Research Article

Approximation of Common Fixed Points of Nonexpansive Semigroups in Hilbert Spaces

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Let H be a real Hilbert space. Consider on H a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$ with a common fixed point, a contraction f with the coefficient $0 < \alpha < 1$, and a strongly positive linear bounded self-adjoint operator A with the coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \bar{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated by the iterative method $x_0 \in H$, $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)(1/s_n) \int_0^{s_n} T(s)x_n ds$, $n \geq 0$ converges strongly to a common fixed point $x^* \in F(S)$, where $F(S)$ denotes the common fixed point of the nonexpansive semigroup. The point x^* solves the variational inequality $\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0$ for all $x \in F(S)$.

1. Introduction and Preliminaries

Let H be a real Hilbert space and T be a nonlinear mapping with the domain $D(T)$. A point $x \in D(T)$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T ; that is, $F(T) = \{x \in D(T) : Tx = x\}$. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(A). \quad (1.1)$$

Recall that a family $S = \{T(s) \mid s \geq 0\}$ of mappings from H into itself is called a one-parameter nonexpansive semigroup if it satisfies the following conditions:

- (i) $T(0)x = x$, for all $x \in H$;
- (ii) $T(s+t)x = T(s)T(t)x$, for all $s, t \geq 0$ and for all $x \in H$;
- (iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$, for all $s \geq 0$ and for all $x, y \in H$;
- (iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(S)$ the set of common fixed points of S , that is, $F(S) = \bigcap_{0 \leq s < \infty} F(T(s))$. It is known that $F(S)$ is closed and convex; see [1]. Let C be a nonempty closed and convex subset of H . One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping; see [2, 3]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \rightarrow C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C, \quad (1.2)$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C . If T enjoys a nonempty fixed point set, Browder [2] proved the following well-known strong convergence theorem.

Theorem B. *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C . Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1 - t)Tz_t$ for $t \in (0, 1)$. Then as $t \rightarrow 0$, $\{z_t\}$ converges strongly to a element of $F(T)$ nearest to u .*

As motivated by Theorem B, Halpern [4] considered the following explicit iteration:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.3)$$

and proved the following theorem.

Theorem H. *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C . Define a real sequence $\{\alpha_n\}$ in $[0, 1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.3). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u .*

In 1977, Lions [5] improved the result of Halpern [4], still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\lim_{n \rightarrow \infty} (\alpha_{n+1} - \alpha_n) / \alpha_{n+1}^2 = 0$.

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n + 1)$. This was overcome in 1992 by Wittmann [6], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Recall that a mapping $f : H \rightarrow H$ is an α -contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H. \quad (1.4)$$

Recall that an operator A is strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \quad (1.5)$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [7–13] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping T on a real Hilbert space H :

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.6)$$

where A is a linear bounded operator on H and b is a given point in H . In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \quad (1.7)$$

strongly converges to the unique solution of the minimization problem (1.6) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [9] studied the following continuous scheme:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \quad (1.8)$$

where f is an α -contraction on a real Hilbert space H , A is a bounded linear strongly positive operator and $\gamma > 0$ is a constant. They showed that $\{x_t\}$ strongly converges to a fixed point $\bar{x}$ of T . Also in [9], they introduced a general explicit iterative scheme by the viscosity approximation method:

$$x_n \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0 \quad (1.9)$$

and proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to a unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \quad (1.10)$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \quad (1.11)$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In this paper, motivated by Li et al. [8], Marino and Xu [9], Plubtieng and Punpaeng [14], Shioji and Takahashi [15], and Shimizu and Takahashi [16], we consider the mapping T_t defined as follows:

$$T_t x = t\gamma f(x) + (I - tA) \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x ds, \quad (1.12)$$

where $\gamma > 0$ is a constant, f is an α -contraction, A is a bounded linear strongly positive self-adjoint operator and $\{\lambda_t\}$ is a positive real divergent net. If $\gamma\alpha < \bar{\gamma}$ for each $0 < t < \|A\|^{-1}$, one can see that T_t is a $(1 - t(\bar{\gamma} - \gamma\alpha))$ -contraction. So, by Banach's contraction mapping principle, there exists a unique solution x_t of the fixed point equation

$$x_t = t\gamma f(x_t) + (I - tA) \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds. \quad (1.13)$$

We show that the sequence $\{x_t\}$ generated by above continuous scheme strongly converges to a common fixed point $x^* \in F(S)$, which is the unique point in $F(S)$ solving the variational inequality $\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0$ for all $x \in F(S)$. Furthermore, we also study the following explicit iterative scheme:

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \quad (1.14)$$

We prove that the sequence $\{x_n\}$ generated by (1.14) converges strongly to the same x^* .

The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [9], Plubtieng and Punpaeng [14], Shioji and Takahashi [15], and Shimizu and Takahashi [16].

In order to prove our main result, we need the following lemmas.

Lemma 1.1 (see [16]). *Let D be a nonempty bounded closed convex subset of a Hilbert space H and let $S = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on D . Then, for any $0 \leq h < \infty$,*

$$\limsup_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(s)x ds - T(h) \frac{1}{t} \int_0^t T(s)x ds \right\| = 0. \quad (1.15)$$

Lemma 1.2 (see [17]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ strongly converges to y , then $(I - T)x = y$.*

Lemma 1.3 (see [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C (i.e., for $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \quad (1.16)$$

Lemma 1.4. Let H be a Hilbert space, f a α -contraction, and A a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\alpha$,

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H. \quad (1.17)$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Proof. From the definition of strongly positive linear bounded operator, we have

$$\langle x - y, A(x - y) \rangle \geq \bar{\gamma}\|x - y\|^2. \quad (1.18)$$

On the other hand, it is easy to see

$$\langle x - y, \gamma f x - \gamma f y \rangle \leq \gamma\alpha\|x - y\|^2. \quad (1.19)$$

Therefore, we have

$$\begin{aligned} \langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle &= \langle x - y, A(x - y) \rangle - \langle x - y, \gamma f x - \gamma f y \rangle \\ &\geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2 \end{aligned} \quad (1.20)$$

for all $x, y \in H$. This completes the proof. $\square$

Remark 1.5. Taking $\gamma = 1$ and $A = I$, the identity mapping, we have the following inequality:

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in H. \quad (1.21)$$

Furthermore, if f is a nonexpansive mapping in Remark 1.5, we have

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq 0, \quad x, y \in H. \quad (1.22)$$

Lemma 1.6 (see [9]). Assume A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 1.7 (see [12]). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following condition:

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, \quad \forall n \geq 0, \quad (1.23)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence of real numbers such that

- (i) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,
- (ii) either $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

2. Main Results

Lemma 2.1. *Let H a real Hilbert space and $S = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup on H such that $F(S) \neq \emptyset$. Let $\{\lambda_t\}_{0 < t < 1}$ be a continuous net of positive real numbers such that $\lim_{t \rightarrow 0} \lambda_t = \infty$. Let $f : H \rightarrow H$ be an α -contraction, A a strongly positive linear bounded self-adjoint operator of H into itself with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_t\}$ be a sequence defined by (1.13). Then*

- (i) $\{x_t\}$ is bounded for all $t \in (0, \|A\|^{-1})$;
- (ii) $\lim_{t \rightarrow 0} \|T(\tau)x_t - x_t\| = 0$ for all $0 \leq \tau < \infty$;
- (iii) x_t defines a continuous curve from $(0, \|A\|^{-1})$ into H .

Proof. (i) Taking $p \in F(S)$, we have

$$\begin{aligned}
 \|x_t - p\| &\leq \left\| t\gamma f(x_t) + (I - tA) \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds - p \right\| \\
 &\leq t\|\gamma f(x_t) - Ap\| + (1 - t\bar{\gamma}) \frac{1}{\lambda_t} \int_0^{\lambda_t} \|T(s)x_t - p\| ds \\
 &\leq t\|\gamma f(x_t) - Ap\| + (1 - t\bar{\gamma}) \|x_t - p\| \\
 &\leq t\gamma \|f(x_t) - f(p)\| + t\|\gamma f(p) - Ap\| + (1 - t\bar{\gamma}) \|x_t - p\| \\
 &\leq [1 - t(\bar{\gamma} - \gamma\alpha)] \|x_t - p\| + t\|\gamma f(p) - Ap\|.
 \end{aligned} \tag{2.1}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\|. \tag{2.2}$$

This implies that $\{x_t\}$ is not only bounded, but also that $\{x_t\}$ is contained in $B(p, 1/(\bar{\gamma} - \gamma\alpha)\|\gamma f(p) - Ap\|)$ of center p and radius $1/(\bar{\gamma} - \gamma\alpha)\|\gamma f(p) - Ap\|$, for all fixed $p \in F(S)$. Moreover for $p \in F(S)$ and $t \in (0, \|A\|^{-1})$,

$$\begin{aligned}
 \left\| \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds - p \right\| &= \left\| \frac{1}{\lambda_t} \int_0^{\lambda_t} (T(s)x_t - T(s)p) ds \right\| \\
 &\leq \|x_t - p\| \\
 &\leq \frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\|.
 \end{aligned} \tag{2.3}$$

(ii) Observe that

$$\begin{aligned}
 \|T(\tau)x_t - x_t\| &\leq \left\| T(\tau)x_t - T(\tau) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right) \right\| \\
 &\quad + \left\| T(\tau) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right) - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right\|
 \end{aligned}$$

$$\begin{aligned}
& + \left\| \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds - x_t \right\| \\
& \leq 2 \left\| x_t - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right\| \\
& \quad + \left\| T(\tau) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right) - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right\| \\
& = 2t \left\| \gamma f(x_t) - A \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right\| \\
& \quad + \left\| T(\tau) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right) - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right\|.
\end{aligned} \tag{2.4}$$

Taking $B(p, 1/(\bar{\gamma} - \gamma\alpha)\|\gamma f(p) - Ap\|)$ as D in Lemma 1.1 and passing to $\lim_{t \rightarrow 0}$ in (2.4), we can obtain (ii) immediately.

(iii) Taking $t_1, t_2 \in (0, \|A\|^{-1})$ and fixing $p \in F(S)$, we see that

$$\begin{aligned}
& \|x_{t_1} - x_{t_2}\| \\
& \leq \left\| (t_1 - t_2)\gamma f(x_{t_1}) + t_2\gamma(f(x_{t_1}) - f(x_{t_2})) - (t_1 - t_2)A \frac{1}{\lambda_{t_1}} \int_0^{\lambda_{t_1}} T(s)x_{t_1} ds \right. \\
& \quad \left. + (I - t_2A) \left(\frac{1}{\lambda_{t_1}} \int_0^{\lambda_{t_1}} T(s)x_{t_1} ds - \frac{1}{\lambda_{t_2}} \int_0^{\lambda_{t_2}} T(s)x_{t_2} ds \right) \right\| \\
& \leq |t_1 - t_2|\gamma\|f(x_{t_1})\| + t_2\gamma\alpha\|x_{t_1} - x_{t_2}\| + |t_1 - t_2|\|A\| \left\| \frac{1}{\lambda_{t_1}} \int_0^{\lambda_{t_1}} T(s)x_{t_1} ds \right\| \\
& \quad + (1 - t_2\bar{\gamma}) \left\| \frac{1}{\lambda_{t_1}} \int_0^{\lambda_{t_1}} T(s)x_{t_1} ds - \frac{1}{\lambda_{t_2}} \int_0^{\lambda_{t_2}} T(s)x_{t_2} ds - \frac{1}{\lambda_{t_2}} \int_{\lambda_{t_1}}^{\lambda_{t_2}} T(s)x_{t_2} ds \right\| \\
& \leq |t_1 - t_2|\gamma\|f(x_{t_1})\| + t_2\gamma\alpha\|x_{t_1} - x_{t_2}\| + |t_1 - t_2|\|A\| \left\| \frac{1}{\lambda_{t_1}} \int_0^{\lambda_{t_1}} T(s)x_{t_1} ds \right\| \\
& \quad + (1 - t_2\bar{\gamma}) \left(\|x_{t_1} - x_{t_2}\| + \left| \frac{1}{\lambda_{t_1}} - \frac{1}{\lambda_{t_2}} \right| \left\| \int_0^{\lambda_{t_1}} T(s)x_{t_2} ds \right\| + \frac{1}{\lambda_{t_2}} \left\| \int_{\lambda_{t_1}}^{\lambda_{t_2}} T(s)x_{t_2} ds \right\| \right).
\end{aligned} \tag{2.5}$$

Thus applying (2.3), we arrive at

$$\begin{aligned}
& \|x_{t_1} - x_{t_2}\| \\
& \leq |t_1 - t_2|\gamma\|f(x_{t_1})\| + t_2\gamma\alpha\|x_{t_1} - x_{t_2}\| + |t_1 - t_2|\|A\| \left(\frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\| + \|p\| \right) \\
& \quad + (1 - t_2\bar{\gamma}) \left(\|x_{t_1} - x_{t_2}\| + \frac{2}{\lambda_{t_2}} |\lambda_{t_2} - \lambda_{t_1}| \left(\frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\| + \|p\| \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq |t_1 - t_2| \left(\gamma \|f(x_{t_1})\| + \|A\| \left(\frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\| + \|p\| \right) \right) \\
&\quad + (1 - t_2(\bar{\gamma} - \gamma\alpha)) \|x_{t_1} - x_{t_2}\| + \frac{2}{\lambda_{t_2}} |\lambda_{t_2} - \lambda_{t_1}| \left(\frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\| + \|p\| \right).
\end{aligned} \tag{2.6}$$

It follows that

$$\|x_{t_1} - x_{t_2}\| \leq M_1 |t_1 - t_2| + M_2 |\lambda_{t_2} - \lambda_{t_1}|, \tag{2.7}$$

where

$$M_1 = \frac{\gamma(\bar{\gamma} - \gamma\alpha) \|f(x_{t_1})\| + \|A\| \|\gamma f(p) - Ap\| + (\bar{\gamma} - \gamma\alpha) \|A\| \|p\|}{t_2(\bar{\gamma} - \gamma\alpha)^2} \tag{2.8}$$

and

$$M_2 = \frac{2(\|\gamma f(p) - Ap\| + (\bar{\gamma} - \gamma\alpha) \|p\|)}{\lambda_{t_2} t_2 (\bar{\gamma} - \gamma\alpha)^2}. \tag{2.9}$$

This inequality, together with the continuity of the net $\{\lambda_t\}$, gives the continuity of the curve $\{x_t\}$. $\square$

Theorem 2.2. Let H be a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup such that $F(S) \neq \emptyset$. Let $\{\lambda_t\}_{0 < t < 1}$ be a net of positive real numbers such that $\lim_{t \rightarrow 0} \lambda_t = \infty$. Let f be an α -contraction and let A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Then sequence $\{x_t\}$ defined by (1.13) strongly converges as $t \rightarrow 0$ to $x^* \in F(S)$, which solves the following variational inequality:

$$\langle (\gamma f - A)x^*, p - x^* \rangle \leq 0, \quad \forall p \in F(S). \tag{2.10}$$

Equivalently, one has

$$P_{F(S)}(I - A + \gamma f)x^* = x^*. \tag{2.11}$$

Proof. The uniqueness of the solution of the variational inequality (2.10) is a consequence of the strong monotonicity of $A - \gamma f$ (Lemma 1.4) and it was proved in [9]. Next, we will use $x^* \in F(S)$ to denote the unique solution of (2.10). To prove that $x_t \rightarrow x^*$ ($t \rightarrow 0$), we write, for a given $p \in F(S)$,

$$x_t - p = t(\gamma f(x_t) - Ap) + (I - tA) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds - p \right). \tag{2.12}$$

Using $x_t - p$ to make inner product, we obtain that

$$\begin{aligned}\|x_t - p\|^2 &= \left\langle (I - tA) \left(\frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds - p \right), x_t - p \right\rangle + t \langle \gamma f(x_t) - Ap, x_t - p \rangle \\ &\leq (1 - t\bar{\gamma}) \|x_t - p\|^2 + t \langle \gamma f(x_t) - Ap, x_t - p \rangle.\end{aligned}\quad (2.13)$$

It follows that

$$\begin{aligned}\|x_t - p\|^2 &\leq \frac{1}{\bar{\gamma}} (\gamma \langle f(x_t) - f(p), x_t - p \rangle + \langle \gamma f(p) - Ap, x_t - p \rangle) \\ &\leq \frac{\gamma\alpha}{\bar{\gamma}} \|x_t - p\|^2 + \frac{1}{\bar{\gamma}} \langle \gamma f(p) - Ap, x_t - p \rangle,\end{aligned}\quad (2.14)$$

which yields that

$$\|x_t - p\|^2 \leq \frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(p) - Ap, x_t - p \rangle. \quad (2.15)$$

Since H is a Hilbert space and $\{x_t\}$ is bounded as $t \rightarrow 0$, we have that if $\{t_n\}$ is a sequence in $(0, 1)$ such that $t_n \rightarrow 0$ and $x_{t_n} \rightarrow \bar{x}$. By (2.15), we see $x_{t_n} \rightarrow \bar{x}$. Moreover, by (ii) of Lemma 2.1 we have $\bar{x} \in F(S)$. We next prove that $\bar{x}$ solves the variational inequality (2.10). From (1.13), we arrive at

$$(A - \gamma f)x_t = -\frac{1}{t}(I - tA) \left[x_t - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right]. \quad (2.16)$$

For $p \in F(S)$, it follows from (1.22) that

$$\begin{aligned}\langle (A - \gamma f)x_t, x_t - p \rangle &= -\frac{1}{t} \left\langle (I - tA) \left[x_t - \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right], x_t - p \right\rangle \\ &= -\frac{1}{t} \left\langle \frac{1}{\lambda_t} \int_0^{\lambda_t} [(I - T(s))x_t - (I - T(s))p] ds, x_t - p \right\rangle \\ &\quad + \left\langle A \frac{1}{\lambda_t} \int_0^{\lambda_t} (I - T(s))x_t ds, x_t - p \right\rangle \\ &= -\frac{1}{t\lambda_t} \int_0^{\lambda_t} \langle (I - T(s))x_t - (I - T(s))p, x_t - p \rangle ds \\ &\quad + \left\langle A \frac{1}{\lambda_t} \int_0^{\lambda_t} (I - T(s))x_t ds, x_t - p \right\rangle \\ &\leq \left\langle A \frac{1}{\lambda_t} \int_0^{\lambda_t} (I - T(s))x_t ds, x_t - p \right\rangle\end{aligned}$$

$$\begin{aligned}
&= \left\langle A \left(t\gamma f(x_t) - tA \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right), x_t - p \right\rangle \\
&= t \left\langle A \left(\gamma f(x_t) - A \frac{1}{\lambda_t} \int_0^{\lambda_t} T(s)x_t ds \right), x_t - p \right\rangle.
\end{aligned} \tag{2.17}$$

Passing to $\lim_{t \rightarrow 0}$, since $\{x_t\}$ is a bounded sequence, we obtain

$$\langle (A - \gamma f)\bar{x}, \bar{x} - p \rangle \leq 0, \tag{2.18}$$

that is, $\bar{x}$ satisfies the variational inequality (2.10). By the uniqueness it follows $\bar{x} = x^*$. In a summary, we have shown that each cluster point of $\{x_t\}$ (as $t \rightarrow 0$) equals x^* . Therefore, $x_t \rightarrow x^*$ as $t \rightarrow 0$. The variational inequality (2.10) can be rewritten as

$$\langle [(I - A + \gamma f)x^*] - x^*, x^* - p \rangle, \quad p \in F(S). \tag{2.19}$$

This, by Lemma 1.3, is equivalent to

$$P_{F(S)}(I - A + \gamma f)x^* = x^*. \tag{2.20}$$

This completes the proof. $\square$

Remark 2.3. Theorem 2.2 which include the corresponding results of Shioji and Takahashi [15] as a special case is reduced to Theorem 3.1 of Plubtieng and Punpaeng [14] when $A = I$, the identity mapping and $\gamma = 1$.

Theorem 2.4. Let H be a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let f be an α -contraction and let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Then sequence $\{x_n\}$ defined by (1.14) strongly converges to $x^* \in F(S)$, which solves the variational inequality (2.10).

Proof. We divide the proof into three parts.

Step 1. Show the sequence $\{x_n\}$ is bounded.

Noticing that $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$, we may assume, with no loss of generality, that $\alpha_n/(1 - \beta_n) < \|A\|^{-1}$ for all $n \geq 0$. From Lemma 1.6, we know that $\|(1 - \beta_n)I - \alpha_n A\| \leq (1 - \beta_n - \alpha_n \bar{\gamma})$. Picking $p \in F(S)$, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \left\| \alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A) \left(\frac{1}{s_n} \int_0^{s_n} T(s)x_n ds - p \right) \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \left\| \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds - p \right\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - p\| \\
&\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned} \tag{2.21}$$

By simple inductions, we see that

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|Ap - \gamma f(p)\|}{\bar{\gamma} - \gamma\alpha} \right\}, \tag{2.22}$$

which yields that the sequence $\{x_n\}$ is bounded.

Step 2. Show that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - A)x^*, y_n - x^* \rangle \leq 0, \tag{2.23}$$

where x^* is obtained in Theorem 2.2 and $y_n = (1/s_n) \int_0^{s_n} T(s)x_n ds$.

Putting $z_0 = P_{F(S)}x_0$, from (2.22) we see that the closed ball M of center z_0 and radius $\max\{\|z_0 - p\|, \|Az_0 - \gamma f(z_0)\|/(\bar{\gamma} - \gamma\alpha)\}$ is $T(s)$ -invariant for each $s \in [0, \infty)$ and contain $\{x_n\}$. Therefore, we assume, without loss of generality, $S = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on M . It follows from Lemma 1.1 that

$$\lim_{n \rightarrow \infty} \|y_n - T(h)y_n\| = 0 \tag{2.24}$$

for all $0 \leq h < \infty$. Taking a suitable subsequence $\{y_{n_i}\}$ of $\{y_n\}$, we see that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - A)x^*, y_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle (\gamma f - A)x^*, y_{n_i} - x^* \rangle. \tag{2.25}$$

Since the sequence $\{y_n\}$ is also bounded, we may assume that $y_{n_i} \rightharpoonup \bar{x}$. From the demiclosedness principle, we have $\bar{x} \in F(S)$. Therefore, we have

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - A)x^*, y_n - x^* \rangle = \langle (\gamma f - A)x^*, \bar{x} - x^* \rangle \leq 0. \tag{2.26}$$

On the other hand, we have

$$\|x_{n+1} - y_n\| \leq \alpha_n \|\gamma f(x_n) - Ax_n\| + \beta_n \|x_n - y_n\|. \tag{2.27}$$

From the assumption $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$, we see that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0, \tag{2.28}$$

which combines with (2.26) gives that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \leq 0. \quad (2.29)$$

Step 3. Show $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Note that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &= \langle \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A)(y_n - x^*), x_{n+1} - x^* \rangle \\ &= \alpha_n \langle \gamma f(x_n) - Ax^*, x_{n+1} - x^* \rangle + \beta_n \langle x_n - x^*, x_{n+1} - x^* \rangle \\ &\quad + \langle ((1 - \beta_n)I - \alpha_n A)(y_n - x^*), x_{n+1} - x^* \rangle \\ &\leq \alpha_n \langle \gamma f(x_n) - f(x^*), x_{n+1} - x^* \rangle + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\ &\quad + \beta_n \|x_n - x^*\| \|x_{n+1} - x^*\| + \|(1 - \beta_n)I - \alpha_n A\| \|y_n - x^*\| \|x_{n+1} - x^*\| \\ &\leq \alpha_n \alpha \gamma \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\ &\quad + \beta_n \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &= [1 - \alpha_n(\bar{\gamma} - \gamma \alpha)] \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\ &\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma \alpha)}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle. \\ &\leq \frac{1 - \alpha_n(\bar{\gamma} - \gamma \alpha)}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|x_{n+1} - x^*\|^2 + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle. \end{aligned} \quad (2.30)$$

It follows that

$$\|x_{n+1} - x^*\|^2 \leq [1 - \alpha_n(\bar{\gamma} - \gamma \alpha)] \|x_n - x^*\|^2 + 2\alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle. \quad (2.31)$$

By using Lemma 1.7, we can obtain the desired conclusion easily. $\square$

Remark 2.5. If $\gamma = 1$ and $A = I$, the identity mapping, then Theorem 2.4 is reduced to Theorem 3.3 of Plubtieng and Punpaeng [14].

If the sequence $\{\beta_n\} \equiv 0$, then Theorem 2.4 is reduced to the following.

Corollary 2.6. Let H be a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let f be a α -contraction and let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \quad (2.32)$$

Then the sequence $\{x_n\}$ defined by above iterative algorithm converges strongly to $x^* \in F(S)$, which solves the variational inequality (2.10).

Remark 2.7. Corollary 2.6 includes Theorem 2 of Shioji and Takahashi [15] as a special case.

Remark 2.8. Theorem 2.2 and Corollary 2.6 improve Theorem 3.2 and Theorem 3.4 of Marino and Xu [9] from a single nonexpansive mapping to a nonexpansive semigroup, respectively.

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