Research Article

Lattices Generated by Orbits of Subspaces under Finite Singular Orthogonal Groups II

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Received 28 March 2012; Accepted 15 June 2012

Academic Editor: Ch Tsitouras

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Let $\mathbb{F}_q^{(2\nu+\delta+l)}$ be a $(2\nu+\delta+l)$ -dimensional vector space over the finite field $\mathbb{F}_q$. In this paper we assume that $\mathbb{F}_q$ is a finite field of odd characteristic, and $O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q)$ the singular orthogonal groups of degree $2\nu + \delta + l$ over $\mathbb{F}_q$. Let $\mathcal{M}$ be any orbit of subspaces under $O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q)$. Denote by $\mathcal{L}$ the set of subspaces which are intersections of subspaces in $\mathcal{M}$, where we make the convention that the intersection of an empty set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is assumed to be $\mathbb{F}_q^{(2\nu+\delta+l)}$. By ordering $\mathcal{L}$ by ordinary or reverse inclusion, two lattices are obtained. This paper studies the questions when these lattices $\mathcal{L}$ are geometric lattices.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is an odd prime power. We choose a fixed nonsquare element z in $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^{(2\nu+\delta+l)}$ be a $(2\nu + \delta + l)$ -dimensional row vector space over the finite field $\mathbb{F}_q$, and let $O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q)$ be one of the singular orthogonal groups of degree $2\nu + \delta + l$ over $\mathbb{F}_q$. There is an action of $O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu+\delta+l)}$ defined as follows:

$$\begin{aligned} \mathbb{F}_q^{(2\nu+\delta+l)} \times O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{(2\nu+\delta+l)}, \\ ((x_1, x_2, \dots, x_{2\nu+\delta+l}), T) &\longmapsto (x_1, x_2, \dots, x_{2\nu+\delta+l})T. \end{aligned} \quad (1.1)$$

Let P be an m -dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta+l)}$ ($1 \leq m \leq 2\nu + \delta + l$), and $v_1, v_2, \dots, v_m$ be

a basis of P . Then, the $m \times (2\nu + \delta + l)$ matrix:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \quad (1.2)$$

is called a *matrix representation* of P . We usually denote a matrix representation of the m -dimensional subspace P still by P . The above action induces an action on the set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$, that is, a subspace P is carried by $T \in O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ into the subspace PT . The set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is partitioned into orbits under $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Clearly, $\{0\}$ and $\{\mathbb{F}_q^{(2\nu+\delta+l)}\}$ are two trivial orbits. Let $\mathcal{M}$ be any orbit of subspaces under $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Denote the set of subspaces which are intersections of subspaces in $\mathcal{M}$ by $\mathcal{L}(\mathcal{M})$ and call $\mathcal{L}(\mathcal{M})$ the set of subspaces generated by $\mathcal{M}$. We agree that the intersection of an empty set of subspaces is $\mathbb{F}_q^{(2\nu+\delta+l)}$. Then, $\mathbb{F}_q^{(2\nu+\delta+l)} \in \mathcal{L}(\mathcal{M})$. Partially ordering $\mathcal{L}(\mathcal{M})$ by ordinary or reverse inclusion, we get two posets and denote them by $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$, respectively. Clearly, for any two elements $P, Q \in \mathcal{L}_O(\mathcal{M})$,

$$P \wedge Q = P \cap Q, \quad P \vee Q = \cap \{R \in \mathcal{L}_O(\mathcal{M}) : R \supseteq \langle P, Q \rangle\}, \quad (1.3)$$

where $\langle P, Q \rangle$ is a subspace generated by P and Q . Therefore, $\mathcal{L}_O(\mathcal{M})$ is a finite lattice.

Similarly, for any two elements $P, Q \in \mathcal{L}_R(\mathcal{M})$,

$$P \wedge Q = \cap \{R \in \mathcal{L}_R(\mathcal{M}) : R \supseteq \langle P, Q \rangle\}, \quad P \vee Q = P \cap Q, \quad (1.4)$$

so $\mathcal{L}_R(\mathcal{M})$ is also a finite lattice. Both $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ are called the lattices generated by $\mathcal{M}$.

The results on the geometricity of lattices generated by subspaces in d -bounded distance-regular graphs can be found in Guo et al. [1]; on the geometricity and the characteristic polynomial of lattices generated by orbits of flats under finite affine-classical groups can be found in Wang and Feng [2], Wang and Guo [3]; on inclusion relations, the geometricity and the characteristic polynomial of lattices generated by orbits of subspaces under finite nonsingular classical groups and a characterization of subspaces contained in lattices can be found in Huo [4–6], Huo and Wan [7, 8]; on inclusion relations, the geometricity and the characteristic polynomial of lattices generated by orbits of subspaces under finite singular symplectic groups, singular unitary groups, and singular pseudosymplectic groups and a characterization of subspaces contained in lattices can be found in Gao and You [9–12]. In [13], the authors studied the various lattices $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ generated by different orbits $\mathcal{M}$ of subspaces under singular orthogonal group $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. The study contents include the inclusion relations between different lattices, the characterization of subspaces contained in a given lattice $\mathcal{L}_R(\mathcal{M})$ (resp., $\mathcal{L}_O(\mathcal{M})$), and the characteristic polynomial of $\mathcal{L}_R(\mathcal{M})$. The purpose of this paper is to study the questions when $\mathcal{L}_R(\mathcal{M})$ (resp., $\mathcal{L}_O(\mathcal{M})$) are geometric lattices.

2. Preliminaries

In the following, we recall some definitions and facts on ordered sets and lattices (see [8, 14]).

Let A be a partially ordered set, and $a, b \in A$. We say that b *covers* a and write $a < \cdot b$, if $a < b$ and there exists no $c \in A$ such that $a < c < b$. An element $m \in A$ is called the *minimal element* if there exists no elements $a \in A$ such that $a < m$. If A has a unique minimal element, denote it by 0 and we say that A is a poset with 0.

Let A be a poset with 0 and $a \in A$. If all maximal ascending chains starting from 0 with endpoint a have the same finite length, this common length is called the *rank* $r(a)$ of a . If rank $r(a)$ is defined for every $a \in A$, A is said to have the rank function $r : A \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set consisting of all positive integers and 0.

A poset A is said to satisfy the *Jordan-Dedekind (JD) condition* if any two maximal chains between the same pair of elements of A have the same finite length.

Proposition 2.1 ([14, Proposition 2.1]). *Let A be a poset with 0. If A satisfies the JD condition then A has the rank function $r : A \rightarrow \mathbb{N}$ which satisfies*

- (i) $r(0) = 0$,
- (ii) $a < \cdot b \Rightarrow r(b) = r(a) + 1$.

Conversely, if A admits a function $r : A \rightarrow \mathbb{N}$ satisfying (i) and (ii), then A satisfies the JD condition with r as its rank function.

Let A be a poset with 0. An element $a \in A$ is called an *atom* of A if $0 < \cdot a$. A lattice L with 0 is called an *atomic lattice* (or a *point lattice*) if every element $a \in L \setminus \{0\}$ is a supremum of atoms, that is, $a = \sup\{b \in L \mid 0 < \cdot b \leq a\}$.

Definition 2.2 ([14, page 46]). A lattice L is called a *semimodular lattice* if for all $a, b \in L$,

$$a \wedge b < \cdot a \Rightarrow b < \cdot a \vee b. \quad (2.1)$$

Proposition 2.3 ([14, Theorem 2.27]). *Let L be a lattice with 0. Then, L is a semimodular lattice if and only if L possesses a rank function r such that for all $x, y \in L$*

$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y). \quad (2.2)$$

Definition 2.4 ([14, page 52]). A lattice L is called a *geometric lattice* if it is

- G'_1 an atomic lattice,
- G'_2 a semimodular lattice,
- G_3 without infinite chains in L .

According to Definition 2.2, Proposition 2.3, and Definition 2.4, we can obtain the following proposition.

Proposition 2.5. *Let L be a lattice with 0. Then, L is a geometric lattice if and only if*

- G_1 for every element $a \in L \setminus \{0\}$, $a = \sup\{b \in L \mid 0 < \cdot b \leq a\}$,
- G_2 L possesses a rank function r and for all $x, y \in L$, (2.2) holds,

G_3 without infinite chains in L .

Let

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \\ & & \Delta \end{pmatrix}, \quad S_l = \begin{pmatrix} S & \\ & 0^{(l)} \end{pmatrix}, \quad (2.3)$$

where $S = S_{2\nu+\delta,\Delta}$, $\delta = 0, 1$, or 2 , and

$$\Delta = \begin{cases} \phi, & \text{if } \delta = 0, \\ 1 \text{ or } z, & \text{if } \delta = 1, \\ \begin{pmatrix} 1 & \\ & -z \end{pmatrix}, & \text{if } \delta = 2. \end{cases} \quad (2.4)$$

The set of all $(2\nu + \delta + l) \times (2\nu + \delta + l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying

$$TS_l T^t = S_l \quad (2.5)$$

forms a group which will be called the *singular orthogonal group* of degree $2\nu + \delta + l$, rank $2\nu + \delta$, and with definite part Δ over $\mathbb{F}_q$ and denoted by $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Clearly, $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ consists of all $(2\nu + \delta + l) \times (2\nu + \delta + l)$ nonsingular matrices of the form:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{matrix} 2\nu + \delta \\ l \\ 2\nu + \delta \end{matrix} \begin{matrix} \\ l \\ \end{matrix}, \quad (2.6)$$

where $T_{11} S T_{11}^t = S$, and T_{22} is nonsingular.

Two $n \times n$ matrices A and B are called to be *cogredient* if there exists a nonsingular matrix P such that $PAP^t = B$.

An m -dimensional subspace P is said to be a *subspace of type* $(m, 2s + \gamma, s, \Gamma)$, if $PS_l P^t$ is cogredient to $M(m, 2s + \gamma, s, \Gamma)$, where the matrix $M(m, 2s + \gamma, s, \Gamma)$, respectively, is as follows

$$M(m, 2s, s) = \begin{pmatrix} 0 & I^{(s)} \\ I^{(s)} & 0 \\ & & 0^{(m-2s)} \end{pmatrix}, \quad \text{if } \gamma = 0, \quad (2.7)$$

$$M(m, 2s + 1, s, 1) = \begin{pmatrix} 0 & I^{(s)} \\ I^{(s)} & 0 \\ & & 1 \\ & & & 0^{(m-2s-1)} \end{pmatrix}, \quad \text{if } \gamma = 1$$

or

$$M(m, 2s+1, s, z) = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & z & \\ & & & 0^{(m-2s-1)} \end{pmatrix}, \quad \text{if } \gamma = 1,$$

$$M(m, 2s+2, s) = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & 1 & \\ & & & -z \\ & & & & 0^{(m-2s-2)} \end{pmatrix}, \quad \text{if } \gamma = 2. \quad (2.8)$$

Let $e_1, e_2, \dots, e_{2\nu+\delta}, e_{2\nu+\delta+1}, \dots, e_{2\nu+\delta+l}$ be a basis of $\mathbb{F}_q^{(2\nu+\delta+l)}$, where

$$e_i = (0, \dots, 0, 1, 0, \dots, 0), \quad (2.9)$$

1 is in the i th position. Denote by E the l -dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta+l)}$ generated by $e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \dots, e_{2\nu+\delta+l}$. An m -dimensional subspace P is called a *subspace of type* $(m, 2s + \gamma, s, \Gamma, k)$ if

(i) P is a subspace of type $(m, 2s + \gamma, s, \Gamma)$,

(ii) $\dim(P \cap E) = k$.

Denote the set of all subspaces of type $(m, 2s + \gamma, s, \Gamma, k)$ in $\mathbb{F}_q^{(2\nu+\delta+l)}$ by $\mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$. By [15, Theorem 6.28], we know that $\mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is nonempty if and only if

$$k \leq l,$$

$$2s + \gamma \leq m - k \leq \begin{cases} \nu + s + \min\{\delta, \gamma\}, \\ \text{if } \gamma \neq \delta \text{ or } \gamma = \delta \text{ and } \Gamma = \Delta, \\ \nu + s, \\ \text{if } \gamma = \delta = 1 \text{ and } \Gamma \neq \Delta, \end{cases} \quad (2.10)$$

or

$$\min\{l, m - 2s - \gamma\} \geq k \geq \begin{cases} \max\{0, m - \nu - s - \min\{\delta, \gamma\}\}, \\ \text{if } \gamma \neq \delta \text{ or } \gamma = \delta \text{ and } \Gamma = \Delta, \\ \max\{0, m - \nu - s\}, \\ \text{if } \gamma = \delta = 1 \text{ and } \Gamma \neq \Delta. \end{cases} \quad (2.11)$$

Moreover, if $\mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is nonempty, then it forms an orbit of subspaces under $O_{2\nu+\delta+l, \Delta}(\mathbb{F}_q)$. Let $\mathcal{L}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ denote the set of subspaces which are intersections of subspaces in $\mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, where we make the

convention that the intersection of an empty set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is assumed to be $\mathbb{F}_q^{(2\nu+\delta+l)}$. Partially ordering $\mathcal{L}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ by ordinary or reverse inclusion, we get two finite lattices and denote them by $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ and $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, respectively.

The case $\mathcal{L}_R(m - l, 2s + \gamma, s, \Gamma; 2\nu + \delta, \Delta)$ has been discussed in [8]. So, we only discuss the case $0 \leq k < l$ in this paper.

By [13], we have the following results.

Theorem 2.6. *Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$, assume that $(m, 2s + \gamma, s, \Gamma, k)$ satisfies conditions (2.10) and (2.11). Then,*

$$\mathcal{L}_R(m, 2s + r, s, \Gamma, k; 2\nu + \delta + l, \Delta) \supset \mathcal{L}_R(m_1, 2s_1 + \gamma_1, s_1, \Gamma_1, k_1; 2\nu + \delta + l, \Delta) \quad (2.12)$$

if and only if

$$k_1 \leq k < l, \quad (2.13)$$

$$2(m - k) - 2(m_1 - k_1) \geq \begin{cases} (2s + \gamma) - (2s_1 + \gamma_1) + |\gamma - \gamma_1| \geq 2|\gamma - \gamma_1|, \\ \text{if } \gamma_1 \neq \gamma \text{ or } \gamma_1 = \gamma \text{ and } \Gamma_1 = \Gamma, \\ (2s + \gamma) - (2s_1 + \gamma_1) + 2 \geq 4, \\ \text{if } \gamma_1 = \gamma = 1 \text{ and } \Gamma_1 \neq \Gamma. \end{cases}$$

Theorem 2.7. *Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$. Assume that $(m, 2s + \gamma, s, \Gamma, k)$ satisfies condition (2.10), then $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ consists of $\mathbb{F}_q^{(2\nu+\delta+l)}$ and all the subspaces of type $(m_1, 2s_1 + \gamma_1, s_1, \Gamma_1, k_1)$, where $(m_1, 2s_1 + \gamma_1, s_1, \Gamma_1, k_1)$ satisfies condition (2.13).*

Theorem 2.8. *Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$, and $(m, 2s + \gamma, s, \Gamma, k)$ satisfy*

$$2s + \gamma \leq m - k \leq \begin{cases} \nu + s + \min\{\delta, \gamma\}, \\ \text{if } \gamma \neq \delta \text{ or } \gamma = \delta \text{ and } \Gamma = \Delta, \\ \nu + s, \\ \text{if } \gamma = \delta = 1 \text{ and } \Gamma \neq \Delta. \end{cases} \quad (2.14)$$

For any $X \in \mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, define

$$r(X) = \begin{cases} \dim X, & \text{if } X \neq \mathbb{F}_q^{(2\nu+\delta+l)}, \\ m + 1, & \text{if } X = \mathbb{F}_q^{(2\nu+\delta+l)}, \end{cases} \quad (2.15)$$

then $r : \mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta) \rightarrow \mathbb{N}$ is a rank function of the lattice $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$.

Theorem 2.9. Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$, and $(m, 2s + \gamma, s, \Gamma, k)$ satisfy (2.14). For any $X \in \mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, define

$$r'(X) = \begin{cases} m + 1 - \dim X, & \text{if } X \neq \mathbb{F}_q^{(2\nu+\delta+l)}, \\ 0, & \text{if } X = \mathbb{F}_q^{(2\nu+\delta+l)}, \end{cases} \quad (2.16)$$

then $r' : \mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta) \rightarrow \mathbb{N}$ is a rank function of the lattice $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$.

3. The Geometricity of Lattices $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$

Theorem 3.1. Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$, assume that $(m, 2s + \gamma, s, \Gamma, k)$ satisfies conditions (2.10) and (2.11). Then

- (i) each of $\mathcal{L}_O(k+1, 0, 0, \phi, k; 2\nu+\delta+l, \Delta)$ and $\mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$ ($\Gamma = 1$ or z) is a finite geometric lattice, when $k = 0$, and is a finite atomic lattice, but not a geometric lattice when $0 < k < l$;
- (ii) when $2 \leq m - k \leq 2\nu + \delta - 1$, $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is a finite atomic lattice, but not a geometric lattice.

Proof. By Theorem 2.8, the rank function of $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is defined by formula (2.15), we will show the condition G_1 of Proposition 2.5 holds for $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$. $\{0\} \in \mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ and it is the minimal element, so all 1-dim subspaces in $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ are atoms of $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$.

Let $U \in \mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta) \setminus \{\{0\}, \mathbb{F}_q^{(2\nu+\delta+l)}\}$, by Theorem 2.7, U is a subspace of type $(m_1, 2s_1 + \gamma_1, s_1, \Gamma_1, k_1)$ and satisfies condition (2.13). If $m_1 = 1$, then U is an atom of $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$. Assume $m_1 \geq 2$, then

$$US_l U^t = [S_{2s_1+\gamma_1, \Gamma_1}, 0^{(m_1-k_1-2s_1-\gamma_1)}, 0^{(k_1)}], \quad (3.1)$$

where $\Gamma_1 = \phi, (1), (z)$, or $[1, -z]$.

Let U_i be an i th ($1 \leq i \leq m_1$) row vector of U , then $\langle U_i \rangle$ is a subspace of type $(1, 0, 0, \phi, 0), (1, 1, 0, 1, 0), (1, 1, 0, z, 0)$, or $(1, 0, 0, 0, 1)$, and $\langle U_i \rangle \subset U$. By Theorem 2.7, we know $\langle U_i \rangle \in \mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, so $\langle U_i \rangle$ is an atom of $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, and $U = \vee_{i=1}^{m_1} \langle U_i \rangle$, hence, U is a union of atoms in $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$. Since $|\mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)| \geq 2$, there exist $W_1, W_2 \in \mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, $W_1 \neq W_2$, such that $\mathbb{F}_q^{(2\nu+\delta+l)} = W_1 \vee W_2$. W_1, W_2 are unions of atoms in $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, hence, $\mathbb{F}_q^{(2\nu+\delta+l)}$ is a union of atoms in $\mathcal{L}_O(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, therefore, G_1 holds.

In the following, we prove (i) and (ii).

The Proof of (i). We only prove the formula (2.2) holds for $\mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$. The other can be obtained in the similar way. We consider two cases:

(a) $k = 0$. $\mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$ consists of $\mathbb{F}_q^{(2\nu+\delta+l)}, \{0\}$ and subspaces of type $(1, 1, 0, \Gamma, 0)$. Let $U, W \in \mathcal{L}_O(1, 1, 0, \Gamma, 0; 2\nu+\delta+l, \Delta)$, if U, W are $\mathbb{F}_q^{(2\nu+\delta+l)}, \{0\}$, respectively, then

$U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$. If $U = W$ is $\{0\}$ or $\mathbb{F}_q^{(2\nu+\delta+l)}$, the other is a subspace of type $(1, 1, 0, \Gamma, 0)$, then $U \wedge W$ is $\{0\}$ or subspace of type $(1, 1, 0, \Gamma, 0)$, $U \vee W$ is a subspace of type $(1, 1, 0, \Gamma, 0)$ or $\mathbb{F}_q^{(2\nu+\delta+l)}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$. If U and W are subspaces of type $(1, 1, 0, \Gamma, 0)$, then $U \wedge W = \{0\}$, $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$.

Hence, (2.2) holds and $\mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$ is a finite geometric lattice when $k = 0$.

(b) $0 < k < l$. Let $U = \langle e_1 + (\Gamma/2)e_{\nu+1} \rangle$, $W = \langle e_{s+1} + (\Gamma/2)e_{\nu+s+1} \rangle$, where $s \leq \nu - 1$, then $U, W \in \mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$. When $q = 3 \pmod{4}$ or $q = 1 \pmod{4}$, then -1 is a nonsquare element or a square element, respectively. Thus, $[\Gamma, \Gamma]$ is cogredient to either $[1, -z]$ or $S_{2,1}$, and $\langle U, W \rangle$ is a subspace of type $(2, 2, 0, \Gamma, 0)$, where $\Gamma = [1, -z]$, or a subspace of type $(2, 2, 1, \phi, 0)$. So $\langle U, W \rangle \notin \mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$, and we have $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$. By the definition of rank function, $r(U \vee W) = k+1+1 = k+2$, $r(U \wedge W) = 0$, $r(U) = r(W) = 1$, we have $r(U \vee W) + r(U \wedge W) = k+2 > r(U) + r(W) = 2$.

Hence, $\mathcal{L}_O(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$ is a finite atomic lattice, but not a geometric lattice when $0 < k < l$.

The Proof of (ii). We will show there exist $U, W \in \mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k; 2\nu+\delta+l, \Delta)$ such that the formula (2.2) does not hold. As to $\gamma = 0, 1$, or 2 , we only show the proof of $\gamma = 1$, others can be obtained in the similar way. We distinguish the following three cases.

(a) $\delta = 0$, or $\delta = 1$, $\Gamma \neq \Delta$. Then, the formula (2.10) is changed into $2s+1 \leq m-k \leq \nu+s$. Let $\sigma = \nu + s - m + k$, we distinguish the following two subcases.

(a.1) $m-k-2s-1 \geq 1$. From $m-k-2s-1 \geq 1$ and $m-k \leq \nu+s$, we have $s+2 \leq \nu$. Let

$$U = \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \Gamma/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \quad (3.2)$$

$s \quad 1 \quad 1 \quad \sigma_1 \quad \sigma \quad s \quad 1 \quad 1 \quad \sigma_1 \quad \sigma \quad k \quad l-k$

$$W = \left\langle e_{s+2} + \left(\frac{\Gamma}{2}\right)e_{\nu+s+2} \right\rangle,$$

where $\sigma_1 = m - k - 2s - 2$, then U is a subspace of type $(m-1, 2s+1, s, \Gamma, k)$, W is a subspace of type $(1, 1, 0, \Gamma, 0)$. When $q = 3 \pmod{4}$ or $q = 1 \pmod{4}$, then -1 is a nonsquare element or a square element, respectively, thus $[\Gamma, \Gamma]$ is cogredient to either $[1, -z]$ or $S_{2,1}$, and $\langle U, W \rangle$ is a subspace of type $(m, 2s+2, s, \Gamma, k)$ or type $(m, 2(s+1), s+1, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu+\delta+l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu+\delta+l, \Delta)$. Thus, we have $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$, $r(U \vee W) = m+1$, $r(U \wedge W) = 0$, $r(U) = m-1$, $r(W) = 1$. Then,

$$r(U \vee W) + r(U \wedge W) = m+1 > r(U) + r(W) = m-1+1 = m. \quad (3.3)$$

(a.2) $m - k - 2s - 1 = 0$. From $2 \leq m - k \leq 2\nu + \delta - 1$, we have $s + 1 \leq \nu$, $s \geq 1$. Let

$$U = \begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \Gamma/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \quad (3.4)$$

$s-1 \quad 1 \quad 1 \quad \sigma \quad s \quad 1 \quad \sigma \quad k \quad l-k$

$$W = \left\langle e_{s+1} - \left(\frac{\Gamma}{2}\right) e_{\nu+s+1} \right\rangle,$$

then U is a subspace of type $(m-1, 2(s-1)+1, s-1, \Gamma, k)$, W is a subspace of type $(1, 1, 0, -\Gamma, 0)$, $\langle U, W \rangle$ is a subspace of type $(m, 2s, s, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$, $r(U \vee W) = m+1$, $r(U \wedge W) = 0$, $r(U) = m-1$, $r(W) = 1$. Then,

$$r(U \vee W) + r(U \wedge W) = m+1 > r(U) + r(W) = m-1+1 = m. \quad (3.5)$$

Therefore, there exist $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ such that formula (2.2) does not hold.

(b) $\delta = 1, \Gamma = \Delta$. Then, the formula (2.10) is changed into $2s+1 \leq m - k \leq \nu + s + 1$. Let $\sigma = \nu + s - m + k + 1$, we distinguish the following two subcases.

(b.1) $m - k - 2s - 1 \geq 1$. From $m - k - 2s - 1 \geq 1$, and $2 \leq m - k \leq 2\nu$, we have $s + 1 \leq \nu$. Let

$$U = \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} \end{pmatrix}, \quad (3.6)$$

$s \quad 1 \quad \sigma_1 \quad \sigma \quad s \quad 1 \quad \sigma_1 \quad \sigma \quad 1 \quad k \quad l-k$

$$W = \left\langle e_{s+1} + \left(\frac{\Delta}{2}\right) e_{\nu+s+1} \right\rangle,$$

where $\sigma_1 = m - k - 2s - 2$, then U is a subspace of type $(m-1, 2s+1, s, \Delta, k)$, W is a subspace of type $(1, 1, 0, \Delta, 0)$. When $q = 3(\bmod 4)$ or $q = 1(\bmod 4)$, similar to the proof of the case (a.1), $\langle U, W \rangle$ is a subspace of type $(m, 2s+2, s, \Gamma, k)$ or $(m, 2(s+1), s+1, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Delta, k; 2\nu + 1 + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_O(m, 2s+1, s, \Delta, k; 2\nu + 1 + l, \Delta)$, and the formula (2.2) does not hold.

(b.2) $m - k - 2s - 1 = 0$. From $2 \leq m - k \leq 2\nu$, we have $s + 1 \leq \nu$. Let

$$U = \begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \quad (3.7)$$

$s-1 \quad 1 \quad 1 \quad \sigma-1 \quad s \quad 1 \quad \sigma-1 \quad 1 \quad k \quad l-k$

$$W = \left\langle e_{s+1} + \left(\frac{\Delta}{2}\right)e_{\nu+s+1} \right\rangle,$$

then U is a subspace of type $(m-1, 2(s-1)+1, s-1, \Delta, k)$, W is a subspace of type $(1, 1, 0, \Delta, 0)$, when $q = 3(\bmod 4)$ or $q = 1(\bmod 4)$, $\langle U, W \rangle$ is subspace of type $(m, 2(s-1) + 2, s-1, \Gamma, k)$ or $(m, 2s, s, \phi, k)$. Similar to the proof of the case (a.1), the formula (2.2) does not hold for U and W .

(c) $\delta = 2$. Then, the formula (2.10) is changed into $2s + 1 \leq m - k \leq \nu + s + 1$. Let $\sigma = \nu + s - m + k + 1$, we distinguish the following two subcases.

(c.1) $m - k - 2s - 1 \geq 1$. From $m - k - 2s - 1 \geq 1$, and $m - k \leq 2\nu + 1$, we have $s + 1 \leq \nu$. Let

$$U = \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \quad (3.8)$$

$s \quad 1 \quad \sigma_1 \quad \sigma \quad s \quad 1 \quad \sigma_1 \quad \sigma \quad 1 \quad 1 \quad k \quad l-k$

$$W = \left\langle e_{s+1} + \left(\frac{\Gamma}{2}\right)e_{\nu+s+1} \right\rangle,$$

where $\sigma_1 = m - k - 2s - 2$ and $x^2 - zy^2 = \Gamma$, then U is a subspace of type $(m-1, 2s+1, s, \Gamma, k)$, W is a subspace of type $(1, 1, 0, \Gamma, 0)$. But when $q = 3(\bmod 4)$ or $q = 1(\bmod 4)$, similar to the proof of the case (a.1), $\langle U, W \rangle$ is a subspace of type $(m, 2s+2, s, \Gamma, k)$ or $(m, 2(s+1), s+1, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_O(m, 2s+1, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, and the formula (2.2) does not hold.

(c.2) $m - k - 2s - 1 = 0$. From $2 \leq m - k \leq 2\nu + 1$, we have $s \geq 1$ and $m \geq 3$. We choose (a, b) and (c, d) being two linearly independent solutions of the equation $x^2 - zy^2 = \Gamma$. Let

$$U = \begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0 \end{pmatrix}, \quad (3.9)$$

$s-1 \quad 1 \quad \sigma \quad s \quad \sigma \quad 1 \quad 1 \quad k \quad l-k$

$$W = \langle ce_{2\nu+1} + de_{2\nu+2} \rangle,$$

then U is a subspace of type $(m-1, 2(s-1)+1, s-1, \Gamma, k)$, W is a subspace of type $(1, 1, 0, \Gamma, 0)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \\ & -z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t, \quad (3.10)$$

because $\det A = -(ad - bc)^2 z$, hence, A is cogredient to $[1, -z]$. Then,

$$\begin{pmatrix} U \\ W \end{pmatrix}_{S_l} \begin{pmatrix} U \\ W \end{pmatrix}^t \quad (3.11)$$

is cogredient to

$$\left[S_{2(s-1)+2, \Delta}, o^{(m-k-2s)}, o^{(k)} \right]. \quad (3.12)$$

Therefore, $\langle U, W \rangle$ is a subspace of type $(m, 2(s-1)+2, s-1, \Gamma, k)$. Similar to the proof of the case (a.2), the formula (2.2) does not hold for U and W . $\square$

4. The Geometricity of Lattices $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$

Theorem 4.1. Let $2\nu + \delta + l > m \geq 1$, $0 \leq k < l$, assume that $(m, 2s + \gamma, s, \Gamma, k)$ satisfies conditions (2.10) and (2.11). Then,

- (i) each of $\mathcal{L}_R(k+1, 0, 0, \phi, k; 2\nu + \delta + l, \Delta)$, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k; 2\nu + \delta + l, \Delta)$ ($\Gamma = 1$ or z) and $\mathcal{L}_R(2\nu + \delta + k - 1, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is a finite geometric lattice when $k = 0$, and is a finite atomic lattice, but not a geometric lattice when $0 < k < l$;
- (ii) when $2 \leq m - k \leq 2\nu + \delta - 2$, $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is a finite atomic lattice, but not a geometric lattice.

Proof. By Theorem 2.9, the rank function of $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ is defined by formula (2.16), $\mathbb{F}_q^{(2\nu + \delta + l)}$ is the minimal element of $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, all subspaces of type $(m, 2s + \gamma, s, \Gamma, k)$ in $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ are atoms of $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$.

The Proof of (i). By [8], $\mathcal{L}_R(k+1, 0, 0, \phi, k; 2\nu + \delta + l, \Delta)$, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k; 2\nu + \delta + l, \Delta)$, and $\mathcal{L}_R(2\nu + \delta + k - 1, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$ are finite geometric lattices when $k = 0$; in the following, we will show that they are finite atomic lattices, but not geometric lattices when $0 < k < l$.

(a) Let

$$\begin{aligned} U &= \langle e_{\nu+1}, e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \dots, e_{2\nu+\delta+k} \rangle, \\ W &= \langle e_1, e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \dots, e_{2\nu+\delta+k+1} \rangle. \end{aligned} \quad (4.1)$$

Then, both U and W are subspaces of type $(k+1, 0, 0, \phi, k)$, and $U \cap W = \langle e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \dots, e_{2\nu+\delta+k} \rangle$, $\langle U, W \rangle$ is a subspace of type $(k+3, 2, 1, \phi, k+1)$. Consequently,

$\langle U, W \rangle \notin \mathcal{L}_R(k+1, 0, 0, \phi, k; 2\nu+\delta+l, \Delta)$, $r'(U \wedge W) = r'(\mathbb{F}_q^{(2\nu+\delta+l)}) = 0$, $r'(U \vee W) = r'(U \cap W) = k+2 - (k-1) = 3$, $r'(U) = r'(W) = k+2 - (k+1) = 1$. Thus,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \quad (4.2)$$

That is, (2.2) does not hold for U and W . Hence, $\mathcal{L}_R(k+1, 0, 0, \phi, k; 2\nu+\delta+l, \Delta)$ are not geometric lattices when $0 < k < l$.

(b) Let

$$\begin{aligned} U &= \left\langle e_1 + \left(\frac{\Gamma}{2}\right)e_{\nu+1}, e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \dots, e_{2\nu+\delta+k} \right\rangle, \\ W &= \left\langle e_{s+1} + \left(\frac{\Gamma}{2}\right)e_{\nu+s+1}, e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \dots, e_{2\nu+\delta+k+1} \right\rangle. \end{aligned} \quad (4.3)$$

Then, both U and W are subspaces of type $(k+1, 1, 0, \Gamma, k)$, and $U \cap W = \langle e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \dots, e_{2\nu+\delta+k} \rangle$, $\langle U, W \rangle$ is a subspace of type $(k+3, 2, 0, \Gamma, k+1)$ or $(k+3, 2, 1, \phi, k+1)$ when $q = 3(\bmod 4)$ or $q = 1(\bmod 4)$. Consequently, $\langle U, W \rangle \notin \mathcal{L}_R(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$, $r'(U \wedge W) = r'(\mathbb{F}_q^{(2\nu+\delta+l)}) = 0$, $r'(U \vee W) = r'(U \cap W) = k+2 - (k-1) = 3$, $r'(U) = r'(W) = k+2 - (k+1) = 1$. Thus,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \quad (4.4)$$

That is, (2.2) does not hold for U and W . Hence, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k; 2\nu+\delta+l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c) From the condition (2.10), the following hold.

- (i) If $\gamma = \delta = 1, \Gamma \neq \Delta$, then $2\nu + \delta - 1 \leq \nu + s$, that is, $\nu \leq s, \nu = s$, hence $2\nu + 1 \leq 2\nu$, and it is a contradiction.
- (ii) If $\gamma = \delta, \Gamma = \Delta$, then $2\nu + \delta - 1 \leq \nu + s + \delta$, that is, $\nu - 1 \leq s$, hence $s = \nu$, or $s = \nu - 1$. When $s = \nu$, from $2s + \gamma \leq 2\nu + \delta - 1$, we obtain $2\nu + \delta \leq 2\nu + \delta - 1$, and it is a contradiction. When $s = \nu - 1$, we have $2\nu + \delta - 2 \leq 2\nu + \delta - 1$. That is, in this situation, $\nu - 1 = s$ holds.
- (iii) If $\gamma \neq \delta$, then $2\nu + \delta - 1 \leq \nu + s + \min\{\delta, \gamma\} \leq \nu + s + \delta$, that is, $\nu - 1 \leq s$, hence $s = \nu$, or $s = \nu - 1$. When $s = \nu$, we have $2\nu + \gamma \leq 2\nu + \delta - 1$, then $\gamma \leq \delta - 1$. When $s = \nu - 1$, we have $2\nu + \gamma - 2 \leq 2\nu + \delta - 1$, then $\gamma - 1 \leq \delta$.

From the discussion above, we know that

(c.1) If $s = \nu$, then $\gamma \leq \delta - 1$, and we have $\delta = 1, \gamma = 0$; $\delta = 2, \gamma = 0$, and $\delta = 2, \gamma = 1$ three possible cases. For $\mathcal{L}_R(2\nu + \delta + k - 1, 2\nu + \gamma, \nu, \Gamma, k; 2\nu + \delta + l, \Delta)$, here we just give the

proof of the case $\delta = 2$, $\gamma = 1$, others can be obtained in the similar way. We choose (a, b) and (c, d) being two linearly independent solutions of the equation $x^2 - zy^2 = \Gamma$. Let

$$U = \begin{pmatrix} I^{(\nu)} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix},$$

$$\begin{matrix} \nu & \nu & 1 & 1 & k & l-k \end{matrix} \quad (4.5)$$

$$W = \begin{pmatrix} 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{matrix} \nu & \nu & 1 & 1 & k & l-k-1 & 1 \end{matrix}$$

then U is a subspace of type $(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k)$, W is a subspace of type $(2, 1, 0, \Gamma, 1)$, and $\langle U, W \rangle$ is a subspace of type $(2\nu + k + 3, 2\nu + 2, \nu, \Gamma, k + 1)$. Consequently, $U, W \in \mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k; 2\nu + \delta + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k; 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $r'(U \vee W) = r'(U \cap W) = 2\nu + k + 2$, $r'(U \wedge W) = 0$, $r'(U) = 2\nu + k + 2 - 2\nu - k - 1 = 1$, $r'(W) = 2\nu + k + 2 - 2 = 2\nu + k$. Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \quad (4.6)$$

That is, (2.2) does not hold for U and W . Hence, $\mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, 1, k; 2\nu + \delta + l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2) If $s = \nu - 1$, then we have $\gamma \neq \delta$, $\gamma - 1 \leq \delta$; or $\gamma = \delta$, $\Gamma = \Delta$. As to $\mathcal{L}_R(2\nu + \delta + k - 1, 2(\nu - 1) + \gamma, \nu - 1, \Gamma, k; 2\nu + \delta + l, \Delta)$, we consider $\delta = 0$, $\delta = 1$, and $\delta = 2$ three cases. Here we just give the proof of the case $\delta = 1$, and we also discuss the following three subcases:

(c.2.1) $\delta = 1$, $\gamma = 0$. For $\mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k; 2\nu + \delta + l, \Delta)$, let

$$U = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix} \quad (4.7)$$

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix}$$

then U is a subspace of type $(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k + 1)$, W is a subspace of type $(2, 1, 0, \Delta, 0)$, and $\langle U, W \rangle$ is a subspace of type $(2\nu + k + 2, 2\nu + 1, \nu, \Delta, k + 1)$. If $\nu = 1$, then $s = 0$, and as to W , from the condition (2.10), we obtain $2 \leq 1$, that is, it is a contradiction. Consequently, $\nu \geq 2$, and $U, W \in \mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k; 2\nu + \delta + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k; 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $r'(U \vee W) = r'(U \cap W)$

$= 2\nu + k + 1, r'(U \wedge W) = 0, r'(U) = 2\nu + k + 1 - 2\nu - k = 1, r'(W) = 2\nu + k + 1 - 2 = 2\nu + k - 1$.
Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \quad (4.8)$$

That is, (2.2) does not hold for U and W . Hence, $\mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k; 2\nu + \delta + l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2.2) $\delta = 1, \gamma = 1, \Gamma = \Delta$. For $\mathcal{L}_R(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k; 2\nu + \delta + l, \Delta)$, let

$$U = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0 \end{pmatrix},$$

$$\begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix} \quad (4.9)$$

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix}$$

then U is a subspace of type $(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k)$, W is a subspace of type $(2, 1, 0, \Delta, 0)$, and $\langle U, W \rangle$ is a subspace of type $(2\nu + k + 2, 2\nu + 1, \nu, \Delta, k + 1)$. Consequently, $U, W \in \mathcal{L}_R(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k; 2\nu + \delta + l, \Delta)$, $\langle U, W \rangle \notin \mathcal{L}_R(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k; 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu + \delta + l)}$, $r'(U \vee W) = r'(U \cap W) = 2\nu + k + 1, r'(U \wedge W) = 0, r'(U) = 2\nu + k + 1 - 2\nu - k = 1, r'(W) = 2\nu + k + 1 - 2 = 2\nu + k - 1$. Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \quad (4.10)$$

That is, (2.2) does not hold for U and W . Hence, $\mathcal{L}_R(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k; 2\nu + \delta + l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2.3) $\delta = 1, \gamma = 2$. See the proof of the Theorem 7 in [12].

The Proof of (ii). Let $U \in \mathcal{M}(m, 2s + \gamma, s, \Gamma, k; 2\nu + \delta + l, \Delta)$, then

$$US_l U^t = [\Lambda_1, 0^{m-k-2s-\gamma}, 0^{(k)}], \quad (4.11)$$

where $\Lambda_1 = S_{2s+\gamma, \Gamma}$. Hence, there exists a $(2\nu + \delta + l - m) \times (2\nu + \delta + l)$ matrix Z such that

$$\begin{pmatrix} U \\ Z \end{pmatrix} S_l \begin{pmatrix} U \\ Z \end{pmatrix}^t = [\Lambda_1, S_{2(m-k-2s-\gamma)}, \Lambda^*, 0^{(k)}, 0^{(l-k)}], \quad (4.12)$$

where Λ^* takes values in Table 1 as follows.

In Table 1 as follows $\sum_i = S_{2(\nu+s-m+k+i)}, i = 0, 1, \text{ or } 2$.

As to $\delta = 0; \delta = 1, \Delta = 1; \delta = 1, \Delta = z$, and $\delta = 2$ four cases, we only show the proof of the case $\delta = 0$, others can be obtained in the similar way. We also distinguish the following three subcases.

Table 1

	$\delta = 0$	$\delta = 1, \Delta = 1$	$\delta = 1, \Delta = z$	$\delta = 2$
$\gamma = 0$	Σ_0	$[\Sigma_0, 1]$	$[\Sigma_0, z]$	$[\Sigma_0, 1, -z]$
$\gamma = 1, \Gamma = 1$	$[\Sigma_0, -1]$	Σ_1	$[\Sigma_0, -1, z]$	$[\Sigma_1, -z]$
$\gamma = 1, \Gamma = z$	$[\Sigma_0, -z]$	$[\Sigma_0, 1, -z]$	Σ_1	$[\Sigma_1, -1]$
$\gamma = 2$	$[\Sigma_0, 1, -z]$	$[\Sigma_1, z]$	$[\Sigma_1, 1]$	Σ_2

(a) If $\gamma = 0$, then $\Lambda_1 = S_{2s}, \Lambda^* = S_{2(v-m+k+s)}$. Let $u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_s, u_{s+1}, \dots, u_{m-k-s}, w_1, \dots, w_k$ and $v_{s+1}, \dots, v_{m-k-s}, u_{m-k-s+1}, \dots, u_v, v_{m-k-s+1}, \dots, v_v, w_{k+1}, \dots, w_l$ be row vectors of U and Z , respectively,

$$W = \langle v_{v-m+k+s+1}, \dots, v_{v-s}, u_{v-s+1}, \dots, u_v, v_{v-s+1}, \dots, v_v, w_1, \dots, w_k \rangle, \quad (4.13)$$

then $W \in \mathcal{M}(m, 2s, s, \phi, k; 2v + l)$.

From $m - k \leq 2v - 2$, we know $s < v$. If $m - k = 2s$, then $m - k - s = s < v$, so $u_v, v_v \notin U$. If $m - k > 2s$, then $s < v - 1$, so $v_{v-1}, v_v \notin U$. In a word, $\dim \langle U, W \rangle \geq m + 2, \dim(U \cap W) \leq m - 2$. That is, $U \wedge W = \mathbb{F}_q^{(2v+l)}, r'(U \wedge W) = 0, r'(U \vee W) \geq m + 1 - (m - 2) = 3, r'(U) = r'(W) = m + 1 - m = 1$. Consequently, $r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W)$.

(b) If $\gamma = 1$, then $\Lambda_1 = S_{2s+1, \Gamma}, \Lambda^* = S_{2(v-m+k+s)+1, -\Gamma}$, and $\Gamma = (1)$ or (z) . Let $u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_s, \omega, u_{s+1}, \dots, u_{m-k-s-1}, w_1, \dots, w_k$ and $v_{s+1}, \dots, v_{m-k-s-1}, u_{m-k-s}, \dots, u_{v-1}, v_{m-k-s}, \dots, v_{v-1}, \omega^*, w_{k+1}, \dots, w_l$ be row vectors of U and Z , respectively

$$W = \left\langle v_{v-m+k+s+1}, \dots, v_{v-s-1}, u_{v-s}, \dots, u_{v-2}, v_{v-s}, \dots, v_{v-2}, \omega, \omega^*, \right. \\ \left. \left(\frac{1}{2} \right) \Gamma u_{v-1} + v_{v-1}, w_1, \dots, w_k \right\rangle, \quad (4.14)$$

because $((1/2)\Gamma u_{v-1} + v_{v-1})S_{2v}((1/2)\Gamma u_{v-1} + v_{v-1})^t = \Gamma$, and

$$\left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \Gamma \quad \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \Gamma \right) \begin{pmatrix} \omega \\ \omega^* \end{pmatrix} S_{2v} \begin{pmatrix} \omega \\ \omega^* \end{pmatrix}^t \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \Gamma \quad \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \Gamma \right)^t = S_{2,1}, \quad (4.15)$$

then $W \in \mathcal{M}(m, 2s + 1, s, \Gamma, k; 2v + l)$. From the conditions $2s + 1 \leq m - k \leq 2v - 2$ and $m - k \leq v + s$, we can obtain $m - k - s - 1 \leq v - 1$ and $s \leq v - 1$, hence $(1/2)\Gamma u_{v-1} + v_{v-1} \notin U$. Obviously, $\omega^* \notin U$. Similar to the proof of the case (a), $r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W)$.

(c) If $\gamma = 2$, then $\Lambda_1 = S_{2s+2, \Gamma}, \Lambda^* = S_{2(v-m+k+s)+2, \Gamma}$, and $\Gamma = [1, -z]$. Let $u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_s, \omega_1, \omega_2, u_{s+1}, \dots, u_{m-k-s-2}, w_1, \dots, w_k$ and $v_{s+1}, \dots, v_{m-k-s-2}, u_{m-k-s-1}, \dots, u_{v-2}, v_{m-k-s-1}, \dots, v_{v-2}, \omega_1^*, \omega_2^*, w_{k+1}, \dots, w_l$ be row vectors of U and Z , respectively,

$$W = \langle v_{v-m+k+s+1}, \dots, v_{v-s-2}, u_{v-s-1}, \dots, u_{v-2}, v_{v-s-1}, \dots, v_{v-2}, \omega_1^*, \omega_2^*, w_1, \dots, w_k \rangle, \quad (4.16)$$

then $W \in \mathcal{M}(m, 2s + 2, s, \Gamma, k; 2v + l)$. Obviously, $\omega_1^*, \omega_2^* \notin U$. Similar to the proof of the case (a), $r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W)$.

From the discussion above, we know that when $2 \leq m - k \leq 2\nu - 2$, $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k; 2\nu + l)$ is a finite atomic lattice, but not a geometric lattice. $\square$

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 61179026 and the Fundamental Research Funds for the Central Universities under Grant no. ZXH2012K003. The authors would like to thank the referee for his suggestions on simplifying the earlier version of the paper.

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