)}{2!} u_2^2(t-1) + \dfrac{w_2 f'''(0)}{3!} u_2^3(t-1) + \cdots \end{pmatrix},
\end{aligned} \tag{3.1}$$

where

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & f'(0) & 0 \\ 0 & 0 & f'(0) \\ w_1 f'(0) & w_2 f'(0) & 0 \end{pmatrix}. \tag{3.2}$$

Setting $\tau = \nu + \tau_j$, we know that $\nu = 0$ is Hopf bifurcation value of system (3.1).

For $\phi = (\phi_1, \phi_2, \phi_3)^T \in \mathcal{C}([-1, 0], \mathbb{R}^3)$, let

$$L_\nu(\phi) = (\tau_j + \nu)(B_1 \phi(0) + B_2 \phi(-1)). \tag{3.3}$$

By the Riesz representation theorem, there exists a function $\eta(\theta,\nu)$ of bounded variation for $\theta \in [-1,0]$, such that

$$L_\nu(\phi) = \int_{-1}^{0} d\eta(\theta,\nu)\phi(\theta) \quad \text{for } \phi \in \mathcal{C}\Big([-1,0],\mathbb{R}^3\Big). \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\nu) = (\tau_j+\nu)B_1\delta(\theta) - (\tau_j+\nu)B_2\delta(\theta+1), \tag{3.5}$$

where δ denotes the Dirac delta function. For $\phi \in \mathcal{C}([-1,0],\mathbb{R}^3)$, define

$$A(\nu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(s,\nu)\phi(s), & \theta = 0, \end{cases}$$
$$R(\nu)\phi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\nu,\phi), & \theta = 0, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} &f(\nu,\phi) \\ &\quad = (\tau_j+\nu) \\ &\quad \times \begin{pmatrix} \dfrac{f''(0)}{2!}u_2^2(t-1) + \dfrac{f'''(0)}{3!}u_2^3(t-1) + \cdots \\ \dfrac{f''(0)}{2!}u_3^2(t-1) + \dfrac{f'''(0)}{3!}u_3^3(t-1) + \cdots \\ \dfrac{w_1 f''(0)}{2!}u_1^2(t-1) + \dfrac{w_1 f'''(0)}{3!}u_1^3(t-1) + \dfrac{w_2 f''(0)}{2!}u_2^2(t-1) + \dfrac{w_2 f'''(0)}{3!}u_2^3(t-1) + \cdots \end{pmatrix}. \end{aligned} \tag{3.7}$$

Then system (3.1) is equivalent to

$$\dot{u}_t = A(\nu)u_t + R(\nu)u_t, \tag{3.8}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$, $u_t = u(t+\theta)$ for $\theta \in [-1,0]$.

For $\psi \in \mathcal{C}^1([0,1],(\mathbb{R}^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0, \end{cases} \tag{3.9}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta) \psi(\xi) d\xi, \tag{3.10}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_0\tau_j$ are eigenvalues of $A(0)$ and A^* corresponding to $i\omega_0\tau_j$ and $-i\omega_0\tau_j$, respectively.

Suppose $q(\theta) = (1, q_1, q_2)^T e^{i\omega_0\tau_j\theta}$ is the eigenvectors of $A(0)$ corresponding to $i\omega_0\tau_j$, then $A(0)q(\theta) = i\omega_0\tau_j q(\theta)$. Then from the definition of $A(0)$ and (3.3)–(3.5), we have

$$\tau_j B_1 q(0) + \tau_j B_2 q(-1) = i\omega_0\tau_j q(0). \tag{3.11}$$

For $q(-1) = q(0)e^{-i\omega_0\tau_j}$, then we obtain

$$\begin{aligned} q_1 &= \frac{(i\omega_0 + 1)e^{i\omega_0\tau_j}}{f'(0)}, \\ q_2 &= \frac{(i\omega_0 + 1)^2 e^{2i\omega_0\tau_j}}{f'^2(0)}. \end{aligned} \tag{3.12}$$

Similarly, we can obtain the eigenvector $q^*(s) = D(1, q_1^*, q_2^*)e^{\omega_0\tau_j s}$ of A^* corresponding to $-i\omega_0\tau_j$, where

$$\begin{aligned} q_1^* &= \frac{(i\omega_0 - 1)^2 e^{-2i\omega_0\tau_j}}{f'^2(0)w_1}, \\ q_2^* &= \frac{(1 - i\omega_0)e^{-i\omega_0\tau_j}}{f'(0)w_1}. \end{aligned} \tag{3.13}$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1$, we need to determine the value of D. By (3.10), we have

$$\begin{aligned} &\langle q^*(s), q(\theta) \rangle \\ &= \overline{D}(1, \overline{q}_1^*, \overline{q}_2^*)(1, q_1, q_2)^T - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{D}(1, \overline{q}_1^*, \overline{q}_2^*) e^{-i\omega_0\tau_j(\xi-\theta)} d\eta(\theta)(1, q_1, q_2)^T e^{i\omega_0\tau_j\xi} d\xi \\ &= \overline{D}\left[1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* - \int_{-1}^{0} (1, \overline{q}_1^*, \overline{q}_2^*)\theta e^{i\omega_0\tau_j\xi} d\eta(\theta)(1, q_1, q_2)^T\right] \\ &= \overline{D}\left[1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + \tau_j f'(0)e^{-i\omega_0\tau_j}\left(w_1\overline{q}_2^* + q_1 + w_2 q_1\overline{q}_2^* + q_2\overline{q}_1^*\right)\right] \\ &= 1. \end{aligned} \tag{3.14}$$

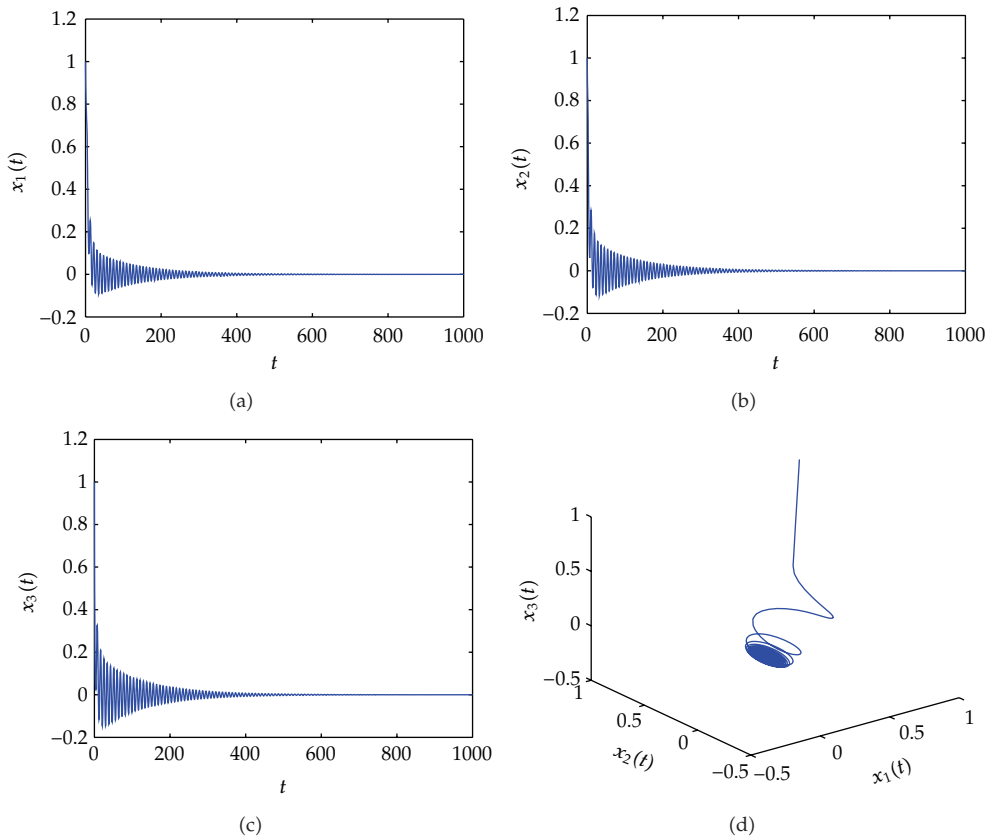

Figure 2: When $\tau = 1.7 < \tau_0 = 1.8434$, the zero equilibrium is asymptotically stable. Here initial value is (1, 1, 1).

Therefore, we can choose D as

$$D = \frac{1}{1 + \overline{q}_1 q_1^* + \overline{q}_2 q_2^* + \tau_j f'(0) e^{i\omega_0 \tau_j} \left(w_1 q_2^* + \overline{q}_1 + w_2 \overline{q}_1 q_2^* + \overline{q}_2 q_1^*\right)}. \tag{3.15}$$

Following the algorithms given in [12] and using similar computation process in [7], we can get that the coefficients which will be used to determine the important quantities:

$$\begin{aligned}
g_{20} &= \tau_j f''(0) \overline{D} e^{-2i\omega_0 \tau_j} \left[w_1 \overline{q}_2^* + q_1^2 + w_2 \overline{q}_2^* q_1^2 + \overline{q}_1^* q_2^2\right], \\
g_{11} &= \tau_j f''(0) \overline{D} \left[w_1 \overline{q}_2^* + (1 + w_2 \overline{q}_2^*) \overline{q}_1 q_1 + \overline{q}_1^* \overline{q}_2 q_2\right], \\
g_{02} &= \tau_j f''(0) \overline{D} e^{2i\omega_0 \tau_j} \left[w_1 \overline{q}_2^* + (1 + w_2 \overline{q}_2^*) \overline{q}_1^2 + \overline{q}_1^* \overline{q}_2^2\right], \\
g_{21} &= \tau_j f''(0) \overline{D} \Big[w_1 \overline{q}_2^* \left(W_{20}^{(1)}(-1) e^{i\omega_0 \tau_j} + 2W_{11}^{(1)}(-1) e^{-i\omega_0 \tau_j}\right) \\
&\qquad + (1 + w_2 \overline{q}_2^*) \left(W_{20}^{(2)}(-1) \overline{q}_1 e^{i\omega_0 \tau_j} + 2W_{11}^{(2)}(-1) q_1 e^{-i\omega_0 \tau_j}\right) \\
&\qquad + \overline{q}_1^* \left(W_{20}^{(3)}(-1) \overline{q}_2 e^{i\omega_0 \tau_j} + 2W_{11}^{(3)}(-1) q_2 e^{-i\omega_0 \tau_j}\right)\Big],
\end{aligned} \tag{3.16}$$

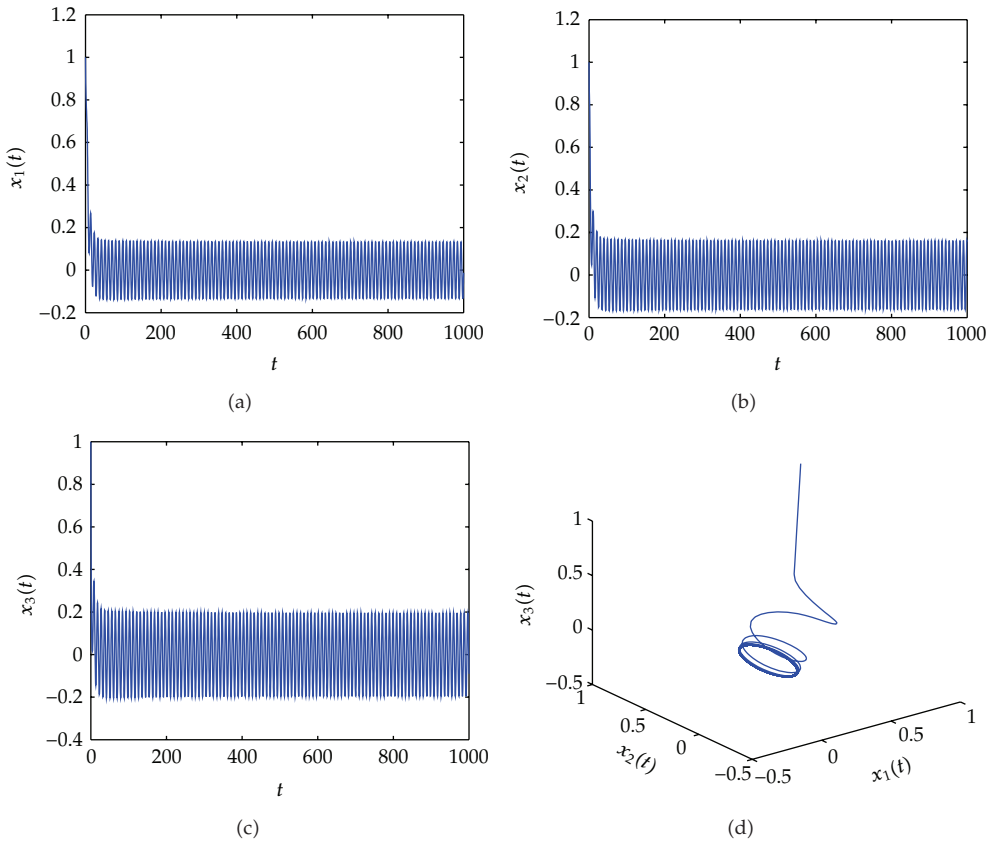

Figure 3: When $\tau = 1.9 > \tau_0 = 1.8434$, a periodic orbit bifurcates from the zero equilibrium. Here, initial value is (1, 1, 1).

where

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_j}q(0)e^{i\omega_0\tau_j\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau_j}\overline{q}(0)e^{-i\omega_0\tau_j\theta} + E_1e^{2i\omega_0\tau_j\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_j}q(0)e^{i\omega_0\tau_j\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau_j}\overline{q}(0)e^{-i\omega_0\tau_j\theta} + E_2, \tag{3.17}$$

moreover E_1, E_2 satisfy the following equations, respectively,

$$\begin{pmatrix} 2i\omega_0 + 1 & -f'(0)e^{-2i\omega_0\tau_j} & 0 \\ 0 & 2i\omega_0 + 1 & -f'(0)e^{-2i\omega_0\tau_j} \\ -w_1f'(0)e^{-2i\omega_0\tau_j} & -w_2f'(0)e^{-2i\omega_0\tau_j} & 2i\omega_0 + 1 \end{pmatrix} E_1 = f''(0)e^{-2i\omega_0\tau_j}\begin{pmatrix} q_1^2 \\ q_2^2 \\ w_1 + w_2q_1^2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & -f'(0) & 0 \\ 0 & 1 & -f'(0) \\ -w_1f'(0) & -w_2f'(0) & 1 \end{pmatrix} E_2 = f''(0)\begin{pmatrix} \overline{q}_1q_1 \\ \overline{q}_2q_2 \\ w_1 + w_2\overline{q}_1q_1 \end{pmatrix}. \tag{3.18}$$

Therefore, all g_{ij} in (3.16) can be expressed in terms of parameters. And we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_j}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2\right) - \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_j)\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= -\frac{\operatorname{im}\{c_1(0)\} + \mu_2 \operatorname{im}\{\lambda'(\tau_j)\}}{\omega_0\tau_j}, \quad j = 0, 1, 2, \ldots, \end{aligned} \tag{3.19}$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical values τ_j, that is, μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_j(\tau < \tau_j)$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$; T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(T_2 < 0)$.

4. Computer Simulation

In this section, we will confirm our theoretical analysis by numerical simulation. We give an example of system (3.1) with $w_1 = 1$, $w_2 = -1$, and $f(\cdot) = \tanh(\cdot)$. Then, $f(0) = 0$ and $f'(0) = 1$.

Equation (1.3) becomes

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \tanh(x_2(t-\tau)), \\ \dot{x}_2(t) &= -x_2(t) + \tanh(x_3(t-\tau)), \\ \dot{x}_3(t) &= -x_3(t) + \tanh(x_1(t-\tau)) - \tanh(x_2(t-\tau)). \end{aligned} \tag{4.1}$$

From (2.6), we have $\Delta = 0.2870$, $\alpha = 1.0118$, $\beta = -0.3295$. By Lemma 2.1, we know (2.7) has roots $z_1 = \alpha + \beta = 0.6823$ and $z_{2,3} = -(\alpha+\beta)/2 \pm i(\sqrt{3}/2)(\alpha - \beta) = -0.3412 \pm 1.1615i$. Clearly, $|z_1| = 0.6823 < 1$, $|z_{2,3}| = 1.4655 > 1$. From (2.12), it follows that $\omega = \sqrt{|z_{2,3}|^2 - 1} = 0.6823$, and, from (2.13), we get $\tau_0 = 1.8434$. Thus, the zero equilibrium is asymptotically stable when $\tau \in [0, \tau_0)$ as is illustrated by the computer simulations (see Figures 2(a)–2(d)). When τ passes through the critical value τ_0, zero equilibrium loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcates from the origins $(0, 0, 0)$, which are depicted in Figures 3(a)–3(d).

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