Research Article

Inequalities between Power Means and Convex Combinations of the Harmonic and Logarithmic Means

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We prove that $\alpha H(a, b) + (1 - \alpha)L(a, b) > M_{(1-4\alpha)/3}(a, b)$ for $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$ if and only if $\alpha \in [1/4, 1)$ and $\alpha H(a, b) + (1 - \alpha)L(a, b) < M_{(1-4\alpha)/3}(a, b)$ if and only if $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$, and the parameter $(1 - 4\alpha)/3$ is the best possible in either case. Here, $H(a, b) = 2ab/(a + b)$, $L(a, b) = (a - b)/(\log a - \log b)$, and $M_p(a, b) = ((a^p + b^p)/2)^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ are the harmonic, logarithmic, and p th power means of a and b , respectively.

1. Introduction

The classical logarithmic mean $L(a, b)$ of two positive real numbers a and b with $a \neq b$ is defined by

$$L(a, b) = \frac{a - b}{\log a - \log b}. \quad (1.1)$$

In the recent past, the bivariate means have been the subject of intensive research. In particular, many remarkable inequalities for $L(a, b)$ can be found in the literature [1–21]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [22–24]. In [22] the authors study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of $L(a, b)$ as an infinite product and an iterative algorithm for computing it as the common limit of two sequences of special geometric and arithmetic means are given in [4].

In [25, 26] it is shown that $L(a, b)$ can be expressed in terms of Gauss hypergeometric function ${}_2F_1$. And, in [26] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive; that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \dots < a_n$ and $0 < b_1 < b_2 < \dots < b_n$, is positive for all $n \geq 1$.

Let $G(a, b) = \sqrt{ab}$, $H(a, b) = 2ab/(a + b)$, $I(a, b) = 1/e(a^a/b^b)^{1/(a-b)}$, $A(a, b) = (a + b)/2$, $M_p(a, b) = ((a^p + b^p)/2)^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$, and $L_p(a, b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the geometric, harmonic, identric, arithmetic, p th power, and p th Lehmer means of two positive numbers a and b , respectively. Then it is well known that both $M_p(a, b)$ and $L_p(a, b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) = L_{-1}(a, b) < G(a, b) = M_0(a, b) = L_{-1/2}(a, b) \\ < L(a, b) < I(a, b) < A(a, b) = M_1(a, b) = L_0(a, b) < \max\{a, b\} \end{aligned} \quad (1.2)$$

hold for all $a, b > 0$ with $a \neq b$.

In [4], Carlson proves that the double inequality

$$\sqrt{\frac{G(a, b)(A(a, b) + G(a, b))}{2}} < L(a, b) < \frac{1}{2}(A(a, b) + G(a, b)) \quad (1.3)$$

holds for all $a, b > 0$ with $a \neq b$.

In [5], Lin finds the best possible upper and lower power bounds for the logarithmic mean as follows:

$$M_0(a, b) < L(a, b) < M_{1/3}(a, b) \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

In [9], Sándor establishes that

$$\sqrt{G(a, b)I(a, b)} < L(a, b) < A(a, b) + G(a, b) - I(a, b) \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

In [27], Alzer gives the optimal Lehmer mean bounds for L , $(LI)^{1/2}$, and $(L + I)/2$ as follows:

$$\begin{aligned} L_{-1/3}(a, b) < L(a, b) < L_0(a, b), \\ L_{-1/4}(a, b) < \sqrt{L(a, b)I(a, b)} < L_0(a, b), \\ L_{-1/4}(a, b) < \frac{1}{2}(L(a, b) + I(a, b)) < L_0(a, b) \end{aligned} \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for $(LI)^{1/2}$ and $(L + I)/2$ in terms of power mean are presented in [28]:

$$\begin{aligned} M_0(a, b) &< \sqrt{L(a, b)I(a, b)} < M_{1/2}(a, b), \\ M_{\log 2 / (1 + \log 2)}(a, b) &< \frac{1}{2}(L(a, b) + I(a, b)) < M_{1/2}(a, b) \end{aligned} \quad (1.7)$$

for all $a, b > 0$ with $a \neq b$.

In [29, 30], the authors obtain the sharp bounds for the products $A^\alpha(a, b)L^{1-\alpha}(a, b)$ and $G^\alpha(a, b)L^{1-\alpha}(a, b)$ and the sum $\alpha A(a, b) + (1 - \alpha)L(a, b)$ in terms of power mean as follows:

$$\begin{aligned} M_0(a, b) &< A^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1+2\alpha)/3}(a, b), \\ M_0(a, b) &< G^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1-\alpha)/3}(a, b), \\ M_{\log 2 / (\log 2 - \log \alpha)}(a, b) &< \alpha A(a, b) + (1 - \alpha)L(a, b) < M_{(1+2\alpha)/3}(a, b) \end{aligned} \quad (1.8)$$

for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$.

In [31], Zhu presents some bounds for $I(a, b)$ in terms of $A(a, b)$ and $L(a, b)$ and $L(a, b)$ in terms of $G(a, b)$ and $I(a, b)$.

In [32], Chu et al. prove that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.

It is the aim of this paper to give the optimal power mean bounds for the convex combination of harmonic and logarithmic means. Our main result is the following theorem.

Theorem 1.1. For $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$, one has

- (1) $\alpha H(a, b) + (1 - \alpha)L(a, b) > M_{(1-4\alpha)/3}(a, b)$ if and only if $\alpha \in [1/4, 1)$;
- (2) $\alpha H(a, b) + (1 - \alpha)L(a, b) < M_{(1-4\alpha)/3}(a, b)$ if and only if $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$.

In particular, the parameter $(1 - 4\alpha)/3$ is the best possible in either case.

2. Lemmas

In order to establish our main result we need to establish four lemmas, which we present in this section.

Lemma 2.1. Let $\alpha \in (1/4, 1)$, $p = (1 - 4\alpha)/3 \in (-1, 0)$, and $g(t) = -4\alpha p(p+1)^2(p+2)t^{p-1} + 2(1 - \alpha)p^2(1 - p^2)t^{p-2} + 2(1 - \alpha)p(1 - p)^2(2 - p)t^{p-3} + 12(1 - \alpha)(1 - p)$. Then $g(t) > 0$ for $t \in [1, +\infty)$.

Proof. Simple computations lead to

$$g(1) = \frac{64}{81}(1 - \alpha)^2(56\alpha^2 + 23\alpha + 11) > 0, \quad (2.1)$$

$$\lim_{t \rightarrow +\infty} g(t) = 12(1 - \alpha)(1 - p) = 8(1 - \alpha)(1 + 2\alpha) > 0, \quad (2.2)$$

$$g'(t) = -2p(1 - p)t^{p-4}g_1(t), \quad (2.3)$$

where

$$g_1(t) = -2\alpha(p+1)^2(p+2)t^2 + (1-\alpha)p(p+1)(2-p)t + (1-\alpha)(1-p)(2-p)(3-p),$$

$$g_1(1) = \frac{4}{27}(1-\alpha)(148\alpha^2 - 11\alpha + 25) > 0, \quad (2.4)$$

$$\lim_{t \rightarrow +\infty} g_1(t) = -\infty, \quad (2.5)$$

$$g'_1(t) = -4\alpha(p+1)^2(p+2)t + (1-\alpha)p(p+1)(2-p)$$

$$= -\frac{4}{27}(1-\alpha)^2[16\alpha(7-4\alpha)t + (4\alpha-1)(4\alpha+5)] < 0 \quad (2.6)$$

for $t \in [1, +\infty)$.

Inequality (2.6) implies that $g_1(t)$ is strictly decreasing in $[1, +\infty)$. Then (2.4) and (2.5) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g_1(t) > 0$ for $t \in [1, \lambda_1]$ and $g_1(t) < 0$ for $t \in (\lambda_1, +\infty)$. It follows from (2.3) that $g(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the piecewise monotonicity of $g(t)$. $\square$

Lemma 2.2. Let $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, and $h(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^p + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^{p-1} + (1-\alpha)p(p^3 - 8p^2 - p + 4)t^{p-2} + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4)t^{p-3} + 4(1-\alpha)(7-4p) - 4p(1-\alpha)t^{-1} + 4\alpha(1+p)t^{-2}$. Then $h(t) > 0$ for $t \in [1, +\infty)$.

Proof. Let

$$h_1(t) = t^{3-p}h(t). \quad (2.7)$$

Then simple computations lead to

$$h_1(1) = \frac{16}{27}(1-\alpha)(80\alpha^2 + 110\alpha - 1) > 0, \quad (2.8)$$

$$h'_1(t) = -3(1-\alpha)(p+1)(p+2)^2(p+3)t^2 + 2(p+1)$$

$$\times (p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t + (1-\alpha)p(p^3 - 8p^2 - p + 4)$$

$$+ 4(1-\alpha)(7-4p)(3-p)t^{2-p} - 4p(1-\alpha)(2-p)t^{1-p} + 4\alpha(1-p^2)t^{-p},$$

$$h'_1(1) = \frac{32}{27}(1-\alpha)(-16\alpha^3 + 38\alpha^2 + 176\alpha - 9) > 0, \quad (2.9)$$

$$\begin{aligned}
h_1''(t) = & -6(1-\alpha)(p+1)(p+2)^2(p+3)t + 2(p+1) \\
& \times (p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha) \\
& + 4(1-\alpha)(7-4p)(3-p)(2-p)t^{1-p} \\
& - 4p(1-\alpha)(2-p)(1-p)t^{-p} - 4\alpha p(1-p^2)t^{-p-1},
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
h_1''(1) = & \frac{8}{81}(1-\alpha)(-128\alpha^4 + 896\alpha^3 + 288\alpha^2 + 5294\alpha - 437) > 0, \\
h_1'''(t) = & -6(1-\alpha)(p+1)(p+2)^2(p+3) + 4(1-\alpha)(7-4p)(3-p)(2-p)(1-p)t^{-p} \\
& + 4p^2(1-\alpha)(2-p)(1-p)t^{-p-1} + 4\alpha p(1+p)^2(1-p)t^{-p-2}, \\
h_1'''(1) = & \frac{8}{81}(1-\alpha)(576\alpha^4 + 3872\alpha^3 + 660\alpha^2 + 6612\alpha - 785) > 0,
\end{aligned} \tag{2.11}$$

$$h_1^{(4)}(t) = -4p(1-p)t^{-p-3}h_2(t), \tag{2.12}$$

where

$$\begin{aligned}
h_2(t) = & (1-\alpha)(7-4p)(3-p)(2-p)t^2 + (1-\alpha)p(2-p)(p+1)t + \alpha(p+1)^2(p+2), \\
h_2(1) = & \frac{4}{27}(1-\alpha)(96\alpha^3 + 232\alpha^2 + 388\alpha + 175) > 0, \\
h_2'(t) = & 2(1-\alpha)(7-4p)(3-p)(2-p)t + (1-\alpha)p(2-p)(p+1) \\
\geq h_2'(1) = & \frac{4}{9}(1-\alpha)(5+4\alpha)(12\alpha^2 + 31\alpha + 23) > 0
\end{aligned} \tag{2.13}$$

for $t \in [1, +\infty)$.

From inequalities (2.13) we clearly see that $h_2(t) > 0$ for $t \in [1, +\infty)$. Then (2.12) leads to the conclusion that $h_1'''(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, Lemma 2.2 follows from (2.7)–(2.11) and the monotonicity of $h_1'''(t)$. $\square$

Lemma 2.3. Let $\alpha \in (0, 1)$, $p = (1 - 4\alpha)/3$, $\lambda_0 = 3\sqrt{345}/80 - 11/16 = 0.00903\dots$, and $f(t) = 2\alpha(1 - t^{p+1})t \log^2 t + (1 - \alpha)(1 + t^{p-1})(1 + t)^2 t \log t + (1 - \alpha)(1 + t)^2(1 - t)(t^p + 1)$. Then the following two statements are true:

(1) if $\alpha \in (1/4, 1)$, then $f(t) > 0$ for $t \in (1, +\infty)$;

(2) if $\alpha \in (0, \lambda_0]$, then $f(t) < 0$ for $t \in (1, +\infty)$.

Proof. Let $f_1(t) = t^{-p}f''(t)$, $f_2(t) = t^{p+2}f'_1(t)$, $f_3(t) = t^{4-p}f''_2(t)$, $f_4(t) = t^{p+2}f'''_3(t)$ and $f_5(t) = t^{4-p}f'''_4(t)$. Then simple computations lead to

$$f(1) = 0, \quad (2.14)$$

$$\begin{aligned} f'(t) &= 2\alpha \left[1 - (p+2)t^{p+1} \right] \log^2 t \\ &\quad + \left[(p+2 - \alpha p - 6\alpha)t^{p+1} + 2(1-\alpha)(p+1)t^p \right. \\ &\quad \left. + (1-\alpha)pt^{p-1} + 3(1-\alpha)t^2 + 4(1-\alpha)t + 3\alpha + 1 \right] \log t \\ &\quad - (1-\alpha) \left[(p+3)t^{p+2} + (p+1)t^{p+1} - (p+3)t^p - (p+1)t^{p-1} + 2t^2 - 2 \right], \\ f'(1) &= 0, \end{aligned} \quad (2.15)$$

$$\begin{aligned} f_1(t) &= -2\alpha(p+1)(p+2)\log^2 t \\ &\quad + \left[(p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2) + 2(1-\alpha)p(p+1)t^{-1} \right. \\ &\quad \left. - (1-\alpha)p(1-p)t^{-2} + 6(1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + 4\alpha pt^{-1-p} \right] \log t \\ &\quad - (1-\alpha)(p+2)(p+3)t + (1-\alpha)(p^2 + 5p + 2)t^{-1} \\ &\quad + (1-\alpha)(p^2 + p - 1)t^{-2} - (1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + (1+3\alpha)t^{-1-p} \\ &\quad - (1-\alpha)p^2 - (1-\alpha)p - 5\alpha + 1, \\ f_1(1) &= 0, \end{aligned} \quad (2.16)$$

$$\begin{aligned} f_2(t) &= - \left[4\alpha(p+1)(p+2)t^{p+1} + 2(1-\alpha)p(p+1)t^p - 2(1-\alpha)p(1-p)t^{p-1} \right. \\ &\quad \left. - 6(1-\alpha)(1-p)t^2 + 4(1-\alpha)pt + 4\alpha(1+p) \right] \log t \\ &\quad - (1-\alpha)(p+2)(p+3)t^{p+2} + (p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2)t^{p+1} \\ &\quad + (1-\alpha)(p^2 - 3p - 2)t^p - (1-\alpha)(p^2 + 3p - 2)t^{p-1} + (1-\alpha)(p+5)t^2 \\ &\quad + 4(1-\alpha)(1-p)t + \alpha - 3\alpha p - p - 1, \\ f_2(1) &= 0, \end{aligned} \quad (2.17)$$

$$\begin{aligned} f'_2(t) &= - \left[4\alpha(p+1)^2(p+2)t^p + 2(1-\alpha)p^2(p+1)t^{p-1} + 2(1-\alpha)p(1-p)^2t^{p-2} \right. \\ &\quad \left. - 12(1-\alpha)(1-p)t + 4(1-\alpha)p \right] \log t - (1-\alpha)(p+2)^2(p+3)t^{p+1} \\ &\quad + (p+1)(p^2 - \alpha p^2 - 15\alpha p + 3p - 22\alpha + 2)t^p + (1-\alpha)p(p^2 - 5p - 4)t^{p-1} \\ &\quad + (1-\alpha)(1-p)(p^2 + 5p - 2)t^{p-2} + 4(1-\alpha)(4-p)t - 4\alpha(1+p)t^{-1} \\ &\quad + 4(1-\alpha)(1-2p), \\ f'_2(1) &= 0, \end{aligned} \quad (2.18)$$

$$\begin{aligned}
f_2''(t) = & - \left[4\alpha p(p+1)^2(p+2)t^{p-1} - 2(1-\alpha)p^2(1-p^2)t^{p-2} - 2(1-\alpha)p(1-p)^2 \right. \\
& \times (2-p)t^{p-3} - 12(1-\alpha)(1-p) \left. \right] \log t - (1-\alpha)(p+1)(p+2)^2 \\
& \times (p+3)t^p + (p+1) \left(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha \right) t^{p-1} \\
& + (1-\alpha)p \left(p^3 - 8p^2 - p + 4 \right) t^{p-2} + (1-\alpha)(1-p) \left(p^3 + 5p^2 - 14p + 4 \right) t^{p-3} \\
& - 4(1-\alpha)pt^{-1} + 4\alpha(1+p)t^{-2} + 4(1-\alpha)(7-4p).
\end{aligned} \tag{2.19}$$

(1) If $\alpha \in (1/4, 1)$, then from (2.19) we note that

$$f_2''(t) = g(t) \log t + h(t), \tag{2.20}$$

where $g(t)$ and $h(t)$ are defined as in Lemmas 2.1 and 2.2, respectively.

Lemmas 2.1 and 2.2 together with (2.20) imply that $f_2'(t)$ is strictly increasing in $[1, +\infty)$. Therefore, $f(t) > 0$ for $t \in (1, +\infty)$ follows from (2.14)–(2.18) and the monotonicity of $f_2'(t)$.

(2) If $\alpha \in (0, \lambda_0]$, then from (2.19) we have

$$\begin{aligned}
f_2''(1) &= \frac{16}{27}(1-\alpha)(80\alpha^2 + 110\alpha - 1) \\
&= \frac{1280}{27}(1-\alpha)(\alpha - \lambda_0) \left(\alpha + \frac{3\sqrt{345}}{80} + \frac{11}{16} \right) \leq 0,
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
f_3(t) = & \left[4\alpha p(1-p)(p+1)^2(p+2)t^2 - 2(1-\alpha)p^2(1-p^2)(2-p)t \right. \\
& \left. - 2(1-\alpha)p(1-p)^2(2-p)(3-p) \right] \log t - (1-\alpha)p(p+1)(p+2)^2(p+3)t^3 \\
& + (p+1) \left(p^4 - \alpha p^4 - 22\alpha p^3 + 2p^3 - 27\alpha p^2 - p^2 - 2p + 18\alpha p + 8\alpha \right) t^2 \\
& + (1-\alpha)p \left(p^4 - 12p^3 + 15p^2 + 8p - 8 \right) t + (1-\alpha)(1-p) \left(p^4 + 4p^3 - 35p^2 + 50p - 12 \right) \\
& + 12(1-\alpha)(1-p)t^{3-p} + 4p(1-\alpha)t^{2-p} - 8\alpha(1+p)t^{1-p},
\end{aligned}$$

$$\begin{aligned}
f_3(1) &= \frac{32}{81}(1-\alpha)(1-4\alpha)(80\alpha^2 + 110\alpha - 1) \\
&= \frac{10240}{81}(1-\alpha) \left(\frac{1}{4} - \alpha \right) (\alpha - \lambda_0) \left(\alpha + \frac{3\sqrt{345}}{80} + \frac{11}{16} \right) \leq 0,
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
f'_3(t) &= \left[8\alpha p(1-p)(p+1)^2(p+2)t - 2(1-\alpha)p^2(1-p^2)(2-p) \right] \log t \\
&\quad - 3(1-\alpha)p(p+1)(p+2)^2(p+3)t - 2(p+1) \\
&\quad \times (3\alpha p^4 - p^4 + 26\alpha p^3 - 2p^3 + 25\alpha p^2 + p^2 - 22\alpha p + 2p - 8\alpha)t \\
&\quad - (1-\alpha)p(p^4 + 8p^3 - 17p^2 - 4p + 8) - 2(1-\alpha)p(1-p)^2(2-p)(3-p)t^{-1} \\
&\quad + 12(1-\alpha)(1-p)(3-p)t^{2-p} + 4(1-\alpha)p(2-p)t^{1-p} - 8\alpha(1-p^2)t^{-p}, \\
f'_3(1) &= \frac{8}{243}(1-\alpha)(3328\alpha^4 + 128\alpha^3 - 7248\alpha^2 + 7118\alpha - 167) \\
&< \frac{8}{243}(1-\alpha)(3328\lambda_0^4 + 128\lambda_0^3 + 7118\lambda_0 - 167) \\
&< \frac{8}{243}(1-\alpha)[3328 \times (0.01)^4 + 128 \times (0.01)^3 + 7118 \times 0.01 - 167] < 0,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
f''_3(t) &= 8\alpha p(1-p)(p+1)^2(p+2) \log t - 6(1-\alpha)p(p+1)(p+2)^2(p+3)t \\
&\quad - 2(1-\alpha)p^2(1-p^2)(2-p)t^{-1} + 2(1-\alpha)p(1-p)^2(2-p)(3-p)t^{-2} \\
&\quad + 12(1-\alpha)(1-p)(2-p)(3-p)t^{1-p} + 4(1-\alpha)p(1-p)(2-p)t^{-p} \\
&\quad + 8\alpha p(1-p^2)t^{-1-p} - 2(p+1) \\
&\quad \times (7\alpha p^4 - p^4 + 34\alpha p^3 - 2p^3 + 21\alpha p^2 + p^2 - 30\alpha p + 2p - 8\alpha), \\
f''_3(1) &= \frac{8}{243}(1-\alpha)(7-4\alpha)(256\alpha^4 - 64\alpha^3 - 1152\alpha^2 + 2066\alpha - 53) \\
&< \frac{8}{243}(1-\alpha)(7-4\alpha)(256\lambda_0^4 + 2066\lambda_0 - 53) \\
&< \frac{8}{243}(1-\alpha)[7-4\alpha](256 \times (0.01)^4 + 2066 \times 0.01 - 53) < 0,
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
f_4(t) &= -6(1-\alpha)p(p+1)(p+2)^2(p+3)t^{p+2} + 8\alpha p(1-p)(p+1)^2(p+2)t^{p+1} \\
&\quad + 2(1-\alpha)p^2(1-p^2)(2-p)t^p - 4(1-\alpha)p(1-p)^2(2-p)(3-p)t^{p-1} \\
&\quad + 12(1-\alpha)(1-p)^2(2-p)(3-p)t^2 - 4(1-\alpha)p^2(1-p)(2-p)t \\
&\quad - 8\alpha p(1-p)(1+p)^2, \\
f_4(1) &= \frac{8}{243}(1-\alpha)(-1024\alpha^4 + 21952\alpha^3 - 10968\alpha^2 + 13474\alpha - 835) \\
&< \frac{8}{243}(1-\alpha)(21952\lambda_0^3 + 13474\lambda_0 - 835) \\
&< \frac{8}{243}(1-\alpha)[21952 \times (0.01)^3 + 13474 \times 0.01 - 835] < 0,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
f'_4(t) &= -6(1-\alpha)p(p+1)(p+2)^3(p+3)t^{p+1} + 8\alpha p(1-p)(p+1)^3(p+2)t^p \\
&\quad + 2(1-\alpha)p^3(1-p^2)(2-p)t^{p-1} + 4(1-\alpha)p(1-p)^3(2-p)(3-p)t^{p-2} \\
&\quad + 24(1-\alpha)(1-p)^2(2-p)(3-p)t - 4(1-\alpha)p^2(1-p)(2-p),
\end{aligned}$$

$$\begin{aligned}
f_4'(1) &= \frac{8}{729}(1-\alpha)(7-4\alpha)\left(-1024\alpha^4 + 21952\alpha^3 - 10968\alpha^2 + 13474\alpha - 835\right) \\
&< \frac{8}{729}(1-\alpha)(7-4\alpha)\left(21952\lambda_0^3 + 13474\lambda_0 - 835\right) \\
&< \frac{8}{729}(1-\alpha)(7-4\alpha)\left[21952 \times (0.01)^3 + 13474 \times 0.01 - 835\right] < 0,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
f_4''(t) &= -6(1-\alpha)p(p+1)^2(p+2)^3(p+3)t^p + 8\alpha p^2(1-p)(p+1)^3(p+2)t^{p-1} \\
&\quad - 2(1-\alpha)p^3(1+p)(1-p)^2(2-p)t^{p-2} - 4(1-\alpha)p(1-p)^3(2-p)^2(3-p)t^{p-3} \\
&\quad + 24(1-\alpha)(1-p)^2(2-p)(3-p), \\
f_4''(1) &= \frac{32}{2187}(1-\alpha)\left(-4096\alpha^6 + 136320\alpha^5 - 241440\alpha^4 + 383672\alpha^3 \right. \\
&\quad \left. - 209850\alpha^2 + 100113\alpha - 7255\right) \\
&< \frac{32}{2187}(1-\alpha)\left(136320\lambda_0^5 + 383672\lambda_0^3 + 100113\lambda_0 - 7255\right) \\
&< \frac{32}{2187}(1-\alpha)\left[136320 \times (0.01)^5 + 383672 \times (0.01)^3 + 100113 \times 0.01 - 7255\right] \\
&< 0,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
f_5(t) &= -6(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t^3 - 8\alpha p^2(1-p)^2(p+1)^3(p+2)t^2 \\
&\quad + 2(1-\alpha)p^3(1+p)(1-p)^2(2-p)^2t + 4(1-\alpha)p(1-p)^3(2-p)^2(3-p)^2, \\
f_5(1) &= \frac{32}{6561}(1-\alpha)(1-4\alpha)\left(-4096\alpha^6 + 173568\alpha^5 - 190368\alpha^4 + 439136\alpha^3 \right. \\
&\quad \left. - 191370\alpha^2 + 96723\alpha - 8665\right) \\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)\left(173568\lambda_0^5 + 439136\lambda_0^3 + 96723\lambda_0 - 8665\right) \\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)\left[173568 \times (0.01)^5 + 439136 \times (0.01)^3 + 96723 \times 0.01 - 8665\right] \\
&< 0,
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
f_5'(t) &= -18(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t^2 - 16\alpha p^2(1-p)^2(p+1)^3(p+2)t \\
&\quad + 2(1-\alpha)p^3(1+p)(1-p)^2(2-p)^2, \\
f_5'(1) &= \frac{32}{6561}(1-\alpha)^2(1-4\alpha)^2\left(-17408\alpha^4 + 69920\alpha^3 - 119136\alpha^2 + 95282\alpha - 30845\right) \\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)^2\left(69920\lambda_0^3 + 95282\lambda_0 - 30845\right) \\
&< \frac{32}{6561}(1-\alpha)(1-4\alpha)^2\left(69920 \times (0.01)^3 + 95282 \times 0.01 - 30845\right) \\
&< 0,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
f_5''(t) &= -36(1-\alpha)p^2(p+1)^2(p+2)^3(p+3)t - 16\alpha p^2(1-p)^2(p+1)^3(p+2), \\
f_5''(1) &= \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160\alpha^3 - 1856\alpha^2 + 3370\alpha - 2205\right) \\
&< \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160\lambda_0^3 + 3370\lambda_0 - 2205\right) \\
&< \frac{128}{6561}(1-\alpha)^3(1-4\alpha)^2(7-4\alpha)\left(160 \times (0.01)^3 + 3370 \times 0.01 - 2205\right) \\
&< 0, \\
f_5'''(t) &= -36(1-\alpha)p^2(p+1)^2(p+2)^3(p+3) \\
&= -\frac{128}{729}(5-2\alpha)(1-4\alpha)^2(1-\alpha)^3(7-4\alpha)^3 < 0.
\end{aligned} \tag{2.30}$$

Inequalities (2.30) imply that $f_5'(t)$ is strictly decreasing in $[1, +\infty)$. Then (2.29) leads to the conclusion that $f_5(t)$ is strictly decreasing in $[1, +\infty)$.

It follows from (2.28) and the monotonicity of $f_5(t)$ that $f_4''(t)$ is strictly decreasing in $[1, +\infty)$. Then inequalities (2.25)–(2.27) lead to the conclusion that $f_4(t) < 0$ for $t \in [1, +\infty)$. Thus, $f_3''(t)$ is strictly decreasing in $[1, +\infty)$.

From inequalities (2.22)–(2.24) and the monotonicity of $f_3''(t)$ we clearly see that $f_3(t) < 0$ for $t \in (1, +\infty)$. Thus, $f_2''(t)$ is strictly decreasing in $[1, +\infty)$.

It follows from (2.17) and (2.18) and inequality (2.21) together with the monotonicity of $f_2''(t)$ that $f_2(t) < 0$ for $t \in (1, +\infty)$, which implies that $f_1(t)$ is strictly decreasing in $[1, +\infty)$.

Therefore, $f(t) < 0$ for $t \in (1, +\infty)$ follows from (2.14)–(2.16) and the monotonicity of $f_1(t)$. $\square$

Lemma 2.4. $3t^4 - 4t(2t^2 - t + 2) \log t - 3 > 0$ for $t > 1$.

Proof. Let

$$J(t) = 3t^4 - 4t(2t^2 - t + 2) \log t - 3. \tag{2.31}$$

Then simple computations lead to

$$\begin{aligned}
J(1) &= 0, \\
J'(t) &= 4(3t^3 - 2t^2 + t - 2) - 8(3t^2 - t + 1) \log t, \\
J'(1) &= 0, \\
J''(t) &= \frac{4}{t} J_1(t),
\end{aligned} \tag{2.32}$$

where $J_1(t) = 9t^3 - 10t^2 + 3t - 2 - 2(6t - 1)t \log t$,

$$J_1(1) = 0, \tag{2.33}$$

$$J_1'(t) = 27t^2 - 32t + 5 - 2(12t - 1) \log t, \tag{2.34}$$

$$J_1'(1) = 0,$$

$$J_1''(t) = \frac{2}{t} J_2(t), \tag{2.35}$$

where $J_2(t) = 27t^2 - 12t \log t - 28t + 1$,

$$\begin{aligned} J_2(1) &= 0, \\ J'_2(t) &= 54t - 12 \log t - 40 > 0 \end{aligned} \quad (2.36)$$

for $t > 1$. □

Therefore, Lemma 2.4 follows from (2.31)–(2.36).

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For all $a, b > 0$ with $a \neq b$, we first prove that

$$\alpha H(a, b) + (1 - \alpha)L(a, b) > M_{(1-4\alpha)/3}(a, b) \quad (3.1)$$

for $\alpha \in [1/4, 1)$,

$$\alpha H(a, b) + (1 - \alpha)L(a, b) < M_{(1-4\alpha)/3}(a, b) \quad (3.2)$$

for $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$.

Without loss of generality, we assume that $a > b$, $t = a/b > 1$ and $p = (1 - 4\alpha)/3$. We divide the proof into two cases.

Case 1 ($\alpha = 1/4$). Let $x = \sqrt{t} = \sqrt{a/b} > 1$. Then we clearly see that

$$\begin{aligned} \alpha H(a, b) + (1 - \alpha)L(a, b) - M_{(1-4\alpha)/3}(a, b) &= \frac{1}{4} [H(a, b) + 3L(a, b)] - \sqrt{ab} \\ &= \frac{b[3x^4 - 4x(2x^2 - x + 2) \log x - 3]}{8(x^2 + 1) \log x}. \end{aligned} \quad (3.3)$$

Therefore, inequality (3.1) follows from (3.3) and Lemma 2.4.

Case 2 ($\alpha \in (0, 3\sqrt{345}/80 - 11/16) \cup (1/4, 1)$). Then we have

$$\begin{aligned} \alpha H(a, b) + (1 - \alpha)L(a, b) - M_{(1-4\alpha)/3}(a, b) &= \alpha H(a, b) + (1 - \alpha)L(a, b) - M_p(a, b) \\ &= b \left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t} - \left(\frac{t^p+1}{2} \right)^{1/p} \right]. \end{aligned} \quad (3.4)$$

Let

$$F(t) = \log \left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t} \right] - \frac{1}{p} \log \left(\frac{t^p+1}{2} \right). \quad (3.5)$$

Then simple computations lead to

$$\lim_{t \rightarrow 1} F(t) = 0,$$

$$F'(t) = \frac{f(t)}{t(t+1)(t^p+1)[2\alpha t \log t + (1-\alpha)(t^2-1)] \log t}, \quad (3.6)$$

where $f(t)$ is defined as in Lemma 2.3.

If $\alpha \in (1/4, 1)$, then inequality (3.1) follows from (3.4)–(3.6) and Lemma 2.3(1). If $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$, then inequality (3.2) follows from (3.4)–(3.6) and Lemma 2.3(2).

Next, we prove that the parameter $(1 - 4\alpha)/3$ in inequalities (3.1) and (3.2) is the best possible.

For any $\alpha \in (0, 3\sqrt{345}/80 - 11/16) \cup (1/4, 1)$, $p \neq 0$, and $x > 0$, one has

$$\alpha H(1, 1+x) + (1-\alpha)L(1, 1+x) - M_p(1, 1+x) = \frac{Q(x)}{2^{1/p}(1+x/2) \log(1+x)}, \quad (3.7)$$

where $Q(x) = 2^{1/p}\alpha(1+x) \log(1+x) + 2^{1/p}(1-\alpha)x(1+x/2) - (1+x/2) \log(1+x)[1+(1+x)^p]^{1/p}$.

Letting $x \rightarrow 0$ and making use of Taylor expansion, we get

$$Q(x) = \frac{2^{1/p}}{8} \left(\frac{1-4\alpha}{3} - p \right) x^3 + o(x^3). \quad (3.8)$$

If $\alpha \in [1/4, 1)$ and $p > (1 - 4\alpha)/3$, then (3.7) and (3.8) imply that there exists $\delta_1 = \delta_1(\alpha, p) > 0$ such that $\alpha H(1, 1+x) + (1-\alpha)L(1, 1+x) < M_p(1, 1+x)$ for $x \in (0, \delta_1)$. If $\alpha \in (0, 3\sqrt{345}/80 - 11/16)$ and $p < (1 - 4\alpha)/3$, then (3.7) and (3.8) imply that there exists $\delta_2 = \delta_2(\alpha, p) > 0$ such that $\alpha H(1, 1+x) + (1-\alpha)L(1, 1+x) > M_p(1, 1+x)$ for $x \in (0, \delta_2)$.

Finally, we prove that there exist $a_1, b_1, a_2, b_2 > 0$ with $a_1 \neq b_1$ and $a_2 \neq b_2$ such that $\alpha H(a_1, b_1) + (1-\alpha)L(a_1, b_1) < M_{(1-4\alpha)/3}(a_1, b_1)$ and $\alpha H(a_2, b_2) + (1-\alpha)L(a_2, b_2) > M_{(1-4\alpha)/3}(a_2, b_2)$ for any $3\sqrt{345}/80 - 11/16 < \alpha < 1/4$.

If $3\sqrt{345}/80 - 11/16 < \alpha < 1/4$, then from the expression of $f_2''(1)$ in (2.21) we clearly see that $f_2''(1) > 0$, which leads to the conclusion that there exists $\lambda > 1$ such that

$$f_2''(t) > 0 \quad (3.9)$$

for $t \in [1, \lambda)$.

From (2.14)–(2.18) and inequality (3.9) we know that

$$f(t) > 0 \quad (3.10)$$

for $t \in (1, \lambda)$. Equations (3.4)–(3.6) and inequality (3.10) lead to the conclusion that $\alpha H(a, b) + (1-\alpha)L(a, b) > M_{(1-4\alpha)/3}(a, b)$ for all $a/b \in (1, \lambda)$.

On the other hand, simple computations lead to

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{M_{(1-4\alpha)/3}(1, x)}{\alpha H(1, x) + (1 - \alpha)L(1, x)} \\ &= 2^{3/(4\alpha-1)} \lim_{x \rightarrow +\infty} \frac{(1 + x^{(4\alpha-1)/3})^{3/(1-4\alpha)}}{2\alpha/(x+1) + (1 - \alpha)(1 - 1/x)/\log x} = +\infty. \end{aligned} \quad (3.11)$$

Equation (3.11) implies that there exists $X = X(\alpha) > 1$ such that $\alpha H(a, b) + (1 - \alpha)L(a, b) < M_{(1-4\alpha)/3}(a, b)$ for all $a/b \in (X, +\infty)$. $\square$

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