Research Article

Uniformly Strong Persistence for a Delayed Predator-Prey Model

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Received 6 July 2012; Accepted 18 September 2012

Academic Editor: Wan-Tong Li

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An asymptotically periodic predator-prey model with time delay is investigated. Some sufficient conditions for the uniformly strong persistence of the system are obtained. Our result is an important complementarity to the earlier results.

1. Introduction

solutions of a predator-prey system with monotone functional responses. One can see \[10–19\] and so forth for more related studies. However, the research work on asymptotically periodic predator-prey model is very few at present.

The so-called asymptotically periodic function is that a function \( \bar{a}(t) \) can be expressed by the form \( \bar{a}(t) = a(t) + \tilde{a}(t) \), where \( a(t) \) is a periodic function and \( \tilde{a}(t) \) satisfies \( \lim_{t \to +\infty} \tilde{a}(t) = 0 \).

In 2006, Kar and Batabyal \[20\] investigated the stability and bifurcation of the following predator-prey model with time delay

\[
\frac{dx}{dt} = x \left[ r - \frac{r}{K} x - \frac{a_1 y}{a_1 + x} - \frac{a_2 z}{a_2 + x} \right],
\]

\[
\frac{dy}{dt} = y \left[ -d_1 + \frac{\beta_1 a_1 x(t-\tau)}{a_1 + x(t-\tau)} - \gamma y \right],
\]

\[
\frac{dz}{dt} = z \left[ -d_2 + \frac{\beta_2 a_2 x(t-\tau)}{a_2 + x(t-\tau)} - \delta z \right],
\]

with initial conditions \( x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \), where \( z(t) \) denotes the densities of prey; \( y(t) \) and \( z(t) \) denote the densities of two predators, respectively, at time \( t \); \( \gamma \) and \( \delta \) denote the intraspecific competition coefficients of the predators; \( \beta_1 \) and \( \beta_2 \) denote the conversion of biomass constant; \( d_1 \) and \( d_2 \) are the death rate of first and second predator species, respectively; \( a_1 \) is the maximum values of per capita reduction rate of \( x \) due to \( y \) and \( a_2 \) is the maximum values of per capita reduction rate of \( x \) due to \( z \); \( a_1 \) and \( a_2 \) are half saturation constants. \( \tau \) is time delay in the prey species. All the parameters are positive constants. For details, one can see \[20\].

It will be pointed out that all biological and environment parameters in model (1.1) are constants in time. However, any biological or environmental parameters are naturally subject to fluctuation in time. Thus the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity the environment (such as seasonal effects of weather, food supplies, and mating habits). Inspired by above considerations and considering the asymptotically periodic function, in this paper, we will modify system (1.1) as follows:

\[
\frac{dx}{dt} = x \left[ r(t) + \bar{r}(t) - \frac{r(t)}{K(t) + \bar{K}(t)} x - \frac{(a_1(t) + \tilde{a}_1(t))y}{a_1(t) + \tilde{a}_1(t) + x} - \frac{(a_2(t) + \tilde{a}_2(t))z}{a_2(t) + \tilde{a}_2(t) + x} \right],
\]

\[
\frac{dy}{dt} = y \left[ -(d_1(t) + \tilde{d}_1(t)) + \frac{\left( \beta_1(t) + \tilde{\beta}_1(t) \right) (a_1(t) + \tilde{a}_1(t)) x(t-\tau)}{a_1(t) + \tilde{a}_1(t) + x(t-\tau)} - \gamma(t) + \tilde{\gamma}(t) \right] y, \tag{1.2}
\]

\[
\frac{dz}{dt} = z \left[ -(d_2(t) + \tilde{d}_2(t)) + \frac{\left( \beta_2(t) + \tilde{\beta}_2(t) \right) (a_2(t) + \tilde{a}_2(t)) x(t-\tau)}{a_2(t) + \tilde{a}_2(t) + x(t-\tau)} - \delta(t) + \tilde{\delta}(t) \right] z,
\]

with initial conditions \( x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \).
The principle object of this paper is to explore the uniformly strong persistence of system (1.2). There are very few papers which deal with this topic, see [10, 21].

In order to obtain our results, we always assume that system (1.2) satisfies (H1) $\alpha_i(t)$, $\beta_i(t)$, $a_i(t)$, $d_i(t)$ ($i = 1, 2$), $r(t)$, $\gamma(t)$, $\delta(t)$, $K(t)$ are continuous, nonnegative periodic functions; $\bar{\alpha}(t)$, $\bar{\beta}(t)$, $\bar{a}(t)$, $\bar{d}(t)$ ($i = 1, 2$), $\bar{R}(t)$, $\bar{\gamma}(t)$, $\bar{\delta}(t)$, $\bar{K}(t)$ are continuous, nonnegative asymptotically items of asymptotically periodic functions.

2. Uniformly Strong Persistence

In this section, we will present some result about the uniformly strong persistence of system (1.2). For convenience and simplicity in the following discussion, we introduce the notations, definition, and Lemmas. Let

$$0 < f^l = \lim_{t \to +\infty} \inf \, f(t) \leq \lim_{t \to +\infty} \sup \, f(t) = f^u < +\infty. \tag{2.1}$$

In view of the definitions of lower limit and upper limit, it follows that for any $\varepsilon > 0$, there exists $T > 0$ such that

$$f^l - \varepsilon \leq f(t) \leq f^u + \varepsilon, \quad \text{for } t \geq T. \tag{2.2}$$

**Definition 2.1.** The system (1.2) is said to be strong persistence, if every solution $x(t)$ of system (1.2) satisfies

$$0 < \lim_{t \to +\infty} \inf \, x(t) \leq \lim_{t \to +\infty} \sup \, x(t) \leq \delta < +\infty. \tag{2.3}$$

**Lemma 2.2.** Both the positive and nonnegative cones of $\mathbb{R}^2$ are invariant with respect to system (1.2).

It follows from Lemma 2.2 that any solution of system (1.2) with a nonnegative initial condition remains nonnegative.

**Lemma 2.3** (see [10]). If $a > 0$, $b > 0$, and $\dot{x}(t) \geq (\leq)x(t)(b - ax^a(t))$, where $a$ is a positive constant, when $t \geq 0$ and $x(0) > 0$, we have

$$x(t) \geq \left( \frac{b}{a} \right)^{1/a} \left[ 1 + \left( \frac{bx^{-a}(0)}{a} - 1 \right) e^{-bat} \right]^{-1/a}. \tag{2.4}$$

In the following, we will be ready to state our result.

**Theorem 2.4.** Let $P_1$, $P_2$, $P_3$, and $Q_1$ be defined by (2.7), (2.10), (2.13), and (2.16), respectively. Assume that conditions (H1) and

(H2) $\alpha_i^u \beta_i^u > d_i^u$ ($i = 1, 2$), $r' \alpha_i^l \alpha_i^l > a_i^u a_i^l P_2 + a_i^l a_i^u P_3$,

(H3) $\alpha_i^l \beta_i^l Q_1 > d_i^u (a_i^u + P_1)$, $\alpha_i^l \beta_i^l Q_1 > d_i^u (a_i^u + P_1)$

hold, then system (1.2) is uniformly strong persistence.
Proof. It follows from (2.2) that for any \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that for \( t \geq T_1 \),

\[
\begin{align*}
    r^l - \varepsilon & \leq r(t) \leq r^u + \varepsilon, \quad -\varepsilon < \bar{r}(t) < \varepsilon, \\
    K^l - \varepsilon & \leq K(t) \leq K^u + \varepsilon, \quad -\varepsilon < \bar{K}(t) < \varepsilon, \\
    a_1^l - \varepsilon & \leq a_1(t) \leq a_1^u + \varepsilon, \quad -\varepsilon < \bar{a}_1(t) < \varepsilon, \\
    a_2^l - \varepsilon & \leq a_2(t) \leq a_2^u + \varepsilon, \quad -\varepsilon < \bar{a}_2(t) < \varepsilon.
\end{align*}
\]

(2.5)

Substitute (2.5) into the first equation of system (1.2), then we have

\[
\begin{align*}
    \frac{dx}{dt} &= x \left[ r(t) + \bar{r}(t) - \frac{r(t) + \bar{r}(t)}{K(t) + \bar{K}(t)} x - \frac{(a_1(t) + \bar{a}_1(t)) y}{a_1(t) + \bar{a}_1(t) + x} - \frac{(a_2(t) + \bar{a}_2(t)) y}{a_2(t) + \bar{a}_2(t) + x} \right] \\
    &\leq x \left[ r(t) + \bar{r}(t) - \frac{r(t) + \bar{r}(t)}{K(t) + \bar{K}(t)} x \right] \leq x(t) \left[ r^u + 2\varepsilon - \frac{r^l - 2\varepsilon}{K^u + 2\varepsilon} x(t) \right].
\end{align*}
\]

(2.6)

By Lemma 2.3, we get

\[
\lim_{t \to +\infty} \sup_{t \geq 0} x(t) \leq \frac{r^u K^u}{r^l} := P_1.
\]

(2.7)

Then for any \( \varepsilon > 0 \), there exists \( T_2 > T_1 > 0 \) such that

\[
x(t) \leq P_1 + \varepsilon, \quad t \geq T_2.
\]

(2.8)

Similarly, from (2.2) and the second equation of system (1.2), we obtain that for any \( \varepsilon > 0 \), there exists \( T_3 > T_2 > 0 \) such that

\[
\begin{align*}
    \dot{y}(t) &= y \left[ -\left( d_1(t) + \bar{d}_1(t) \right) + \frac{\left( \beta_1(t) + \bar{\beta}_1(t) \right) (a_1(t) + \bar{a}_1(t)) x(t - \tau)}{a_1(t) + \bar{a}_1(t) + x(t - \tau)} - (\gamma(t) + \bar{\gamma}(t)) \right] \\
    &\leq y(t) \left[ -\left( d_1^l - 2\varepsilon \right) - \left( \gamma^l - 2\varepsilon \right) y(t) + (\beta_1^u + 2\varepsilon) (a_1^u + 2\varepsilon) \right].
\end{align*}
\]

(2.9)

In view of Lemma 2.3, we derive

\[
\lim_{t \to +\infty} \sup_{t \geq 0} y(t) \leq \frac{\alpha_1^u \beta_1^u - d_1^l}{\gamma^l} := P_2.
\]

(2.10)
Then for any $\varepsilon > 0$, there exists $T_4 > T_3 > 0$ such that

$$y(t) \leq P_2 + \varepsilon, \quad t \geq T_4.$$  \hspace{1cm} (2.11)

From (2.2) and the third equation of system (1.2), we obtain that for any $\varepsilon > 0$, there exists $T_5 > T_4 > 0$ such that

$$z(t) = z \left[ - (d_2(t) + \bar{a}_{2}(t)) + \frac{\left( \beta_2(t) + \tilde{\beta}_2(t) \right) (a_2(t) + \bar{a}_{2}(t)) x(t - \tau)}{a_2(t) + \bar{a}_{2}(t) + x(t - \tau)} - \left( \delta(t) + \tilde{\delta}(t) \right) z \right]$$

$$\leq z(t) \left[ - (d^*_2 - 2\varepsilon) - (\delta^* - 2\varepsilon) z(t) + \left( \beta^*_2 + 2\varepsilon \right) \left( a^*_2 + 2\varepsilon \right) \right].$$  \hspace{1cm} (2.12)

In view of Lemma 2.3, we derive

$$\lim_{t \to +\infty} \sup z(t) \leq \frac{a^*_2 \beta^*_2 - d^*_2}{\delta^*} := P_3.$$  \hspace{1cm} (2.13)

Then for any $\varepsilon > 0$, there exists $T_6 > T_5 > 0$ such that

$$z(t) \leq P_3 + \varepsilon, \quad t \geq T_6.$$  \hspace{1cm} (2.14)

According (2.8), (2.11), (2.14) and the first equation of system (1.2), we obtain that for any $\varepsilon > 0$, there exists $T_7 > T_6 > 0$ such that

$$\frac{dx}{dt} = x \left[ r(t) + \bar{r}(t) - \frac{r(t) + \tilde{r}(t)}{K(t) + \tilde{K}(t)} x - \frac{(a_1(t) + \bar{a}_1(t)) y}{a_1(t) + \bar{a}_1(t) + x} - \frac{(a_2(t) + \bar{a}_2(t)) z}{a_2(t) + \bar{a}_2(t) + x} \right]$$

$$\geq x(t) \left[ r^* + 2\varepsilon \right] - \frac{r^* + 2\varepsilon}{K^* - 2\varepsilon} x(t) - \frac{(a^*_1 + 2\varepsilon)(P_2 + \varepsilon)}{a^*_1 - 2\varepsilon} - \frac{(a^*_2 + 2\varepsilon)(P_3 + \varepsilon)}{a^*_2 - 2\varepsilon}.$$  \hspace{1cm} (2.15)

Using Lemma 2.3 again, we have

$$\lim_{t \to +\infty} \inf x(t) \geq \frac{K^* (r^* a^*_1 a^*_2 - a^*_1 a^*_2 P_2 - a^*_1 a^*_2 P_3)}{a^*_1 a^*_2 r^*} := Q_1.$$  \hspace{1cm} (2.16)

Thus for any $\varepsilon > 0$, there exists $T_8 > T_7 > 0$ such that

$$x(t) \geq Q_1 - \varepsilon.$$  \hspace{1cm} (2.17)
According (2.8), (2.11), (2.14) and the second equation of system (1.2), we obtain that for any $\varepsilon > 0$, there exists $T_9 > T_8 > 0$ such that

$$\dot{y}(t) = y \left[ -\left( d_1(t) + \tilde{d}_1(t) \right) + \frac{\left( \beta_1(t) + \tilde{\beta}_1(t) \right) (a_1(t) + \tilde{a}_1(t)) x(t - \tau) }{a_1(t) + \tilde{a}_1(t) + x(t - \tau)} - (\gamma(t) + \tilde{\gamma}(t)) y \right]$$

$$\geq y(t) \left[ -\left( d_1^u + 2\varepsilon \right) - (\gamma^u + 2\varepsilon) y(t) + \frac{\left( \beta_1^l - 2\varepsilon \right) \left( a_1^l - 2\varepsilon \right) (Q_1 - \varepsilon) }{a_1^u + 2\varepsilon + P_1 + \varepsilon} \right].$$

(2.18)

Using Lemma 2.3 again, we have

$$\lim_{t \to +\infty} \inf y(t) \geq \frac{\beta_1^l a_1^l Q_1 - d_1^u (a_1^u + P_1) }{\gamma^u (a_1^u + P_1)} := Q_2.$$

(2.19)

Thus for any $\varepsilon > 0$, there exists $T_{10} > T_9 > 0$ such that

$$y(t) \geq Q_2 - \varepsilon.$$

(2.20)

According (2.8), (2.11), (2.14) and the third equation of system (1.2), we obtain that for any $\varepsilon > 0$, there exists $T_{11} > T_{10} > 0$ such that

$$\dot{z}(t) = z \left[ -\left( d_2(t) + \tilde{d}_2(t) \right) + \frac{\left( \beta_2(t) + \tilde{\beta}_2(t) \right) (a_2(t) + \tilde{a}_2(t)) x(t - \tau) }{a_2(t) + \tilde{a}_2(t) + x(t - \tau)} - (\delta(t) + \tilde{\delta}(t)) z \right]$$

$$\geq z(t) \left[ -\left( d_2^u + 2\varepsilon \right) - (\delta^u + 2\varepsilon) z(t) + \frac{\left( \beta_2^l - 2\varepsilon \right) \left( a_2^l - 2\varepsilon \right) (Q_1 - \varepsilon) }{a_2^u + 2\varepsilon + P_1 + \varepsilon} \right].$$

(2.21)

Using Lemma 2.3 again, we have

$$\lim_{t \to +\infty} \inf z(t) \geq \frac{\beta_2^l a_2^l Q_1 - d_2^u (a_2^u + P_1) }{\delta^u (a_2^u + P_1)} := Q_3.$$

(2.22)

Thus the proof of Theorem 2.4 is complete.

\[\square\]

**Acknowledgments**

This work is supported by National Natural Science Foundation of China (no. 11261010 and no. 11161015), Soft Science and Technology Program of Guizhou Province (no. 2011LKC2030), Natural Science and Technology Foundation of Guizhou Province ([J[2012]2100), Governor Foundation of Guizhou Province ([2012]53), and Doctoral Foundation of Guizhou University of Finance and Economics (2010).
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