Research Article

Convergence Theorems for Equilibrium Problems and Fixed-Point Problems of an Infinite Family of $k_i$-Strictly Pseudocontractive Mapping in Hilbert Spaces

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1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset of $H$ and let $F : C \times C \rightarrow R$ be a bifunction. We consider the following equilibrium problem (EP) which is to find $z \in C$ such that

$$\text{EP} : F(z, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

Denote the set of solutions of EP by $\text{EP}(F)$. Given a mapping $T : C \rightarrow H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in \text{EP}(F)$ if and only if $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$, that is, $z$ is a solution of the variational inequality. Numerous problems in physics, optimization, and
economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem [1–13].

A mapping \( B : C \to C \) is called \( \theta \)-Lipschitzian if there exists a positive constant \( \theta \) such that

\[
\| Bx - By \| \leq \theta \| x - y \|, \quad \forall x, y \in C. \tag{1.2}
\]

\( B \) is said to be \( \eta \)-strongly monotone if there exists a positive constant \( \eta \) such that

\[
\langle Bx - By, x - y \rangle \geq \eta \| x - y \|^2, \quad \forall x, y \in C. \tag{1.3}
\]

A mapping \( S : C \to C \) is said to be \( k \)-strictly pseudocontractive mapping if there exists a constant \( 0 \leq k < 1 \) such that

\[
\| Sx - Sy \|^2 \leq \| x - y \|^2 + k \| (I - S)x - (I - S)y \|^2, \tag{1.4}
\]

for all \( x, y \in C \) and \( F(S) \) denotes the set of fixed point of the mapping \( S \), that is \( F(S) = \{ x \in C : Sx = x \} \).

If \( k = 1 \), then \( S \) is said to a pseudocontractive mapping, that is,

\[
\| Sx - Sy \|^2 \leq \| x - y \|^2 + \| (I - S)x - (I - S)y \|^2, \tag{1.5}
\]

is equivalent to

\[
\langle (I - S)x - (I - S)y, x - y \rangle \geq 0, \tag{1.6}
\]

for all \( x, y \in C \).

The class of \( k \)-strict pseudo-contractive mappings extends the class of nonexpansive mappings (A mapping \( T \) is said to be nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \), for all \( x, y \in C \)). That is, \( S \) is nonexpansive if and only if \( S \) is a 0-strict pseudocontractive mapping. Clearly, the class of \( k \)-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mapping.

In 2006, Marino and Xu [14] introduced the general iterative method and proved that for a given \( x_0 \in H \), the sequence \( \{ x_n \} \) generated by the algorithm

\[
x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n B)Tx_n, \quad n \in \mathbb{N}, \tag{1.7}
\]

where \( T \) is a self-nonexpansive mapping on \( H \), \( f \) is an \( \alpha \)-contraction of \( H \) into itself (i.e., \( \| f(x) - f(y) \| \leq \alpha \| x - y \| \), for all \( x, y \in H \) and \( \alpha \in (0,1) \)), \( \{ \alpha_n \} \subset (0,1) \) satisfies certain conditions, \( B \) is strongly positive bounded linear operator on \( H \), and converges strongly to fixed point \( x^\star \) of \( T \) which is the unique solution to the following variational inequality:

\[
\langle (y - B)x^\star, x^\star - x \rangle \leq 0, \quad \forall x \in F(T). \tag{1.8}
\]
Tian [15] considered the following iterative method, for a nonexpansive mapping $T : H \to H$ with $F(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F)Tx_n, \quad n \in \mathbb{N}, \quad (1.9)$$

where $F$ is $k$-Lipschitzian and $\eta$-strongly monotone operator. The sequence $\{x_n\}$ converges strongly to fixed-point $q$ in $F(T)$ which is the unique solution to the following variational inequality:

$$\langle (\gamma f - \mu F) q, p - q \rangle \leq 0, \quad p \in F(T). \quad (1.10)$$

For finding a common element of $EP(F) \cap F(S)$, S. Takahashi and W. Takahashi [16] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial point $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (1.11)$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}.$$  

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} f(z)$.

Liu [17] introduced the following scheme: $x_1 \in H$ and

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$y_n = \beta_n u_n + (1 - \beta_n)Su_n,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad \forall n \in \mathbb{N}, \quad (1.12)$$

where $S$ is a $k$-strict pseudo-contractive mapping and $B$ is a strongly positive bounded linear operator. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$, the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} (I - B + \gamma f)(z)$.

In [18], the concept of $W$ mapping had been modified for a countable family $\{T_n\}_{n \in \mathbb{N}}$ of nonexpansive mappings by defining the sequence $\{W_n\}_{n \in \mathbb{N}}$ of $W$-mappings generated by $\{T_n\}_{n \in \mathbb{N}}$ and $\{\lambda_n\} \subset (0, 1)$, proceeding backward

$$U_{n+1} := I,$$

$$U_n := \lambda_n T_n U_{n+1} + (1 - \lambda_n) I,$$

...
\[ U_{n,k} := \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \]
\[ \ldots \]
\[ U_{n,2} := \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \]
\[ W_n = U_{n,1} := \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I. \]

(1.13)

Yao et al. [19] using this concept, introduced the following algorithm: \( x_1 \in H \) and

\[ F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \]
\[ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)W_n u_n, \quad \forall n \in N. \]

(1.14)

They proved that under certain appropriate conditions imposed on \( \{\alpha_n\} \) and \( \{r_n\} \), the sequences \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \text{EP}(F) \).

Colao and Marino [20] considered the following explicit viscosity scheme

\[ F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \]
\[ x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n u_n, \quad \forall n \in N, \]

(1.15)

where \( A \) is a strongly positive operator on \( H \). Under certain appropriate conditions, the sequences \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \text{EP}(F) \).

Motivated and inspired by these facts, in this paper, we first extend the definition of \( W_n \) from an infinite family of nonexpansive mappings to an infinite family of strictly pseudo-contractive mappings, and then propose the iteration scheme (3.2) for finding an element of \( \text{EP}(F) \cap \bigcap_{i=1}^{\infty} F(S_i) \), where \( \{S_i\} \) is an infinite family of \( k_i \)-strictly pseudo-contractive mappings of \( C \) into itself. Finally, the convergence theorem of the iteration scheme is obtained. Our results include Yao et al. [19], Colao and Marino [20] as some special cases.

**2. Preliminaries**

Throughout this paper, we always assume that \( C \) is a nonempty closed convex subset of a Hilbert space \( H \). We write \( x_n \to x \) to indicate that the sequence \( \{x_n\} \) converges weakly to \( x \). \( x_n \to x \) implies that \( \{x_n\} \) converges strongly to \( x \). We denote by \( N \) and \( R \) the sets of positive integers and real numbers, respectively. For any \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \), such that

\[ \|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \]

(2.1)

Such a \( P_C \) is called the metric projection of \( H \) onto \( C \). It is known that \( P_C \) is nonexpansive. Furthermore, for \( x \in H \) and \( u \in C \),

\[ u = P_C x \iff (x - u, u - y) \geq 0, \quad \forall y \in C. \]

(2.2)
It is widely known that $H$ satisfies Opial’s condition [21], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality
\[
\lim_{n \to \infty} \inf \|x_n - x\| < \lim_{n \to \infty} \inf \|x_n - y\|
\] (2.3)
holds for every $y \in H$ with $y \neq x$.

In order to solve the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, we assume that $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$.

(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$.

(A3) $\lim_{t \to 0} F(tx + (1-t)y, y) \leq F(x, y)$, for all $x, y, z \in C$.

(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

**Lemma 2.1** (see [22]). Let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that
\[
F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C.
\] (2.4)

Furthermore, if $T_r x = \{z \in C : F(z, y) + (1/r)(y - z, z - x) \geq 0, \forall y \in C\}$, then the following hold:

(1) $T_r$ is single-valued.

(2) $T_r$ is firmly nonexpansive, that is,
\[
\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H.
\] (2.5)

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

**Lemma 2.2** (see [23]). Let $S : C \to H$ be a $k$-strictly pseudo-contractive mapping. Define $T : C \to H$ by $Tx = \lambda x + (1 - \lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, $T$ is nonexpansive mapping such that $F(T) = F(S)$.

**Lemma 2.3** (see [24]). In a Hilbert space $H$, there holds the inequality
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.
\] (2.6)

**Lemma 2.4** (see [25]). Let $H$ be a Hilbert space and $C$ be a closed convex subset of $H$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ weakly converging to $x$ and if $\{(I - T)x_n\}$ converges strongly to $y$, then $(I - T)x = y$. 
Lemma 2.5 (see [26]). Let \( \{x_n\} \) and \( \{z_n\} \) be bounded sequences in a Banach space \( E \) and \( \{\gamma_n\} \) be a sequence in \([0,1]\) satisfying the following condition

\[
0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1.
\]  

(2.7)

Suppose that \( x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n, n \geq 0 \) and \( \lim_{n \to \infty} \sup(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then \( \lim_{n \to \infty} \|z_n - x_n\| = 0 \).

Lemma 2.6 (see [27]). Assume that \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - b_n) a_n + b_n \delta_n, \quad n \geq 0,
\]

(2.8)

where \( \{b_n\} \) is a sequence in \((0,1)\) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \), such that

(i) \( \sum_{i=1}^{\infty} b_i = \infty \).

(ii) \( \lim_{n \to \infty} \sup \delta_n \leq 0 \) or \( \sum_{i=1}^{\infty} |b_i \delta_i| < \infty \).

Then, \( \lim_{n \to \infty} a_n = 0 \).

Let \( \{S_i\} \) be an infinite family of \( k_i \)-strictly pseudo-contractive mappings of \( C \) into itself, we define a mapping \( W_n \) of \( C \) into itself as follows,

\[
U_{n,n+1} := I,
\]

\[
U_{n,n} := \tau_n S_n' U_{n,n+1} + (1 - \tau_n) I,
\]

\[
\quad \cdots
\]

\[
U_{n,k} := \tau_k S_k' U_{n,k+1} + (1 - \tau_k) I,
\]

\[
\quad \cdots
\]

\[
U_{n,2} := \tau_2 S_2' U_{n,3} + (1 - \tau_2) I,
\]

\[
W_n = U_{n,1} := \tau_1 S_1' U_{n,2} + (1 - \tau_1) I,
\]

where \( 0 \leq \tau_i \leq 1 \), \( S_i' = \sigma_i I + (1 - \sigma_i) S_i \) and \( \sigma_i \in [k_i, 1) \) for \( i \in N \). We can obtain \( S_i' \) is a nonexpansive mapping and \( F(S_i) = F(S_i') \) by Lemma 2.2. Furthermore, we obtain that \( W_n \) is a nonexpansive mapping.

Remark 2.7. If \( k_i = 0 \), \( \sigma_i = 0 \) for \( i \in N \), then the definition of \( W_n \) in (2.9) reduces to the definition of \( W_n \) in (1.13).

To establish our results, we need the following technical lemmas.

Lemma 2.8 (see [18]). Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space. Let \( \{S_i\} \) be an infinite family of nonexpansive mappings of \( C \) into itself and let \( \{\tau_i\} \) be a real sequence such that \( 0 < \tau_i \leq b < 1 \) for every \( i \in N \). Then, for every \( x \in C \) and \( k \in N \), the limit \( \lim_{n \to \infty} U_{n,k} x \) exists.
In view of the previous lemma, we will define
\[
Wx := \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x, \quad x \in C. \tag{2.10}
\]

**Lemma 2.9** (see [18]). Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space. Let \( \{S_i\} \) be an infinite family of nonexpansive mappings of \( C \) into itself such that \( \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset \) and let \( \{\tau_i\} \) be a real sequence such that \( 0 < \tau_i \leq b < 1 \) for every \( i \in \mathbb{N} \). Then, \( F(W) = \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset \).

The following lemmas follow from Lemmas 2.2, 2.8, and 2.9.

**Lemma 2.10.** Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space. Let \( \{S_i\} \) be an infinite family of \( k_i \)-strictly pseudo-contractive mappings of \( C \) into itself such that \( \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset \). Define \( S'_i = \sigma_i I + (1 - \sigma_i) S_i \) and \( \sigma_i \in [k_i, 1) \) and let \( \{\tau_i\} \) be a real sequence such that \( 0 < \tau_i \leq b < 1 \) for every \( i \in \mathbb{N} \). Then, \( F(W) = \bigcap_{i=1}^{\infty} F(S_i) = \bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset \).

**Lemma 2.11** (see [28]). Let \( C \) be a nonempty closed convex subset of a Hilbert space. Let \( \{S'_i\} \) be an infinite family of nonexpansive mappings of \( C \) into itself such that \( \bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset \) and let \( \{\tau_i\} \) be a real sequence such that \( 0 < \tau_i \leq b < 1 \) for every \( i \in \mathbb{N} \). If \( K \) is any bounded subset of \( C \), then
\[
\lim_{n \to \infty} \sup_{x \in K} \|Wx - W_n x\| = 0. \tag{2.11}
\]

3. Main Results

Let \( H \) be a real Hilbert space and \( F \) be a \( k \)-Lipschitzian and \( \eta \)-strongly monotone operator with \( k > 0 \), \( \eta > 0 \), \( 0 < \mu < 2\eta/k^2 \) and \( 0 < t < 1 \). Then, for \( t \in [0, \{1, 1/\tau\}] \), \( S = (I - t\mu F) : H \to H \) is a contraction with contractive coefficient \( 1 - t\tau \) and \( \tau = (1/2)\mu(2\eta - \mu k^2) \).

In fact, from (1.2) and (1.3), we obtain
\[
\|Sx - Sy\|^2 = \|x - y - t\mu(Fx - Fy)\|^2
\leq \|x - y\|^2 + k^2 t^2 \mu^2 \|Fx - Fy\|^2 - 2t\mu\langle Fx - Fy, x - y \rangle
\leq \|x - y\|^2 + k^2 t^2 \mu^2 \|x - y\|^2 - 2t\eta \mu \|x - y\|^2 \tag{3.1}
\leq (1 - t\mu(2\eta - \mu k^2)) \|x - y\|^2
\leq (1 - t\tau)^2 \|x - y\|^2.
\]

Thus, \( S = (1 - t\mu F) \) is a contraction with contractive coefficient \( 1 - t\tau \in (0, 1) \).

Now, we show the strong convergence results for an infinite family \( k_i \)-strictly pseudo-contractive mappings in Hilbert space.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( F \) be a bifunction from \( C \times C \to \mathbb{R} \) satisfying (A1)–(A4). Let \( S_i : C \to C \) be a \( k_i \)-strictly pseudo-contractive mapping with \( \bigcap_{i=1}^{\infty} F(S_i) \cap EP \neq \emptyset \) and \( \{\tau_i\} \) be a real sequence such that \( 0 < \tau_i \leq b < 1 \), \( i \in \mathbb{N} \). Let \( f \) be a contraction of \( H \) into itself with \( \beta \in (0, 1) \) and \( B \) be \( k \)-Lipschitzian and \( \eta \)-strongly monotone...
Step 1

Proof.

We divide the proof into five steps.

\(x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + (1 - \beta_n) I - \mu \alpha_n B) y_n, \quad \forall n \in N,
\)

where \(u_n = T_{\lambda_n} x_n\) and \(\{W_n : C \rightarrow C\}\) is the sequence defined by (2.9). If \(\{\alpha_n\}\), \(\{\beta_n\}\), \(\{\delta_n\}\), and \(\{\lambda_n\}\) satisfy the following conditions:

(i) \(\{\alpha_n\} \subset (0, 1)\), \(\lim_{n \rightarrow \infty} \alpha_n = 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\),

(ii) \(0 < \lim_{n \rightarrow \infty} \inf \beta_n \leq \lim_{n \rightarrow \infty} \sup \beta_n < 1\),

(iii) \(0 < \lim_{n \rightarrow \infty} \inf \delta_n \leq \lim_{n \rightarrow \infty} \sup \delta_n < 1\), \(\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0\),

(iv) \(\{\lambda_n\} \subset (0, \infty)\), \(\lim_{n \rightarrow \infty} \lambda_n = 0\), \(\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0\).

Then \(\{x_n\}\) converges strongly to \(z \in \bigcap_{i=1}^{\infty} F(S_i) \cap EP \neq \emptyset\), where \(z\) is the unique solution of variational inequality

\[
\lim_{n \rightarrow \infty} \sup \langle (r f - \mu B) z, p - z \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(S_i) \cap EP \neq \emptyset,
\]

that is, \(z = P_{\bigcap_{i=1}^{\infty} F(S_i) \cap EP}(I - \mu B + r f) z\), which is the optimality condition for the minimization problem

\[
\min_{z \in \bigcap_{i=1}^{\infty} F(S_i) \cap EP} \frac{1}{2} \langle \mu B z, z \rangle - h(z),
\]

where \(h\) is a potential function for \(r f\) (i.e., \(h'(z) = r f(z)\) for \(z \in H\)).

Proof. We divide the proof into five steps.

Step 1. We prove that \(\{x_n\}\) is bounded.

Noting the conditions (i) and (ii), we may assume, without loss of generality, that \(\frac{\alpha_n}{1 - \beta_n} \leq \min \{1, 1/\tau\}\). For \(x, y \in C\), we obtain

\[
\|((1 - \beta_n) I - \alpha_n \mu B)x - ((1 - \beta_n) I - \alpha_n \mu B)y\| \\
\leq (1 - \beta_n) \|\left(I - \frac{\alpha_n}{1 - \beta_n} \mu B\right)x - \left(I - \frac{\alpha_n}{1 - \beta_n} \mu B\right)y\| \\
\leq (1 - \beta_n) \left(1 - \frac{\alpha_n}{1 - \beta_n} \tau\right) \|x - y\| \\
= (1 - \beta_n - \alpha_n \tau) \|x - y\|.
\]
Take \( p \in \bigcap_{i=1}^{\infty} F(S_i) \cap \text{EP} \neq \emptyset \). Since \( u_n = T_{\lambda_n} x_n \) and \( p = T_{\lambda_n} p \), then from Lemma 2.1, we know that, for any \( n \in \mathbb{N} \),

\[
\|u_n - p\| = \|T_{\lambda_n}x_n - T_{\lambda_n}p\| \leq \|x_n - p\|. \tag{3.6}
\]

Furthermore, since \( W_n p = p \) and (3.6), we have

\[
\|y_n - p\| = \|\delta_n u_n + (1 - \delta_n) W_n u_n - p\|
= \|\delta_n (u_n - p) + (1 - \delta_n) (W_n u_n - p)\|
\leq \delta_n \|u_n - p\| + (1 - \delta_n) \|W_n u_n - p\|
\leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.7}
\]

Thus, it follows from (3.7) that

\[
x_{n+1} - p = \| \alpha_n r f(x_n) + \beta_n x_n + \left( (1 - \beta_n) I - \mu \alpha_n B \right) y_n - p \|
= \| \alpha_n r f(x_n) - f(p) + \alpha_n (r f(p) - \mu B p) + \beta_n (x_n - p) + (1 - \beta_n) I - \mu \alpha_n B \right) (y_n - p) \|
\leq \alpha_n r \| f(x_n - p) + \alpha_n \| r f(p) - \mu B p \| + \beta_n \| x_n - p \|
+ (1 - \beta_n - \tau \alpha_n) \| y_n - p \|
\leq (1 - \alpha_n (\tau - r \beta)) \| x_n - p \| + \alpha_n \| r f(p) - \mu B p \|
\leq \max \left\{ \| x_n - p \|, \frac{\| r f(p) - \mu B p \|}{\tau - r \beta} \right\}. \tag{3.8}
\]

By induction, we have

\[
\| x_n - p \| \leq \max \left\{ \| x_1 - p \|, \frac{\| r f(p) - \mu B p \|}{\tau - r \beta} \right\}, \quad n \geq 1. \tag{3.9}
\]

Hence, \( \{x_n\} \) is bounded and we also obtain that \( \{u_n\}, \{W_n u_n\}, \{y_n\}, \{B y_n\}, \) and \( \{f(x_n)\} \) are all bounded. Without loss of generality, we can assume that there exists a bounded set \( K \subset C \) such that \( \{u_n\}, \{W_n u_n\}, \{y_n\}, \{B y_n\}, \{f(x_n)\} \in K \) for all \( n \in \mathbb{N} \).

**Step 2.** We show that \( \lim_{n \to \infty} \| x_n - x_{n+1} \| = 0 \).

Let \( x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n \). We note that

\[
z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n r f(x_n) + \left( (1 - \beta_n) I - \mu \alpha_n B \right) y_n}{1 - \beta_n}, \tag{3.10}
\]
and then
\[ z_{n+1} - z_n = \frac{\alpha_{n+1}r f(x_{n+1}) + ((1 - \beta_{n+1}) I - \mu \alpha_{n+1} B) y_{n+1}}{1 - \beta_{n+1}} \]
\[ - \frac{\alpha_n r f(x_n) + ((1 - \beta_n) I - \mu \alpha_n B) y_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (r f(x_{n+1}) - \mu B y_{n+1}) - \frac{\alpha_n}{1 - \beta_n} (r f(x_n) - \mu B y_n) + y_{n+1} - y_n. \]  

Therefore,
\[ \| z_{n+1} - z_n \| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\| r f(x_{n+1}) \| + \| \mu B y_{n+1} \|) \]
\[ + \frac{\alpha_n}{1 - \beta_n} (\| r f(x_n) \| + \| \mu B y_n \|) + \| y_{n+1} - y_n \|. \]  

It follows from (3.2) that
\[ \| y_{n+1} - y_n \| = \| \delta_{n+1} u_{n+1} + (1 - \delta_{n+1}) W_{n+1} u_{n+1} - (\delta_n u_n + (1 - \delta_n) W_n u_n) \| \]
\[ \leq |\delta_{n+1} - \delta_n| \| u_n \| + \delta_{n+1} \| u_{n+1} - u_n \| + (1 - \delta_{n+1}) \| W_{n+1} u_{n+1} - W_n u_n \| \]
\[ + |\delta_{n+1} - \delta_n| \| W_n u_n \|. \]  

We will estimate \( \| u_{n+1} - u_n \| \). From \( u_{n+1} = T_{\lambda_{n+1}} x_{n+1} \) and \( u_n = T_{\lambda_n} x_n \), we obtain
\[ F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}} \langle y - u_{n+1}, u_{n+1} - y_{n+1} \rangle \geq 0, \quad \forall y \in C, \]  
\[ F(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C. \]  

Taking \( y = u_n \) in (3.14) and \( y = u_{n+1} \) in (3.15), we have
\[ F(u_{n+1}, u_n) + \frac{1}{\lambda_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \]
\[ F(u_n, u_{n+1}) + \frac{1}{\lambda_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0. \]  

So, from (A2), one has
\[ \langle u_{n+1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}} \rangle \geq 0, \]  

in Equation (3.17).
furthermore,
\[\left\langle u_{n+1} - u_n, u_n - u_{n+1} - x_n - \frac{\lambda_n}{\lambda_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.18}\]

Since \(\lim_{n \to \infty} \lambda_n > 0\), we assume that there exists a real number such that \(\lambda_n > a > 0\) for all \(n \in \mathbb{N}\). Thus, we obtain
\[
\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right) (u_{n+1} - x_{n+1}) \right\rangle
\]
\[
\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \tag{3.19}\]

which means
\[
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \left|\frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\|
\]
\[
\leq \|x_{n+1} - x_n\| + \frac{1}{a} |\lambda_{n+1} - \lambda_n| \|u_{n+1} - x_{n+1}\|, \tag{3.20}\]

where \(L_1 = \sup \{\|u_{n+1} - x_{n+1}\| : n \in \mathbb{N}\}\).

Next, we estimate \(\|W_{n+1}u_{n+1} - W_n u_n\|\). Notice that
\[
\|W_{n+1}u_{n+1} - W_n u_n\| = \|W_{n+1}u_{n+1} - W_{n+1}u_n + W_{n+1}u_n - W_n u_n\|
\]
\[
\leq \|u_{n+1} - u_n\| + \|W_{n+1}u_n - W_n u_n\|. \tag{3.21}\]

From (2.9), we obtain
\[
\|W_{n+1}u_n - W_n u_n\| = \|\tau_1 S_1^t U_{n+1,2} u_n - \tau_1 S_1^t U_{n,2} u_n\|
\]
\[
\leq \tau_1 \|U_{n+1,2} u_n - U_{n,2} u_n\|
\]
\[
= \tau_1 \|\tau_2 S_2^t U_{n+1,3} u_n - \tau_2 S_2^t U_{n,3} u_n\|
\]
\[
\leq \tau_1 \tau_2 \|U_{n+1,3} u_n - U_{n,3} u_n\|
\]
\[
\leq \cdots
\]
\[
\leq \tau_1 \tau_2 \cdots \tau_n \|U_{n+1,n+1} u_n - U_{n,n+1} u_n\|
\]
\[
\leq L_2 \prod_{i=1}^{n} \tau_i, \tag{3.22}\]

where \(L_2 \geq 0\) is a constant such that \(\|U_{n+1,n+1} u_n - U_{n,n+1} u_n\| \leq L_2\), for all \(n \in \mathbb{N}\).
Substituting (3.20) and (3.22) into (3.21), we obtain
\[
\|W_{n+1}u_{n+1} - W_n u_n\| \leq \|x_{n+1} - x_n\| + L_1|\lambda_{n+1} - \lambda_n| + L_2 \prod_{i=1}^{n} \tau_i. \tag{3.23}
\]
Hence, we have
\[
\|y_{n+1} - y_n\| \leq |\delta_{n+1} - \delta_n| (\|u_n\| + \|W_n u_n\|) + \|x_{n+1} - x_n\|
+ (1 - \delta_{n+1})L_2 \prod_{i=1}^{n} \tau_i + L_1|\lambda_{n+1} - \lambda_n|
\]
\[
\leq L_3|\delta_{n+1} - \delta_n| + \|x_{n+1} - x_n\| + (1 - \delta_{n+1})L_2 \prod_{i=1}^{n} \tau_i + L_1|\lambda_{n+1} - \lambda_n|,
\tag{3.24}
\]
where \(L_3 = \sup \{\|u_n\| + \|W_n u_n\| : n \in N\}\).
Furthermore,
\[
\|z_{n+1} - z_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|rf(x_{n+1})\| + \|\mu By_{n+1}\|)
+ \frac{\alpha_n}{1 - \beta_n} (\|rf(x_n)\| + \|\mu By_n\|)
\]
\[
+ \|x_{n+1} - x_n\| + L_1|\lambda_{n+1} - \lambda_n| + L_2(1 - \delta_{n+1}) \prod_{i=1}^{n} \tau_i
\]
\[
+ L_3|\delta_{n+1} - \delta_n|.
\tag{3.25}
\]
It follows from (3.25) that
\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|rf(x_{n+1})\| + \|\mu By_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n} (\|rf(x_n)\| + \|\mu By_n\|)
\]
\[
+ L_1|\lambda_{n+1} - \lambda_n| + L_2(1 - \delta_{n+1}) \prod_{i=1}^{n} \tau_i + L_3|\delta_{n+1} - \delta_n|.
\tag{3.26}
\]
By the conditions (i), (iii), and (iv), we obtain
\[
\lim_{n \to \infty} \sup (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.27}
\]
Hence, by Lemma 2.5, one has
\[
\lim_{n \to \infty} \|z_n - x_n\| = 0, \tag{3.28}
\]
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which implies

\[ \lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - \beta_n) \| z_n - x_n \| = 0. \]  

(3.29)

**Step 3.** We claim that \( \lim_{n \to \infty} \| W u_n - u_n \| = 0. \)

Notice that

\[ \| W u_n - u_n \| = \| W u_n - W u_n + W u_n - u_n \| \]
\[ \leq \| W u_n - W u_n \| + \| W u_n - u_n \| \]
\[ \leq \sup_{u \in K} \| W u - W u_n \| + \| W u_n - u_n \|. \]  

(3.30)

It follows from (3.2) that

\[ \| W u_n - u_n \| = \| W u_n - y_n + y_n - u_n \| \]
\[ \leq \| y_n - u_n \| + \| W u_n - y_n \| \]
\[ = \| y_n - u_n \| + \delta_n \| W u_n - u_n \| \]
\[ \leq \| x_n - u_n \| + \| y_n - x_n \| + \delta_n \| W u_n - u_n \|. \]  

(3.31)

By the condition (iii), we obtain

\[ \| W u_n - u_n \| \leq \frac{1}{1 - \delta_n} (\| x_n - u_n \| + \| y_n - x_n \|). \]  

(3.32)

First, we show \( \lim_{n \to \infty} \| x_n - u_n \| = 0. \) From (3.2), for all \( p \in \cap_{i=1}^{\infty} F(S_i) \cap \text{EP}(F) \), applying Lemma 2.3 and noting that \( \| \cdot \| \) is convex, we obtain

\[ \| x_{n+1} - p \|^2 = \| \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n) I - \mu \alpha_n B \| y_n - p \|^2 \]
\[ = \| \alpha_n (f(x_n) + \mu By_n) + \beta_n (x_n - p) + (1 - \beta_n) (y_n - p) \|^2 \]
\[ \leq \| \beta_n (x_n - p) + (1 - \beta_n) (y_n - p) \|^2 + 2\alpha_n \| f(x_n) + \mu By_n \| \| x_{n+1} - p \| \]
\[ \leq \beta_n \| x_n - p \|^2 + (1 - \beta_n) \| y_n - p \|^2 + 2\alpha_n \| f(x_n) + \mu By_n \| \| x_{n+1} - p \| \]
\[ \leq \beta_n \| x_n - p \|^2 + (1 - \beta_n) \| u_n - p \|^2 + 2\alpha_n \| f(x_n) + \mu By_n \| \| x_{n+1} - p \|. \]  

(3.33)

Since \( u_n = T_{\lambda_n} x_n, p = T_{\lambda_n} p \), we have

\[ \| u_n - p \|^2 = \| T_{\lambda_n} x_n - T_{\lambda_n} p \|^2 \leq \langle x_n - p, u_n - p \rangle \]
\[ = \frac{1}{2} (\| x_n - p \|^2 + \| u_n - p \|^2 - \| x_n - u_n \|^2). \]  

(3.34)
which implies
\[ \|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \] (3.35)

Substituting (3.35) into (3.33), we have
\[ \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - \beta_n)\|x_n - u_n\|^2 + 2\alpha_n\|rf(x_n) + \mu By_n\|\|x_{n+1} - p\|, \] (3.36)
which means
\[ (1 - \beta_n)\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|rf(x_n) + \mu By_n\|\|x_{n+1} - p\| \]
\[ \leq \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\|rf(x_n) + \mu By_n\|\|x_{n+1} - p\|. \] (3.37)

Noticing \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \lim_{n \to \infty} \inf(1 - \beta_n) > 0 \), we have
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0. \] (3.38)

Second, we show \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \). It follows from (3.2) that
\[ \|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \]
\[ = \|\alpha_n rf(x_n) + \beta_n x_n + ((1 - \beta_n)I - \mu \alpha_n B)y_n - y_n\| + \|x_{n+1} - x_n\| \] (3.39)
\[ \leq \alpha_n\|rf(x_n) + \mu By_n\| + \beta_n\|x_n - y_n\| + \|x_{n+1} - x_n\|. \]
This implies that
\[ (1 - \beta_n)\|y_n - x_n\| \leq \alpha_n\|rf(x_n) + \mu By_n\| + \|x_{n+1} - x_n\|. \] (3.40)

Noticing \( \lim_{n \to \infty} \alpha_n = 0 \), \( \lim_{n \to \infty} \inf(1 - \beta_n) > 0 \) and (3.30), we have
\[ \lim_{n \to \infty} \|y_n - x_n\| = 0. \] (3.41)

Thus, substituting (3.41) and (3.38) into (3.32), we obtain
\[ \lim_{n \to \infty} \|W_n u_n - u_n\| = 0. \] (3.42)

Furthermore, (3.42), (3.30), and Lemma 2.11 lead to
\[ \lim_{n \to \infty} \|W u_n - u_n\| = 0. \] (3.43)
Step 4. Letting \( z = P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf)z \), we show

\[
\lim_{n \to \infty} \sup (rf - \mu B)z_n \leq 0. \tag{3.44}
\]

We know that \( P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf) \) is a contraction. Indeed, for any \( x, y \in H \), we have

\[
\| P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf)x - P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf)y \|
\leq \| (I - \mu B + rf)x - (I - \mu B + rf)y \|
\leq (1 - \tau + r\beta)\| x - y \|,
\]

and hence \( P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf) \) is a contraction due to \( (1 - \tau + r\beta) \in (0, 1) \). Thus, Banach’s Contraction Mapping Principle guarantees that \( P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf) \) has a unique fixed point, which implies \( z = P_{\mathcal{W} \cap \text{EP}(F)}(I - \mu B + rf)z \).

Since \( \{u_n\} \subset \mathcal{C} \), without loss of generality, we can assume that \( \{u_n\} \to \omega \), it follows from (3.43) that \( W u_n \to \omega \). Since \( \mathcal{C} \) is closed and convex, \( \mathcal{C} \) is weakly closed. Thus we have \( \omega \in \mathcal{C} \).

Let us show \( \omega \in F(W) \). For the sake of contradiction, suppose that \( \omega \notin F(W) \), that is, \( W\omega \neq \omega \). Since \( \{u_n\} \to \omega \), by the Opial condition, we have

\[
\liminf_{n \to \infty} \|u_n - \omega\| < \liminf_{n \to \infty} \|u_n - W\omega\|
\leq \liminf_{n \to \infty} \left\{ \|u_n - Wu_n\| + \|Wu_n - W\omega\| \right\}
\leq \liminf_{n \to \infty} \left\{ \|u_n - Wu_n\| + \|u_n - \omega\| \right\}. \tag{3.46}
\]

It follows (3.43) that

\[
\liminf_{n \to \infty} \|u_n - \omega\| < \liminf_{n \to \infty} \|u_n - \omega\|. \tag{3.47}
\]

This is a contradiction, which shows that \( \omega \in F(W) \).

Next, we prove that \( \omega \in \text{EP}(F) \). By (3.2), we obtain

\[
F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n \rangle \geq 0. \tag{3.48}
\]

It follows from (A2) that

\[
\frac{1}{\lambda_n}\langle y - u_n, u_n - x_n \rangle \geq F(y, u_n). \tag{3.49}
\]

Replacing \( n \) by \( n_i \), we have

\[
\left\langle y - u_{n_i}, \frac{1}{\lambda_n}(u_n - x_n) \right\rangle \geq F(y, u_{n_i}). \tag{3.50}
\]
Since \((1/\lambda_n)(u_n - x_n) \to 0\) and \(\{u_n\} \to \omega\), it follows from (A4) that \(F(y, \omega) \geq 0\) for all \(y \in C\). Put \(z_t = ty + (1 - t)\omega\) for all \(t \in (0, 1)\) and \(y \in C\). Then, we have \(z_t \in C\) and then \(F(z_t, \omega) \geq 0\). Hence, from (A1) and (A4), we have

\[
0 = F(z_t, z_t) \leq tF(z_t, y) + (1 - t)F(z_t, y) \leq tF(z_t, y),
\]

(3.51)

which means \(F(z_t, y) \geq 0\). From (A3), we obtain \(F(\omega, y) \geq 0\) for \(y \in C\) and then \(\omega \in \text{EP}(F)\). Therefore, \(\omega \in F(W) \cap \text{EP}(F)\).

Since \(z = P_{F(W) \cap \text{EP}(F)}(I - \mu B + rf)z\), it follows from (3.38), (3.42), and Lemma 2.11 that

\[
\lim_{i \to \infty} \sup \langle (rf - \mu B)z, x_n - z \rangle = \lim_{i \to \infty} \langle (rf - \mu B)z, x_n - z \rangle \\
= \lim_{i \to \infty} \langle (rf - \mu B)z, x_n - u_n \rangle \\
+ \lim_{i \to \infty} \langle (rf - \mu B)z, u_n - W_nu_n \rangle \\
+ \lim_{i \to \infty} \langle (rf - \mu B)z, W_nu_n - Wu_n \rangle \\
= \langle (rf - \mu B)z, \omega - z \rangle \leq 0.
\]

Step 5. Finally we prove that \(x_n \to \omega\) as \(n \to \infty\). In fact, from (3.2) and (3.7), we obtain

\[
\|x_{n+1} - \omega\|^2 = \|a_nr f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \mu \alpha_n B)y_n - \omega\|^2 \\
= \|a_nr(f(x_n) - f(\omega)) + \alpha_n(rf(\omega) - \mu B\omega) \\
+ \beta_n(x_n - \omega) + ((1 - \beta_n)I - \mu \alpha_n B)(y_n - \omega)\|^2 \\
= \alpha_n r \langle f(x_n) - f(\omega), x_n - \omega \rangle + \alpha_n \langle rf(\omega) - \mu B\omega, x_{n+1} - \omega \rangle \\
+ \beta_n \langle x_n - \omega, x_{n+1} - \omega \rangle + \langle ((1 - \beta_n)I - \mu \alpha_n B)(y_n - \omega), x_{n+1} - \omega \rangle \\
\leq \alpha_n r \bar{\beta} \|x_n - \omega\|^2 + \|x_{n+1} - \omega\|^2 + \alpha_n \langle rf(\omega) - \mu B\omega, x_{n+1} - \omega \rangle \\
+ \beta_n \|x_n - \omega\|^2 + \|x_{n+1} - \omega\|^2 + \frac{(1 - \beta_n - \alpha_n \tau)}{2} \|y_n - \omega\|^2 + \|x_{n+1} - \omega\|^2 \\
\leq \frac{1 - \alpha_n (\tau - r \bar{\beta})}{2} \left(\|x_n - \omega\|^2 + \|x_{n+1} - \omega\|^2\right) + \alpha_n \langle rf(\omega) - \mu B\omega, x_{n+1} - \omega \rangle, \\
\]

(3.53)
which implies

\[
\|x_{n+1} - \omega\|^2 \leq \frac{1 - \alpha_n (\tau - r \beta)}{1 + \alpha_n (\tau - r \beta)} \|x_n - \omega\|^2 \\
+ \frac{2 \alpha_n (\tau - r \beta)}{(1 + \alpha_n (\tau - r \beta))(\tau - r \beta)} \langle rf(\omega)^2 - \mu B \omega, x_{n+1} - \omega\rangle \\
\leq (1 - \alpha_n (\tau - r \beta)) \|x_n - \omega\|^2 \\
+ \frac{2 \alpha_n (\tau - r \beta)}{(1 + \alpha_n (\tau - r \beta))(\tau - r \beta)} \langle rf(\omega)^2 - \mu B \omega, x_{n+1} - \omega\rangle.
\]

From condition (i) and (3.7), we know that \(\sum_{n=1}^{\infty} \alpha_n (\tau - r \beta) = \infty\) and \(\lim_{n \to \infty} \sup (2/(1 + \alpha_n (\tau - r \beta))(\tau - r \beta)) \langle rf(\omega)^2 - \mu B \omega, x_{n+1} - \omega\rangle = 0\). We can conclude from Lemma 2.6 that \(x_n \to \omega\) as \(n \to \infty\). This completes the proof of Theorem 3.1.

**Remark 3.2.** If \(r = 1, \mu = 1, B = I\) and \(\delta_i = 0, k_i = 0, \sigma_i = 0\) for \(i \in N\), then Theorem 3.1 reduces to Theorem 3.5 of Yao et al. [19]. Furthermore, we extend the corresponding results of Yao et al. [19] from one infinite family of nonexpansive mapping to an infinite family of strictly pseudo-contractive mappings.

**Remark 3.3.** If \(\mu = 1\) and \(\delta_i = 0, k_i = 0, \sigma_i = 0\) for \(i \in N\), then Theorem 3.1 reduces to Theorem 10 of Colao and Marino [20]. Furthermore, we extend the corresponding results of Colao and Marino [20] from one infinite family of nonexpansive mapping to an infinite family of strictly pseudo-contractive mappings, and from a strongly positive bounded linear operator \(A\) to a \(k\)-Lipschitzian and \(\eta\)-strongly monotone operator \(B\).

**Theorem 3.4.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(F\) be a bifunction from \(C \times C \to \mathbb{R}\) satisfying (A1)–(A4). Let \(S : C \to C\) be a nonexpansive mapping with \(F(S) \cap EP \neq \emptyset\). Let \(f\) be a contraction of \(H\) into itself with \(\beta \in (0, 1)\) and \(B\) be \(k\)-Lipschitzian and \(\eta\)-strongly monotone operator on \(H\) with coefficients \(k, \eta > 0\). Let \(0 < \mu < 2\eta/k^2\), \(0 < r < (1/2)\mu(2\eta - \mu k^2)/\beta = \tau/\beta\) and \(\tau < 1\). Let \(\{x_n\}\) be sequence generated by

\[
F(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \delta_n u_n + (1 - \delta_n) S u_n, \\
x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \mu A_n B) y_n, \quad \forall n \in N,
\]

where \(u_n = T_{\lambda_n} x_n\). If \(\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \) and \(\{\lambda_n\}\) satisfy the following conditions:

(i) \(\{\alpha_n\} \subset (0, 1)\), \(\lim_{n \to \infty} \alpha_n = 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\),

(ii) \(0 < \lim_{n \to \infty} \inf \beta_n \leq \lim_{n \to \infty} \sup \beta_n < 1\),

(iii) \(0 < \lim_{n \to \infty} \inf \delta_n \leq \lim_{n \to \infty} \sup \delta_n < 1\), \(\lim_{n \to \infty} \delta_{n+1} - \delta_n = 0\),

(iv) \(\{\lambda_n\} \subset (0, \infty)\), \(\lim_{n \to \infty} \lambda_n > 0\), \(\lim_{n \to \infty} \lambda_{n+1} - \lambda_n = 0\).
Then \( \{x_n\} \) converges strongly to \( z \in F(S) \cap EP \neq \emptyset \), where \( z \) is the unique solution of variational inequality

\[
\lim_{n \to \infty} \sup (rf - \mu B)z, p - z) \leq 0, \quad \forall p \in F(S) \cap EP \neq \emptyset,
\]

that is, \( z = P_{F(S) \cap EP}(I - \mu B + rf)z \).

**Proof.** By Theorem 3.1, letting \( k_i = 0, \sigma_i = 0, \tau_i = 1 \) and \( S_i = S \) for \( i \in \mathbb{N} \), we can obtain Theorem 3.4.

### 4. Numerical Example

Now, we present a numerical example to illustrate our theoretical analysis results obtained in Section 3.

**Example 4.1.** Let \( H = \mathbb{R}, C = [-1, 1], S_n = I, \tau_n = \tau \in (0, 1), \lambda_n = 1, n \in \mathbb{N}, F(x, y) = 0, \) for all \( x, y \in C, B = I, r = \mu = 1, f(x) = (1/10)x, \) for all \( x, \) with contraction coefficient \( \beta = 1/5, \delta_n = 1/2, \alpha_n = 1/n, \beta_n = 1/4 + 1/2n \) for every \( n \in \mathbb{N} \). Then \( \{x_n\} \) is the sequence generated by

\[
x_{n+1} = \left(1 - \frac{9}{10n}\right)x_n,
\]

and \( \{x_n\} \to 0 \), as \( n \to \infty \), where \( 0 \) is the unique solution of the minimization problem

\[
\min_{x \in C} \frac{9}{20} x^2 + c.
\]

**Proof.** We divide the proof into four steps.

**Step 1.** We show

\[
T_{\lambda_n}x = P_Cx, \quad \forall x \in H,
\]

where

\[
P_Cx = \begin{cases} 
\frac{x}{|x|}, & x \not\in C, \\
x, & x \in C.
\end{cases}
\]

Since \( F(x, y) = 0 \), for all \( x, y \in C \), due to the definition of \( T_{\lambda_n}(x) \), for all \( x \in H \), by Lemma 2.1, we obtain

\[
T_{\lambda_n}x = \{z \in C : (y - z, z - x) \geq 0, \forall y \in C\}.
\]

By the property of \( P_C \), for \( x \in C \), we have \( T_{\lambda_n}x = P_Cx = Ix \). Furthermore, it follows from (3) in Lemma 2.1 that

\[
EP(F) = C.
\]
Step 2. We show that

\[ W_n = I. \]  \hspace{1cm} (4.7)

It follows from (2.9) that

\[ W_1 = U_{1,1} = \tau_1 S'_1 U_{1,2} + (1 - \tau_1)I = \tau_1 S'_1 + (1 - \tau_1)I, \]
\[ W_2 = U_{2,1} = \tau_1 S'_1 U_{2,2} + (1 - \tau_1)I \]
\[ = \tau_1 S'_1 (\tau_2 S'_2 U_{2,3} + (1 - \tau_2)I) + (1 - \tau_1)I \]
\[ = \tau_1 \tau_2 S'_1 S'_2 + \tau_1 (1 - \tau_2)S'_1 + (1 - \tau_1)I, \]
\[ W_3 = U_{3,1} = \tau_1 S'_1 U_{3,2} + (1 - \tau_1)I \]
\[ = \tau_1 S'_1 (\tau_2 S'_2 U_{3,3} + (1 - \tau_2)I) + (1 - \tau_1)I \]
\[ = \tau_1 \tau_2 S'_1 S'_2 U_{3,3} + \tau_1 (1 - \tau_2)S'_1 + (1 - \tau_1)I \]
\[ = \tau_1 \tau_2 \tau_3 S'_1 S'_2 S'_3 + \tau_1 \tau_2 \tau_3 S'_1 S'_2 + \tau_1 (1 - \tau_2)S'_1 + (1 - \tau_1)I. \]

Furthermore, we obtain

\[ W_n = U_{n,1} = \tau_1 \tau_2 \tau_3 \cdots \tau_n S'_1 S'_2 S'_3 \cdots S'_n + \tau_1 \tau_2 \cdots \tau_{n-1} (1 - \tau_n) S'_1 S'_2 \cdots S'_{n-1} \]
\[ + \tau_1 \tau_2 \cdots \tau_{n-2} (1 - \tau_{n-1}) S'_1 S'_2 S'_3 \cdots S'_{n-2} + \cdots + \tau_1 (1 - \tau_2) S'_1 + (1 - \tau_1)I. \]  \hspace{1cm} (4.9)

Since \( S'_i = I, \tau_i = \tau \) for \( i \in N \), one has

\[ W_n = \left[ \tau^n + \tau^{n-1}(1 - \tau) + \cdots + \tau(1 - \tau) + (1 - \tau) \right] I = I. \]  \hspace{1cm} (4.10)

Step 3. We show that

\[ x_{n+1} = \left( 1 - \frac{9}{10n} \right) x_n, \]  \hspace{1cm} (4.11)

\( \{ x_n \} \to 0 \), as \( n \to \infty \), where 0 is the unique solution of the minimization problem

\[ \min_{x \in \mathbb{C}} \frac{9}{20} x^2 + c. \]  \hspace{1cm} (4.12)
Table 1: This table shows the value of sequence \( \{x_n\} \) on each iteration step (initial value \( x_1 = 0.2 \)).

<table>
<thead>
<tr>
<th>n</th>
<th>( x_n )</th>
<th>n</th>
<th>( x_n )</th>
</tr>
</thead>
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<tr>
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<td>0.2000</td>
<td>17</td>
<td>0.0017</td>
</tr>
<tr>
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<td>0.0015</td>
</tr>
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<td>21</td>
<td>0.0014</td>
</tr>
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<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>26</td>
<td>0.0012</td>
</tr>
<tr>
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<tr>
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<td>...</td>
<td>...</td>
<td>...</td>
</tr>
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<td>0.0021</td>
<td>30</td>
<td>0.0010</td>
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<td>0.0019</td>
<td>31</td>
<td>0.0009</td>
</tr>
<tr>
<td>16</td>
<td>0.0018</td>
<td>32</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

In fact, we can see that \( B = I \) is \( k \)-Lipschitzian and \( \eta \)-strongly monotone operator on \( H \) with coefficient \( k = 1, \eta = 3/4 \) such that \( 0 < \mu < 2\eta/k^2, 0 < r < (1/2)\mu(2\eta - \mu k^2)/\beta = \tau/\beta \), so we take \( r = \mu = 1 \). Since \( S_n' = I, n \in N \), we have

\[
\bigcap_{i=1}^{\infty} F(S_i) = H. \tag{4.13}
\]

Furthermore, we obtain

\[
\bigcap_{i=1}^{\infty} F(S_i) \in EP(F) = C = [-1, 1]. \tag{4.14}
\]

Next, we need prove \( \{x_n\} \to 0, n \to \infty \). Since \( y_n = u_n \) for all \( n \in N \), we have

\[
x_{n+1} = \alpha_n r f(x_n) + \beta_n x_n + \left( (1 - \beta_n)I - \mu x_n B \right) y_n
\]

\[
= \left( 1 - \frac{9}{10n} \right) x_n, \tag{4.15}
\]

for all \( n \in N \).

Thus, we obtain a special sequence \( \{x_n\} \) of (3.2) in Theorem 3.1 as follows

\[
x_{n+1} = \left( 1 - \frac{9}{10n} \right) x_n. \tag{4.16}
\]

By Lemma 2.6, it is obviously that \( x_n \to 0 \), 0 is the unique solution of the minimization problem

\[
\min_{x \in C} \frac{9}{20} x^2 + c, \tag{4.17}
\]

where \( c \) is a constant number.
Step 4. Finally, we use software Matlab 7.0 to give the numerical experiment results and then obtain Table 1 which show that the iteration process of the sequence \( \{x_n\} \) is a monotone decreasing sequence and converges to 0. From Table 1 and the corresponding graph Figure 1, we show that the more the iteration steps are, the more slowly the sequence \( \{x_n\} \) converges to 0.

\begin{figure}[h]
\centering
\includegraphics[width=\columnwidth]{graph.png}
\caption{The corresponding graph at \( x = 0.2 \).}
\end{figure}

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**References**


