Research Article

Sharp Bounds by the Generalized Logarithmic Mean for the Geometric Weighted Mean of the Geometric and Harmonic Means

Wei-Mao Qian¹ and Bo-Yong Long²

¹ School of Distance Education, Huzhou Broadcast and TV University, Huzhou 313000, China
² School of Mathematics Science, Anhui University, Hefei 230039, China

Correspondence should be addressed to Wei-Mao Qian, qwm661977@126.com

Received 29 January 2012; Revised 19 February 2012; Accepted 12 March 2012

Academic Editor: Yuri Sotskov

Copyright © 2012 W.-M. Qian and B.-Y. Long. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present sharp upper and lower generalized logarithmic mean bounds for the geometric weighted mean of the geometric and harmonic means.

1. Introduction

For \( p \in \mathbb{R} \) the generalized logarithmic mean \( L_p(a, b) \) of two positive numbers \( a \) and \( b \) is defined by

\[
L_p(a, b) = \begin{cases}
  a, & a = b, \\
  \left( \frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)} \right)^{1/p}, & p \neq 0, \ p \neq -1, \ a \neq b, \\
  \frac{1}{e} \left( \frac{b}{a} \right)^{1/(b-a)}, & p = 0, \ a \neq b, \\
  \frac{b - a}{\log b - \log a'}, & p = -1, \ a \neq b.
\end{cases}
\]  

(1.1)

It is well-known that \( L_p(a, b) \) is continuous and strictly increasing with respect to \( p \in \mathbb{R} \) for fixed \( a \) and \( b \) with \( a \neq b \). In the recent past, the generalized logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for \( L_p \) can be
Proposition 1.2. For all positive real numbers \( a \) and \( b \), where \( a \neq b \), consider the variant of Jensen’s functional equation involving \( L_p \), which appear in a heat conduction problem. Let \( A(a, b) = (a+b)/2 \), \( I(a, b) = (1/e)(b^p / a^q)^{1/(b−a)} \), \( L(a, b) = (b−a)/(\log b−\log a) \), \( G(a, b) = \sqrt{ab} \), and \( H(a, b) = 2ab/(a + b) \) be the arithmetic, geometric, logarithmic, and harmonic means of two positive numbers \( a \) and \( b \) with \( a \neq b \), respectively. Then it is well known that

\[
\begin{align*}
\min\{a, b\} < H(a, b) < G(a, b) = L_{-2}(a, b) < L(a, b) = L_{-1}(a, b) \\
< I(a, b) = L_0(a, b) < A(a, b) = L_1(a, b) < \max\{a, b\}.
\end{align*}
\]  

(1.2)

In [28–30], the authors present bounds for \( L \) and \( I \) in terms of \( G \) and \( A \).

**Proposition 1.1.** For all positive real numbers \( a \) and \( b \) with \( a \neq b \), one has

\[
A^{1/3}(a, b)G^{2/3}(a, b) < L(a, b) < \frac{1}{3} A(a, b) + \frac{2}{3} G(a, b),
\]  

\[
\frac{1}{3} G(a, b) + \frac{2}{3} A(a, b) < I(a, b).
\]  

(1.3)

The proof of the following Proposition 1.2 can be found in [31].

**Proposition 1.2.** For all positive real numbers \( a \) and \( b \) with \( a \neq b \), we have

\[
\sqrt{G(a, b)A(a, b)} < \sqrt{L(a, b)I(a, b)} < \frac{1}{2}(L(a, b) + I(a, b)) < \frac{1}{2}(G(a, b) + A(a, b)).
\]  

(1.4)

For \( r \in \mathbb{R} \) the \( r \)th power mean \( M_r(a, b) \) of two positive numbers \( a \) and \( b \) is defined by

\[
M_r(a, b) = \begin{cases} 
\left( \frac{a^r + b^r}{2} \right)^{1/r}, & r \neq 0, \\
\sqrt{ab}, & r = 0.
\end{cases}
\]  

(1.5)

The main properties of these means are given in [32]. Several authors discussed the relationship of certain means to \( M_r \). The following sharp bounds for \( L, I, (IL)^{1/2} \), and \( (I + L)/2 \) in terms of power means are proved in [31, 33–37].

**Proposition 1.3.** For all positive real numbers \( a \) and \( b \) with \( a \neq b \) one has

\[
\begin{align*}
M_0(a, b) < L(a, b) < M_{1/3}(a, b), & \quad M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b), \\
M_0(a, b) < L^{1/2}(a, b)L^{1/2}(b, a) < M_{1/2}(a, b), & \quad \frac{1}{2}[I(a, b) + L(a, b)] < M_{1/2}(a, b).
\end{align*}
\]  

(1.6)

The following three results were established by Alzer and Qiu in [38].
Journal of Applied Mathematics

**Proposition 1.4.** The inequalities

\[ aA(a,b) + (1 - a)G(a,b) < I(a,b) < \beta A(a,b) + (1 - \beta)G(a,b) \]  

hold for all positive real numbers \(a\) and \(b\) with \(a \neq b\) if and only if

\[ \alpha \leq \frac{2}{3}, \quad \beta \geq \frac{2}{e} = 0.73575 \ldots \]  

(1.8)

**Proposition 1.5.** Let \(a\) and \(b\) be real numbers with \(a \neq b\). If \(0 < a, b \leq e\), then

\[ [G(a,b)]^{A(ab)} < [L(a,b)]^{I(ab)} < [A(a,b)]^G(a,b). \]  

(1.9)

And, if \(a, b \geq e\), then

\[ [A(a,b)]^G(a,b) < [I(a,b)]^{I(ab)} < [G(a,b)]^{A(ab)}. \]  

(1.10)

**Proposition 1.6.** For all positive real numbers \(a\) and \(b\) with \(a \neq b\), one has

\[ M_{c}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) \]  

(1.11)

with the best possible parameter \(c = \log 2/(1 + \log 2) = 0.40938 \ldots \)

In [39] the authors presented inequalities between the generalized logarithmic mean and the product \(A^\alpha(a,b)G^\beta(a,b)H^\gamma(a,b)\) for all \(a, b > 0\) with \(a \neq b\) and \(\alpha, \beta > 0\) with \(\alpha + \beta < 1\).

It is the aim of this paper to give a solution to the problem: for \(\alpha \in (0, 1)\), what are the greatest value \(p\) and the least value \(q\), such that the inequality

\[ L_{p}(a,b) \leq G^\alpha(a,b)H^{1-\alpha}(a,b) \leq L_{q}(a,b) \]  

(1.12)

holds for all \(a, b > 0\)?

### 2. Main Result

**Theorem 2.1.** For \(\alpha \in (0, 1)\) and all \(a, b > 0\), one has the following:

1. \(L_{3a-5}(a,b) = G^\alpha(a,b)H^{1-\alpha}(a,b) = L_{-(2/a)}(a,b)\) for \(\alpha = 2/3\),
2. \(L_{3a-5}(a,b) > G^\alpha(a,b)H^{1-\alpha}(a,b) \geq L_{-(2/a)}(a,b)\) for \(0 < \alpha < 2/3\), and \(L_{3a-5}(a,b) \leq G^\alpha(a,b)H^{1-\alpha}(a,b) \leq L_{-(2/a)}(a,b)\) for \(2/3 < \alpha < 1\), with equality if and only if \(a = b\), and the parameters \(3\alpha - 5\) and \(-2/a\) in each inequality cannot be improved.

**Proof.** (1) If \(\alpha = 2/3\) and \(a = b\), then (1.1) implies that \(L_{3a-5}(a,b) = G^\alpha(a,b)H^{1-\alpha}(a,b) = L_{-(2/a)}(a,b) = a\).
If $a = 2/3$ and $a \neq b$, then (1.1) leads to

$$L_{3\alpha - 5}(a,b) = L_{(2/3)}(a,b) = L_3(a,b) = \left[ \frac{a^2 - b^2}{2(b - a)} \right]^{-1/3}$$

(2.1)

$$= (ab)^{-1/3} \left( \frac{2ab}{a + b} \right)^{1/3} = G^{2/3}(a,b)H^{1/3}(a,b) = G^x(a,b)H^{1-x}(a,b).$$

(2) If $a = b$, then from (1.1) we clearly see that $L_{3\alpha - 5}(a,b) = G^x(a,b)H^{1-x}(a,b) = L_{(2/3)}(a,b) = a$ for any $x \in (0, 1)$.

If $a \neq b$, without loss of generality, we assume $a > b$. Let $a/b = t > 1$ and

$$f(t) = \log L_{3\alpha - 5}(a,b) - \log \left[ G^x(a,b)H^{1-x}(a,b) \right].$$

Then (1.1) and simple computations yield

$$f(t) = \frac{1}{3\alpha - 5} \log \frac{t^{3\alpha - 4} - 1}{(3\alpha - 4)(t - 1)} - \frac{\alpha}{2} \log t - (1 - \alpha) \log \frac{2t}{1 + t},$$

(2.3)

$$\lim_{t \to 1} f(t) = 0,$$

$$f'(t) = \frac{t^{4-3\alpha}}{t(t^2 - 1)(t^{4-3\alpha} - 1)} g(t),$$

(2.4)

where $g(t) = \frac{(2 - \alpha/2)t^{3\alpha - 2} - ((2 - \alpha)(2 - 3\alpha)/5 - 3\alpha)t^{3\alpha - 3} + ((1 - \alpha)(2 - 3\alpha)/2(5 - 3\alpha))t^{3\alpha - 4} - ((1 - \alpha)(2 - 3\alpha)/2(5 - 3\alpha))^2 + ((2 - \alpha)(2 - 3\alpha)/5 - 3\alpha)t - (2 - \alpha)/2}{(2 - 2\alpha)(3\alpha - 2)t^{3\alpha - 3} - 3(2 - \alpha)(2 - 3\alpha)(\alpha - 1)t^{3\alpha - 4}}$

(2.5)

$$g(1) = 0,$$

$$g'(t) = \frac{(2 - \alpha)(3\alpha - 2)}{2} t^{3\alpha - 3} - \frac{3(2 - \alpha)(2 - 3\alpha)(\alpha - 1)}{5 - 3\alpha} t^{3\alpha - 4}$$

$$+ \frac{(1 - \alpha)(2 - 3\alpha)(3\alpha - 4)}{2(5 - 3\alpha)} t^{3\alpha - 5} - \frac{(1 - \alpha)(2 - 3\alpha)}{(5 - 3\alpha)} t$$

$$+ \frac{(2 - \alpha)(2 - 3\alpha)}{(5 - 3\alpha)},$$

$$g''(1) = 0,$$

$$g''(t) = \frac{3(2 - \alpha)(3\alpha - 2)(\alpha - 1)}{2} t^{3\alpha - 4} - \frac{3(2 - \alpha)(2 - 3\alpha)(\alpha - 1)(3\alpha - 4)}{5 - 3\alpha} t^{3\alpha - 5}$$

$$- \frac{(1 - \alpha)(2 - 3\alpha)(3\alpha - 4)}{2} t^{3\alpha - 6} - \frac{(1 - \alpha)(2 - 3\alpha)}{(5 - 3\alpha)} t$$

$$+ \frac{1}{2},$$

$$g''(t) = \frac{3}{2} (1 - \alpha)(2 - \alpha)(4 - 3\alpha)(3\alpha - 2) t^{3\alpha - 7} (t - 1)^2.$$
For $0 < \alpha < 2/3$, then (2.7) implies

$$g''(t) < 0$$

for $t > 1$.

From (2.3)–(2.6) and (2.8) we know that $f(t) > 0$ for $t > 1$.

If $2/3 < \alpha < 1$, then (2.7) leads to

$$g''(t) > 0$$

for $t > 1$. Therefore $f(t) < 0$ for $t > 1$ follows from (2.3)–(2.6) and (2.9).

Let

$$h(t) = \log L_{-(2/\alpha)}(a,b) - \log \left[G^\alpha(a,b) H^{1-\alpha}(a,b)\right]$$

for $t = a/b > 1$; then (1.1) and elementary calculations lead to

$$h(t) = -\frac{\alpha}{2} \log \left(\frac{t^{(\alpha-2)/\alpha} - 1}{(\alpha-2)/\alpha} - 1\right) - \frac{\alpha}{2} \log t - (1 - \alpha) \log \frac{2t}{1 + t},$$

$$\lim_{t \to 1^-} h(t) = 0,$$

$$h'(t) = -\frac{t^{(2-\alpha)/\alpha}}{t(t^2 - 1)(t^{(2-\alpha)/\alpha} - 1)} v(t),$$

where $v(t) = ((2 - \alpha)/2)t^{(3\alpha-2)/\alpha} + ((3\alpha - 2)/2)t^{(2\alpha-2)/\alpha} - ((3\alpha - 2)/2)t - (2 - \alpha)/2$,

$$v(1) = 0,$$

$$v'(t) = \frac{(2 - \alpha)(3\alpha - 2)}{2\alpha} t^{(2-\alpha)/\alpha} + \frac{(3\alpha - 2)(\alpha - 1)}{\alpha} t^{(2-\alpha)/\alpha} - \frac{3\alpha - 2}{2},$$

$$v'(1) = 0,$$

$$v''(t) = \frac{(2 - \alpha)(1 - \alpha)(2 - 3\alpha)}{\alpha^2} t^{-2/\alpha}(t - 1).$$

If $\alpha \in (0, 2/3)$, then (2.15) implies

$$v''(t) > 0$$

for $t > 1$.

From (2.11)–(2.14) and (2.16) we know that $h(t) < 0$ for $t > 1$.

If $\alpha \in (2/3, 1)$, then (2.15) leads to

$$v''(t) < 0$$

for $t > 1$. Therefore, $h(t) > 0$ for $t > 1$ follows from (2.11)–(2.14) and (2.17).
Next, we prove that the parameters \(-2/\alpha\) and \(3\alpha - 5\) in either case cannot be improved. The proof is divided into two cases.

**Case 1** \((\alpha \in (0, 2/3))\). For any \(\epsilon > 0\) and \(x \in (0, 1)\), from (1.1) one has

\[
\left[ G^\alpha (1, 1 + x) H^{0-\alpha} (1, 1 + x) \right]^{5-3\alpha+\epsilon} \leq \left[ L_{3\alpha-5-\epsilon} (1, 1 + x) \right]^{5-3\alpha+\epsilon} - \frac{f_1(x)}{1 + x/2} (1-\alpha)(5-3\alpha+\epsilon) \left[ (1 + x)^{4-3\alpha+\epsilon} - 1 \right].
\]

(2.18)

where \( f_1(x) = (1 + x)^{(1-\alpha)(5-3\alpha+\epsilon)} \left[ (1 + x)^{4-3\alpha+\epsilon} - 1 \right] - (4 - 3\alpha + \epsilon)x(1 + x)^{4-3\alpha+\epsilon} \).

Let \( x \to 0\); making use of the Taylor expansion, we get

\[
f_1(x) = \frac{\epsilon(4 - 3\alpha + \epsilon)(5 - 3\alpha + \epsilon)}{24} x^3 + o(x^3).
\]

(2.19)

Equations (2.18) and (2.19) imply that for any \(\alpha \in (0, 2/3)\) and \(\epsilon > 0\) there exists \(\delta = \delta(\epsilon, \alpha) \in (0, 1)\), such that \(L_{3\alpha-5-\epsilon} (1, 1 + x) < G^\alpha (1, 1 + x) H^{0-\alpha} (1, 1 + x)\) for \(x \in (0, \delta)\).

On the other hand, for any \(\epsilon \in (0, (2/\alpha) - 1)\) we have

\[
L_{-(2/\alpha)+\epsilon} (1, t) - G^\alpha (1, t) H^{0-\alpha} (1, t)
\]

\[
= \frac{\epsilon t^{(2-\alpha)/2 - 2\alpha+\epsilon}}{(2\alpha - \epsilon - 1)(1 - 1/t)} \left[ 1 - t^{-2/\alpha+\epsilon+1} \left( \frac{2t}{1+t} \right)^{1-\alpha} \right] - \frac{t^{-\alpha\epsilon} 2/\alpha-\epsilon}{(2\alpha - \epsilon - 1)(1 - 1/t)} \left[ 1 - t^{-2/\alpha+\epsilon+1} \left( \frac{2t}{1+t} \right)^{1-\alpha} \right].
\]

(2.20)

From (2.20) we know that for any \(\alpha \in (0, 2/3)\) and \(\epsilon \in (0, (2/\alpha) - 1)\) there exists \(T = T(\epsilon, \alpha) > 1\), such that \(L_{-(2/\alpha)+\epsilon} (1, t) > G^\alpha (1, t) H^{0-\alpha} (1, t)\) for \(t \in (T, \infty)\).

**Case 2** \((\alpha \in (2/3, 1))\). For any \(\epsilon \in (0, 4 - 3\alpha)\) and \(x \in (0, 1)\), from (1.1) one has

\[
\left[ L_{3\alpha-5+\epsilon} (1, 1 + x) \right]^{5-3\alpha-\epsilon} - \left[ G^\alpha (1, 1 + x) H^{0-\alpha} (1, 1 + x) \right]^{5-3\alpha-\epsilon} \leq \frac{f_2(x)}{1 + x/2} (1-\alpha)(5-3\alpha-\epsilon) \left[ (1 + x)^{4-3\alpha-\epsilon} - 1 \right].
\]

(2.21)

where \( f_2(x) = (4 - 3\alpha - \epsilon)x(1 + x)^{4-3\alpha-\epsilon} - (1 + x/2)^{(1-\alpha)(5-3\alpha-\epsilon)} \left[ (1 + x)^{4-3\alpha-\epsilon} - 1 \right]. \)

Let \( x \to 0\); making use of the Taylor expansion, we have

\[
f_2(x) = \frac{\epsilon}{24} (4 - 3\alpha - \epsilon)(5 - 3\alpha - \epsilon)x^3 + o(x^3).
\]

(2.22)
Equations (2.21) and (2.22) imply that for any \( \alpha \in (2/3, 1) \) and \( \epsilon \in (0, 4 - 3\alpha) \) there exists \( \delta = \delta(\epsilon, \alpha) \in (0, 1) \), such that \( L_{3\alpha-5\epsilon}(1, 1 + x) > G^\alpha(1, 1 + x)H^{1-\alpha}(1, 1 + x) \) for \( x \in (0, \delta) \).

On the other hand, for any \( \epsilon > 0 \), we have

\[
G^\alpha(1,t)H^{1-\alpha}(1, t) - L_{-(2/\alpha)-\epsilon}(1, t) = t^{1/2} \left\{ \frac{2t}{1+t} \right\}^{1-\alpha} - t^{-\epsilon \alpha^2/2(2+\epsilon \alpha)} \left[ \frac{1 - t^{-(2/\alpha+\epsilon-1)}}{(2/\alpha + \epsilon - 1)(1 - 1/t)} \right]^{-\alpha/(2+\epsilon \alpha)}.
\]

\[
\lim_{t \to +\alpha} \left\{ \frac{2t}{1+t} \right\}^{1-\alpha} - t^{-\epsilon \alpha^2/2(2+\epsilon \alpha)} \left[ \frac{1 - t^{-(2/\alpha+\epsilon-1)}}{(2/\alpha + \epsilon - 1)(1 - 1/t)} \right]^{-\alpha/(2+\epsilon \alpha)} = 2^{1-\alpha} > 0.
\]

From (2.23) we know that for any \( \alpha \in (2/3, 1) \) and \( \epsilon > 0 \) there exists \( T = T(\epsilon, \alpha) > 1 \), such that \( L_{-(2/\alpha)-\epsilon}(1, t) < G^\alpha(1, t)H^{1-\alpha}(1, t) \) for \( t \in (T, \infty) \).

\[\square\]

**Acknowledgment**

This work was supported by the Natural Science Foundation of Zhejiang Broad-cast and TV University under Grant XKT-09G21.

**References**


