Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 407409, 11 pages
doi:10.1155/2012/407409

*Research Article*

# Asymptotic Stability of the Golden-Section Control Law for Multi-Input and Multi-Output Linear Systems

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Received 16 September 2011; Accepted 19 December 2011

Academic Editor: Mark A. Petersen



This paper is concerned with the problem of the asymptotic stability of the characteristic model-based golden-section control law for multi-input and multi-output linear systems. First, by choosing a set of polynomial matrices of the objective function of the generalized least-square control, we prove that the control law of the generalized least square can become the characteristic model-based golden-section control law. Then, based on both the stability result of the generalized least-square control system and the stability theory of matrix polynomial, the asymptotic stability of the closed loop system for the characteristic model under the control of the golden-section control law is proved for minimum phase system.

## 1. Introduction

Since the late 1950s, the first adaptive control system was presented by the Massachusetts Institute of Technology, the adaptive control has been attracted extensively. Fruitful results have been achieved in theory and application. However, the adaptive control in practice has not been widely used. The reason is that there are some problems of the existing adaptive control theory in practical engineering applications, such as the following: the transient response is very poor, the number of parameters need to be estimated is too many, the convergence of parameter estimation is difficult to be guaranteed, and parameters that need to be adjusted artificially are too many [1].

To solve the above problems, Wu et al. [1, 2] presented an integrated and practical all-coefficient adaptive control theory and method based on characteristic models, which

has been gradually improved in the application course of more than 20 years. This theory and method provide a new approach for the modeling and control of the complex systems. It should be mentioned that the theory and method have already been applied successfully to more than 400 systems belonging to nine kinds of engineering plants in the field of astronautics and industry. In particular, the engineering key points of the method are creatively applied to the reentry adaptive control of a manned spaceship, which the accuracy of the parachute-opening point of the spaceship reaches the advanced level of the world.

The method given by Wu is simple in design, easy to adjust and with strong robustness, and in some ways solves the above-mentioned problems. This method includes the following three aspects: (1) all-coefficient adaptive control method; (2) golden-section adaptive control law; (3) characteristic model [1].

It is worth mentioning that the golden-section control law is a new control law, which can solve the problem that adaptive control cannot guarantee the stability of a closed-loop system during the transient process when the parameters have not converged to their "true value." The so-called golden-section control law means that the golden section ratio (0.382/0.618) is used to controller designs, see [1, 2] or Section 3 of this paper in detail.

For a second-order single-input and single-output (SISO) invariant linear system, Xie et al. [3, 4] proved that the golden-section control law has strong robust stability. The sufficient conditions for the stability of the closed-loop system based on golden-section control law for SISO and 2-input-2-output invariant linear systems have been reported; see, Qi et al. [5], Sun and Wu [6, 7], and Meng et al. [8]. Recently, Sun [9] gave sufficient conditions for the stability of the golden-section control system for 3-input-3-output invariant linear systems, but these conditions are difficult to verify. Among these references, we notice that the closed-loop control properties based on the golden-section control law are given by using the stability results of the generalized least-square control system and Jury stability criteria, aiming at the characteristic model of a second-order continuous SISO invariant linear system.

In summary, for the MIMO system, the stability of golden-section control systems is still an open question.

In this paper, for the MIMO linear system, we investigate the stability of the characteristic model-based golden-section control law by using the stability results of the multivariable generalized least-square control system.

## 2. Preliminaries

### *2.1. Introduction to Generalized Least-Square Controller*

Consider the following system described by the linear vector difference equation:

$$A\left(z^{-1}\right)Y(k) = z^{-d}B\left(z^{-1}\right)U(k) + C\left(z^{-1}\right)\xi(k), \tag{2.1}$$

where $U(k)$ and $Y(k)$ are the $n \times 1$ input and output vectors, respectively, and $\xi(k)$ is the $n \times 1$ zero-mean white-noise vector with covariance matrix $E(\xi(k)\xi^T(k)) = r_\xi$, $d$ denotes the system

time delay, and $A$, $B$, and $C$ are polynomial matrices in backward shift operator $z^{-1}$ given by

$$\begin{aligned} A\left(z^{-1}\right) &= I + A_1 z^{-1} + \cdots + A_{n_a} z^{-n_a}, \\ B\left(z^{-1}\right) &= B_0 + B_1 z^{-1} + \cdots + B_{n_b} z^{-n_b}, \quad B_0 \text{ nonsingular}, \\ C\left(z^{-1}\right) &= I + C_1 z^{-1} + \cdots + C_{n_c} z^{-n_c}, \end{aligned} \tag{2.2}$$

here $A_i$, $B_i$, and $C_i$ are matrix coefficients. The cost function to be considered is described by

$$J = E\left(\left\|P\left(z^{-1}\right)Y(k+d) - R\left(z^{-1}\right)Y_r(k)\right\|^2 + \left\|Q'\left(z^{-1}\right)U(k)\right\|^2\right), \tag{2.3}$$

where $Y_r(k)$ is an $n$-vector defining the known reference signal. $P$, $R$, and $Q'$ are $n \times n$ weighting polynomial matrices. The notation $\|X\|^2 = X^{\mathrm{T}}X$ has been used.

The optimal control law is [10]

$$H\left(z^{-1}\right)U(k) = -E\left(z^{-1}\right)Y_r(k) - \tilde{G}\left(z^{-1}\right)Y(k), \tag{2.4}$$

where

$$H\left(z^{-1}\right) = \tilde{F}\left(z^{-1}\right)B\left(z^{-1}\right) + \tilde{C}\left(z^{-1}\right)Q\left(z^{-1}\right), \tag{2.5}$$

$$E\left(z^{-1}\right) = -\tilde{C}\left(z^{-1}\right)R\left(z^{-1}\right), \tag{2.6}$$

$$Q\left(z^{-1}\right) = \left[(P_0B_0)^{\mathrm{T}}\right]^{-1}\left[Q'(0)\right]^{\mathrm{T}}Q'\left(z^{-1}\right), \quad B_0 = B(0), \quad P_0 = P(0), \tag{2.7}$$

$$\tilde{C}\left(z^{-1}\right)P\left(z^{-1}\right) = \tilde{F}\left(z^{-1}\right)A\left(z^{-1}\right) + z^{-d}\tilde{G}\left(z^{-1}\right), \tag{2.8}$$

$$\tilde{C}\left(z^{-1}\right)F\left(z^{-1}\right) = \tilde{F}\left(z^{-1}\right)C\left(z^{-1}\right), \tag{2.9}$$

here the order of the polynomial matrix $F$ is equal to $d-1$.

### 2.2. Two Important Lemmas

Let an $(m \times m)$ nonsingular real matrix polynomial of $n$th-order $P(z)$ be given by

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0, \tag{2.10}$$

where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$, and $a_0$ are $(m \times m)$ real matrices, $m \geq 1$.

We can construct an $(mn \times mn)$ symmetric matrix $C = [c_{ij}]$ by the Christoffel-Darboux formula as follows. $c_{ij}$'s are defined as

$$c_{ij} = \sum_{k=1}^{i} a_{n+k-i}^{\mathrm{T}} a_{n+k-j} - a_{j-k}^{\mathrm{T}} a_{i-k}, \qquad i \le j,$$
$$c_{ji} = c_{ij}^{\mathrm{T}}, \qquad i > j \tag{2.11}$$

for $i, j = 1, 2, \ldots, n$.

For example, when $n = 2$ [11],

$$\mathrm{C} = \begin{bmatrix} a_2^{\mathrm{T}} a_2 - a_0^{\mathrm{T}} a_0 & a_2^{\mathrm{T}} a_1 - a_1^{\mathrm{T}} a_0 \\ a_1^{\mathrm{T}} a_2 - a_0^{\mathrm{T}} a_1 & a_2^{\mathrm{T}} a_2 - a_0^{\mathrm{T}} a_0 \end{bmatrix}. \tag{2.12}$$

**Lemma 2.1** (see [11, Theorem 1]). *If $C = [c_{ij}]$ defined in* (2.11) *is positive definite, then all the roots of the determinant of the matrix polynomial* (2.10) *lie inside the unit circle.*

It is well known that the following lemma holds.

**Lemma 2.2.** *Let $A$ be an $(n \times n)$ real symmetric matrix and $I$ be an $(n \times n)$ identity matrix. Then there exists a sufficiently small positive number $\varepsilon$ such that $I + \varepsilon A$ becomes a positive definite matrix.*

## 3. Problem Formulation

Suppose that a second-order multi-input and multi-output dynamic process can be expressed as

$$Y^{(2)}(t) + A_1 Y^{(1)}(t) + A_0 Y(t) = B_1 U^{(1)}(t) + B_0 U(t), \tag{3.1}$$

where $U(k)$ and $Y(k)$ are the $n \times 1$ input and output vectors, respectively, and $A_0$, $A_1$, $B_0$, and $B_1$ are polynomial matrices. By using forward difference method, the corresponding difference equation can be given as

$$Y(k+1) = \overline{A}_1 Y(k) + \overline{A}_2 Y(k-1) + \overline{B}_0 U(k) + \overline{B}_1 U(k-1) + e(k+1), \tag{3.2}$$

where

$$\overline{A}_1 = 2I - A_1 \Delta t, \qquad \overline{A}_2 = -I + A_1 \Delta t - A_0 (\Delta t)^2,$$
$$\overline{B}_0 = -B_1 \Delta t, \qquad \overline{B}_1 = -B_1 \Delta t + B_0 \Delta t^2, \tag{3.3}$$

and $e(k)$ is the modeling error vector.

For a linear multi-input and multi-output constant high-order plant, if sampling period $\Delta t$ is sufficiently small, when the control requirement is position keeping or tracking, we can prove that its characteristic model can be also expressed with (3.2).

It is easily to be seen that (3.2) can be written as (2.1), that is,

$$A\left(z^{-1}\right)Y(k) = z^{-1}B\left(z^{-1}\right)U(k) + C\left(z^{-1}\right)e(k), \tag{3.4}$$

where

$$\begin{aligned} A\left(z^{-1}\right) &= I - \overline{A}_1 z^{-1} - \overline{A}_2 z^{-2}, \\ B\left(z^{-1}\right) &= \overline{B}_0 + \overline{B}_1 z^{-1}, \\ C\left(z^{-1}\right) &= I. \end{aligned} \tag{3.5}$$

The characteristic model-based golden-section control law is designed as follows:

$$U(k) = (1+\lambda)^{-1}\overline{B}_0^{-1}\left[Y_r(k) - \overline{B}_1 U(k-1) - l_1\overline{A}_1 Y(k) - l_2\overline{A}_2 Y(k-1)\right], \tag{3.6}$$

where $\lambda \geq 0$ is a constant; $l_1 = 0.382$ and $l_2 = 0.618$ are golden-section coefficients.

## 4. Main Result and Proof

In this section, by selecting a set of polynomial matrices in the objective function (2.3), for the characteristic model (3.2), we find the generalized least-square control law under the objective function. By comparison, we can see that this control law is just the same law as the golden-section control law based on characteristic model. Finally, based on the stability result of generalized least-square control law, we will give the stability result of the closed-loop of the golden-section control law.

For the objective function (2.3), we choose that

$$P\left(z^{-1}\right) = I - l_2\overline{A}_1 z^{-1} - l_1\overline{A}_2 z^{-2}, \qquad R\left(z^{-1}\right) = I, \qquad Q'\left(z^{-1}\right) = \sqrt{\lambda}\,\overline{B}_0, \qquad \widetilde{F}\left(z^{-1}\right) = I. \tag{4.1}$$

Note that $d = 1$, and hence $F(z^{-1}) = I$. By $C(z^{-1}) = I$, $\widetilde{F}(z^{-1}) = I$, and (2.9), we get $\widetilde{C}(z^{-1}) = I$. Using $\widetilde{C}(z^{-1}) = I$, $R(z^{-1}) = I$, and (2.6), $E(z^{-1}) = -I$.

From $P_0 = I$, $B(0) = \overline{B}_0$, $Q'(z^{-1}) = \sqrt{\lambda}\,\overline{B}_0$, and (2.7), it follows that

$$Q\left(z^{-1}\right) = \left[(P_0B_0)^{\mathrm{T}}\right]^{-1}\left[Q'(0)\right]^{\mathrm{T}}Q'\left(z^{-1}\right) = \left[\overline{B}_0^{\mathrm{T}}\right]^{-1}\left[\sqrt{\lambda}\,\overline{B}_0\right]^{\mathrm{T}}\sqrt{\lambda}\,\overline{B}_0 = \left[\overline{B}_0^{\mathrm{T}}\right]^{-1}\overline{B}_0^{\mathrm{T}}\lambda\overline{B}_0 = \lambda\overline{B}_0. \tag{4.2}$$

By using Diophantine equation (2.8), we obtain that

$$I - l_2\overline{A}_1 z^{-1} - l_1\overline{A}_2 z^{-2} = I - \overline{A}_1 z^{-1} - \overline{A}_2 z^{-2} + z^{-1}\tilde{G}\left(z^{-1}\right). \tag{4.3}$$

Therefore,

$$\tilde{G}\left(z^{-1}\right) = l_1\overline{A}_1 + l_2\overline{A}_2 z^{-1}. \tag{4.4}$$

Using (2.5), we have

$$\begin{aligned} H\left(z^{-1}\right) &= \tilde{F}\left(z^{-1}\right)B\left(z^{-1}\right) + \tilde{C}\left(z^{-1}\right)Q\left(z^{-1}\right) \\ &= \overline{B}_0 + \overline{B}_1 z^{-1} + I\lambda\overline{B}_0 \\ &= \overline{B}_0 + \overline{B}_1 z^{-1} + \lambda\overline{B}_0 \\ &= (1+\lambda)\overline{B}_0 + \overline{B}_1 z^{-1}. \end{aligned} \tag{4.5}$$

According to (2.4), it follows that

$$\left[(1+\lambda)\overline{B}_0 + \overline{B}_1 z^{-1}\right]U(k) = IY_r(k) - \left(l_1\overline{A}_1 + l_2\overline{A}_2 z^{-1}\right)Y(k). \tag{4.6}$$

That is,

$$U(k) = (1+\lambda)^{-1}\overline{B}_0^{-1}\left[Y_r(k) - \overline{B}_1 U(k-1) - l_1\overline{A}_1 Y(k) - l_2\overline{A}_2 Y(k-1)\right]. \tag{4.7}$$

The above control law obtained by generalized least-square control law is exactly the characteristic model-based golden-section control law designed in (3.6). Hence, the stability of the closed-loop system based on the characteristic model-based golden-section control law is determined by the distribution of zero points on $z$-plane of the following equation:

$$\det H\left(z^{-1}\right)\det\left[A\left(z^{-1}\right) + z^{-d}B\left(z^{-1}\right)H^{-1}\left(z^{-1}\right)\tilde{G}\left(z^{-1}\right)\right] = 0. \tag{4.8}$$

We note that

$$H\left(z^{-1}\right) = \tilde{F}\left(z^{-1}\right)B\left(z^{-1}\right) + \tilde{C}\left(z^{-1}\right)Q\left(z^{-1}\right) = IB\left(z^{-1}\right) + I\lambda\overline{B}_0 = B\left(z^{-1}\right) + \lambda\overline{B}_0. \tag{4.9}$$

Now, taking $\lambda = 0$, we have $H(z^{-1}) = B(z^{-1})$, and, furthermore,

$$\begin{aligned}
&A\left(z^{-1}\right) + z^{-d}B\left(z^{-1}\right)H^{-1}\left(z^{-1}\right)\widetilde{G}\left(z^{-1}\right) \\
&\quad = A\left(z^{-1}\right) + B\left(z^{-1}\right)\left[B\left(z^{-1}\right)\right]^{-1}\left[\widetilde{C}\left(z^{-1}\right)P\left(z^{-1}\right) - \widetilde{F}\left(z^{-1}\right)A\left(z^{-1}\right)\right] \\
&\quad = A\left(z^{-1}\right) + \left[IP\left(z^{-1}\right) - IA\left(z^{-1}\right)\right] \\
&\quad = A\left(z^{-1}\right) + \left[P\left(z^{-1}\right) - A\left(z^{-1}\right)\right] \\
&\quad = P\left(z^{-1}\right).
\end{aligned} \tag{4.10}$$

Then, when $B(z^{-1})$ is stable, the stability of the closed-loop system formed by the characteristic model-based golden-section control law is determined by the stability of $P(z^{-1})$.

**Theorem 4.1.** *Assume that* (3.2) *is a minimum phase system and* $\lambda = 0$. *Then, the closed-loop system involving the characteristic model-based golden-section control law* (3.6) *is asymptotic stable.*

*Remark 4.2.* Since the corresponding relationship between the zero positions of continuous controlled objects and that of the discrete-time systems is complex, here the minimum phase system means that the zero points of the characteristic model (3.2) lie inside the unit circle.

*Proof.* First, we notice that $P(z^{-1}) = I - l_2\overline{A}_1 z^{-1} - l_1\overline{A}_2 z^{-2}$, and take

$$a_2 = I, \qquad a_1 = -l_2\overline{A}_1, \qquad a_0 = -l_1\overline{A}_2. \tag{4.11}$$

By (2.12), we have

$$C = \begin{bmatrix} I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 & -l_2\overline{A}_1 - l_1 l_2\overline{A}_1^{\mathrm{T}}\overline{A}_2 \\ -l_2\overline{A}_1^{\mathrm{T}} - l_1 l_2\overline{A}_2^{\mathrm{T}}\overline{A}_1 & I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 \end{bmatrix}. \tag{4.12}$$

Now, we show that $C$ is positive definite when $\Delta t$ is sufficiently small.

It is easy to see that

$$C = \begin{bmatrix} B & N \\ N^{\mathrm{T}} & \alpha I \end{bmatrix} + \begin{bmatrix} \overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 & O \\ O & \overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 \end{bmatrix}, \tag{4.13}$$

where $B = \alpha I$, $N = -l_2\overline{A}_1 - l_1 l_2\overline{A}_1^{\mathrm{T}}\overline{A}_2$, $0.764 = 2l_1 < \alpha < 1 - l_1^2 = 0.8541$, and $\overline{\alpha} = 1 - \alpha$.

To show $C$ be positive definite, we will prove that

$$\begin{bmatrix} B & N \\ N^{\mathrm{T}} & \alpha I \end{bmatrix}, \tag{4.14}$$

and $\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2$ are all positive definite when $\Delta t$ is sufficiently small as follows.

First, we notice that

$$\begin{aligned}
&\begin{bmatrix} I & O \\ -N^{\mathrm{T}}B^{-1} & I \end{bmatrix}\begin{bmatrix} B & N \\ N^{\mathrm{T}} & \alpha I \end{bmatrix}\begin{bmatrix} I & -(B^{-1})^{\mathrm{T}}N \\ O & I \end{bmatrix} \\
&\quad = \begin{bmatrix} B & -B(B^{-1})^{\mathrm{T}}N + N \\ O & \alpha I - N^{\mathrm{T}}B^{-1}N \end{bmatrix} = \begin{bmatrix} B & O \\ O & \alpha I - N^{\mathrm{T}}B^{-1}N \end{bmatrix} = \begin{bmatrix} \alpha I & O \\ O & \alpha I - N^{\mathrm{T}}B^{-1}N \end{bmatrix}.
\end{aligned} \tag{4.15}$$

We claim that $\alpha I - N^{\mathrm{T}}B^{-1}N$ is positive definite when $\Delta t$ is sufficiently small. In fact, since

$$\begin{aligned}
N^{\mathrm{T}}B^{-1}N &= \left[-l_2\overline{A}_1^{\mathrm{T}} - l_1l_2\overline{A}_2^{\mathrm{T}}\overline{A}_1\right]B^{-1}\left[-l_2\overline{A}_1 - l_1l_2\overline{A}_1^{\mathrm{T}}\overline{A}_2\right] \\
&= \left[l_2\overline{A}_1^{\mathrm{T}} + l_1l_2\overline{A}_2^{\mathrm{T}}\overline{A}_1\right]B^{-1}\left[l_2\overline{A}_1 + l_1l_2\overline{A}_1^{\mathrm{T}}\overline{A}_2\right] \\
&= \frac{1}{\alpha}\left[l_2\overline{A}_1^{\mathrm{T}} + l_1l_2\overline{A}_2^{\mathrm{T}}\overline{A}_1\right]\left[l_2\overline{A}_1 + l_1l_2\overline{A}_1^{\mathrm{T}}\overline{A}_2\right], \\
l_2\overline{A}_1^{\mathrm{T}} + l_1l_2\overline{A}_2^{\mathrm{T}}\overline{A}_1 &= l_2\left(2I - A_1^{\mathrm{T}}\Delta t\right) + l_1l_2\left[-I + A_1^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}(\Delta t)^2\right][2I - A_1\Delta t] \\
&= 2l_2I - l_2A_1^{\mathrm{T}}\Delta t - 2l_1l_2I + l_1l_2A_1\Delta t + l_1l_2\left[A_1^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}(\Delta t)^2\right][2I - A_1\Delta t] \\
&= 2l_2(1 - l_1)I - l_2A_1^{\mathrm{T}}\Delta t + l_1l_2A_1\Delta t + l_1l_2\left[A_1^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}(\Delta t)^2\right][2I - A_1\Delta t] \\
&= 2l_2^2I - l_2A_1^{\mathrm{T}}\Delta t + l_1l_2A_1\Delta t + l_1l_2\left[A_1^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}(\Delta t)^2\right][2I - A_1\Delta t] \\
&= 2l_2^2I - l_2A_1^{\mathrm{T}}\Delta t + l_1l_2A_1\Delta t + \left[l_1l_2A_1^{\mathrm{T}}\Delta t - l_1l_2A_0^{\mathrm{T}}(\Delta t)^2\right][2I - A_1\Delta t] \\
&= 2l_2^2I - l_2A_1^{\mathrm{T}}\Delta t + l_1l_2A_1\Delta t + 2l_1l_2A_1^{\mathrm{T}}\Delta t - 2l_1l_2A_0^{\mathrm{T}}(\Delta t)^2 \\
&\quad - l_1l_2A_1^{\mathrm{T}}A_1(\Delta t)^2 + l_1l_2A_0^{\mathrm{T}}A_1(\Delta t)^3 \\
&= 2l_2^2I + (-l_2 + 2l_1l_2)A_1^{\mathrm{T}}\Delta t + l_1l_2A_1\Delta t - \left(2l_1l_2A_0^{\mathrm{T}} + l_1l_2A_1^{\mathrm{T}}A_1\right)(\Delta t)^2 \\
&\quad + l_1l_2A_0^{\mathrm{T}}A_1(\Delta t)^3,
\end{aligned} \tag{4.16}$$

we get

$$N^{\mathrm{T}}B^{-1}N = \frac{1}{\alpha}\left[4l_2^4 I + 2l_2^2(-l_2 + 3l_1l_2)\left(A_1 + A_1^{\mathrm{T}}\right)\Delta t + \cdots + (l_1l_2)^2 A_0^{\mathrm{T}}A_1A_1^{\mathrm{T}}A_0(\Delta t)^6\right],$$

$$\begin{aligned}\alpha I - N^{\mathrm{T}}B^{-1}N &= \alpha I - \frac{1}{\alpha}\left[4l_2^4 I + 2l_2^2(-l_2 + 3l_1l_2)\left(A_1 + A_1^{\mathrm{T}}\right)\Delta t + \cdots + (l_1l_2)^2 A_0^{\mathrm{T}}A_1A_1^{\mathrm{T}}A_0(\Delta t)^6\right] \\ &= \left(\alpha - \frac{4l_2^4}{\alpha}\right)I - \frac{1}{\alpha}\left[2l_2^2(-l_2 + 3l_1l_2)\left(A_1 + A_1^{\mathrm{T}}\right)\Delta t + \cdots + (l_1l_2)^2 A_0^{\mathrm{T}}A_1A_1^{\mathrm{T}}A_0(\Delta t)^6\right].\end{aligned} \tag{4.17}$$

Using $\alpha > 2l_1 = 2l_2^2$ yields $\alpha - 4l_2^4/\alpha = (\alpha^2 - 4l_2^4)/\alpha > 0$. Thus, we have

$$\begin{aligned}&\alpha I - N^{\mathrm{T}}B^{-1}N \\ &= \left(\frac{\alpha^2 - 4l_2^4}{\alpha}\right)\left\{I + \frac{\Delta t}{\alpha^2 - 4l_2^4}\left[-2l_2^2(-l_2 + 3l_1l_2)\left(A_1 + A_1^{\mathrm{T}}\right) - \cdots - (l_1l_2)^2 A_0^{\mathrm{T}}A_1A_1^{\mathrm{T}}A_0(\Delta t)^5\right]\right\}.\end{aligned} \tag{4.18}$$

By Lemma 2.2, it is follows that $\alpha I - N^{\mathrm{T}}B^{-1}N$ is positive definite when $\Delta t$ is sufficiently small. Therefore, there exists nonsingular matrix $D$ such that $D^{\mathrm{T}}(\alpha I - N^{\mathrm{T}}B^{-1}N)D = I$. Furthermore, we have

$$\begin{aligned}&\begin{bmatrix} \frac{1}{\sqrt{\alpha}}I & O \\ O & D^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \alpha I & O \\ O & \alpha I - N^{\mathrm{T}}B^{-1}N \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\alpha}}I & O \\ O & D \end{bmatrix} \\ &\quad = \begin{bmatrix} \frac{\alpha}{\sqrt{\alpha}}I & O \\ O & D^T(\alpha I - N^{\mathrm{T}}B^{-1}N) \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\alpha}}I & O \\ O & D \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.\end{aligned} \tag{4.19}$$

That is, by the congruent transformation twice,

$$\begin{bmatrix} B & N \\ N^{\mathrm{T}} & \alpha I \end{bmatrix} \tag{4.20}$$

can be transformed into the identity matrix. From this, it is positive definite when $\Delta t$ is sufficiently small.

We also claim that $\overline{\alpha}I - {l_1}^2\overline{A}_2^{\mathrm{T}}\overline{A}_2$ is positive definite when $\Delta t$ is sufficiently small. In fact, it can be seen that

$$\begin{aligned}
&\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 \\
&\quad = \overline{\alpha}I - l_1^2\left[-I + A_1^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}(\Delta t)^2\right]\left[-I + A_1\Delta t - A_0(\Delta t)^2\right] \\
&\quad = \left(\overline{\alpha} - l_1^2\right)I - l_1^2\left[-A_1\Delta t + A_0(\Delta t)^2 - A_1^{\mathrm{T}}\Delta t + A_1^{\mathrm{T}}A_1(\Delta t)^2 - A_1^{\mathrm{T}}A_0(\Delta t)^3\right. \\
&\qquad\qquad \left. + A_0^{\mathrm{T}}(\Delta t)^2 - A_0^{\mathrm{T}}A_1(\Delta t)^3 + A_0^{\mathrm{T}}A_0(\Delta t)^4\right].
\end{aligned} \tag{4.21}$$

Using $\alpha < 1 - l_1^2$ and $\overline{\alpha} = 1 - \alpha$, we have $\overline{\alpha} - l_1^2 > 0$. Thus,

$$\begin{aligned}
&\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2 \\
&\quad = \left(\overline{\alpha} - l_1^2\right)\left\{I - \frac{l_1^2\Delta t}{\overline{\alpha} - l_1^2}\left[-A_1 - A_1^{\mathrm{T}} + A_0\Delta t + A_1^{\mathrm{T}}A_1\Delta t - A_1^{\mathrm{T}}A_0(\Delta t)^2\right.\right. \\
&\qquad\qquad \left.\left. + A_0^{\mathrm{T}}\Delta t - A_0^{\mathrm{T}}A_1(\Delta t)^2 + A_0^{\mathrm{T}}A_0(\Delta t)^3\right]\right\}.
\end{aligned} \tag{4.22}$$

By Lemma 2.2, $\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2$ is also positive definite when $\Delta t$ is sufficiently small.

Thus, it follows from (4.13) that $C$ is positive definite when $\Delta t$ is sufficiently small. By Lemma 2.1, $P(z^{-1})$ is a stable matrix polynomial. Hence, the asymptotic stability of the closed-loop system follows immediately. This completes the proof. □

*Remark 4.3.* It is well known that an eigenvalue of a matrix is a continuous function with respect to elements of the matrix. From this, (4.18), and (4.22), we can also see that $\alpha I - N^{\mathrm{T}}B^{-1}N$ and $\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2$ are positive definite when $\Delta t$ is sufficiently small.

*Remark 4.4.* By using a constructive proof of Lemma 2.2, we can determine minimum sampling period $\Delta t$ so that $\alpha I - N^{\mathrm{T}}B^{-1}N$ and $\overline{\alpha}I - l_1^2\overline{A}_2^{\mathrm{T}}\overline{A}_2$ are all positive definite.

*Conjecture 4.5.* If (3.2) is a nonminimum phase system, we may investigate the stability of the closed-loop system involving the characteristic model-based golden-section control law (3.6) by using the approach of the root locus in linear multivariable systems.

## Acknowledgment

This work was supported by the National Nature Science Foundation of China under Grant 60874055, the Key Subject Construction Project, and the Funds for Creative Research Groups of Hebei Normal University of Science and Technology (no. CXTD2010-05).

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