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Research Article

Refinements of Kantorovich Inequality for Hermitian Matrices

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Some new Kantorovich-type inequalities for Hermitian matrix are proposed in this paper. We consider what happens to these inequalities when the positive definite matrix is allowed to be invertible and provides refinements of the classical results.

1. Introduction and Preliminaries

We first state the well-known Kantorovich inequality for a positive definite Hermite matrix (see [1, 2]), let $A \in M_n$ be a positive definite Hermitian matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$1 \leq x^* A x x^* A^{-1} x \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n}, \tag{1.1}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$, where A^* denotes the conjugate transpose of matrix A. A matrix $A \in M_n$ is Hermitian if $A = A^*$. An equivalent form of this result is incorporated in

$$0 \leq x^* A x x^* A^{-1} x - 1 \leq \frac{(\lambda_n - \lambda_1)^2}{4\lambda_1 \lambda_n}, \tag{1.2}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

Attributed to Kantorovich, the inequality has built up a considerable literature. This typically comprises generalizations. Examples are [3–5] for matrix versions. Operator

versions are developed in [6, 7]. Multivariate versions have been useful in statistics to assess the robustness of least squares, see [8, 9] and the references therein.

Due to the important applications of the original Kantorovich inequality for matrices [10] in Statistics [8, 11, 12] and Numerical Analysis [13, 14], any new inequality of this type will have a flow of consequences in the areas of applications.

Motivated by the interest in both pure and applied mathematics outlined above we establish in this paper some improvements of Kantorovich inequalities. The classical Kantorovich-type inequalities are modified to apply not only to positive definite but also to invertible Hermitian matrices. As natural tools in deriving the new results, the recent Grüss-type inequalities for vectors in inner product in [6, 15–19] are utilized.

To simplify the proof, we first introduce some lemmas.

2. Lemmas

Let B be a Hermitian matrix with real eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, if $A - B$ is positive semidefinite, we write

$$A \geq B, \tag{2.1}$$

that is, $\lambda_i \geq \mu_i$, $i = 1, 2, \ldots, n$. On $\mathbb{C}^n$, we have the standard inner product defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$, where $x = (x_1, \ldots, x_n)^* \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n)^* \in \mathbb{C}^n$.

Lemma 2.1. *Let a, b, c, and d be real numbers, then one has the following inequality:*

$$\left(a^2 - b^2\right)\left(c^2 - d^2\right) \leq (ac - bd)^2. \tag{2.2}$$

Lemma 2.2. *Let A and B be Hermitian matrices, if $AB = BA$, then*

$$AB \leq \frac{(A + B)^2}{4}. \tag{2.3}$$

Lemma 2.3. *Let $A \geq 0$, $B \geq 0$, if $AB = BA$, then*

$$AB \geq 0. \tag{2.4}$$

3. Some Results

The following lemmas can be obtained from [16–19] by replacing Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with inner product spaces $\mathbb{C}^n$, so we omit the details.

Lemma 3.1. *Let u, v, and e be vectors in $\mathbb{C}^n$, and $\|e\| = 1$. If α, β, δ, and γ are real or complex numbers such that*

$$\operatorname{Re}\langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re}\langle \delta e - v, v - \gamma e \rangle \geq 0, \tag{3.1}$$

then

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \le \frac{1}{4}(\beta-\alpha)(\delta-\gamma) - \left[\operatorname{Re}\langle \beta e - u, u - \alpha e\rangle \operatorname{Re}\langle \delta e - v, v - \gamma e\rangle\right]^{1/2}. \quad (3.2)$$

Lemma 3.2. *With the assumptions in Lemma 3.1, one has*

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \le \frac{1}{4}(\beta-\alpha)(\gamma-\delta) - \left|\langle u,e\rangle - \frac{\alpha+\beta}{2}\right|\left|\langle v,e\rangle - \frac{\gamma+\delta}{2}\right|. \quad (3.3)$$

Lemma 3.3. *With the assumptions in Lemma 3.1, if* $\operatorname{Re}(\beta\overline{\alpha}) \ge 0$, $\operatorname{Re}(\delta\overline{\gamma}) \ge 0$, *one has*

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \le \frac{|\beta-\alpha||\delta-\gamma|}{4\left[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})\right]^{1/2}}|\langle u,e\rangle\langle e,v\rangle|. \quad (3.4)$$

Lemma 3.4. *With the assumptions in Lemma 3.3, one has*

$$\begin{aligned} &|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \\ &\quad \le \left\{\left(|\alpha+\beta| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{1/2}\right)\left(|\delta+\gamma| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{1/2}\right)\right\}^{1/2}\left[|\langle u,e\rangle\langle e,v\rangle|\right]^{1/2}. \end{aligned} \quad (3.5)$$

4. New Kantorovich Inequalities for Hermitian Matrices

For a Hermitian matrix A, as in [6], we define the following transform:

$$C(A) = (\lambda_n I - A)(A - \lambda_1 I). \quad (4.1)$$

When A is invertible, if $\lambda_1\lambda_n > 0$, then,

$$C\left(A^{-1}\right) = \left(\frac{1}{\lambda_1}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_n}I\right). \quad (4.2)$$

Otherwise, $\lambda_1\lambda_n < 0$, then,

$$C\left(A^{-1}\right) = \left(\frac{1}{\lambda_{k+1}}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_k}I\right), \quad (4.3)$$

where

$$\lambda_1 \le \cdots \le \lambda_k < 0 < \lambda_{k+1} \le \cdots \le \lambda_n. \quad (4.4)$$

From Lemma 2.3 we can conclude that $C(A) \ge 0$ and $C(A^{-1}) \ge 0$.

For two Hermitian matrices A and B, and $x \in \mathbb{C}^n$, $\|x\| = 1$, we define the following functional:

$$G(A, B; x) = \langle Ax, Bx \rangle - \langle Ax, x \rangle \langle x, Bx \rangle. \tag{4.5}$$

When $A = B$, we denote

$$G(A; x) = \|Ax\|^2 - \langle Ax, x \rangle^2, \tag{4.6}$$

for $x \in \mathbb{C}^n$, $\|x\| = 1$.

Lemma 4.1. *With notations above, and for $x \in \mathbb{C}^n$, $\|x\| = 1$, then*

$$0 \le \langle C(A)x, x \rangle \le \frac{(\lambda_n - \lambda_1)^2}{4}, \tag{4.7}$$

if $\lambda_n \lambda_1 > 0$,

$$0 \le \left\langle C\left(A^{-1}\right)x, x \right\rangle \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n \lambda_1)^2}, \tag{4.8}$$

if $\lambda_n \lambda_1 < 0$,

$$0 \le \left\langle C\left(A^{-1}\right)x, x \right\rangle \le \frac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_{k+1} \lambda_k)^2}. \tag{4.9}$$

Proof. From $C(A) \ge 0$, then

$$\langle C(A)x, x \rangle \ge 0. \tag{4.10}$$

While, from Lemma 2.2, we can get

$$C(A) = (\lambda_n I - A)(A - \lambda_1 I) \le \frac{(\lambda_n - \lambda_1)^2}{4} I. \tag{4.11}$$

Then $\langle C(A)x, x \rangle \le (\lambda_n - \lambda_1)^2/4$ is straightforward. The proof for $C(A^{-1})$ is similar. □

Lemma 4.2. *With notations above, and for $x \in \mathbb{C}^n$, $\|x\| = 1$, then*

$$\left| x^* A x x^* A^{-1} x - 1 \right|^2 \le G(A; x) G\left(A^{-1}; x\right). \tag{4.12}$$

Proof. Thus,

$$\begin{aligned}\left|x^*Axx^*A^{-1}x-1\right|^2 &= \left|x^*\left(\left(x^*A^{-1}x\right)I-A^{-1}\right)((x^*Ax)I-A)x\right|^2 \\ &\le \left\|\left(\left(x^*A^{-1}x\right)I-A^{-1}\right)x\right\|^2\|((x^*Ax)I-A)x\|^2,\end{aligned} \tag{4.13}$$

while

$$\begin{aligned}\|((x^*Ax)I-A)x\|^2 &= x^*\left((x^*Ax)^2I-2(x^*Ax)A+A^2\right)x \\ &= x^*A^2x-(x^*Ax)^2 \\ &= \|Ax\|^2-\langle Ax,x\rangle^2 \\ &= G(A;x).\end{aligned} \tag{4.14}$$

Similarly, we can get $\|((x^*A^{-1}x)I-A^{-1})x\|^2 = G(A^{-1};x)$, then we complete the proof. □

Theorem 4.3. *Let A, B be two Hermitian matrices, and $C(A)\ge 0$, $C(B)\ge 0$ are defined as above, then*

$$\begin{aligned}|G(A,B;x)| &\le \frac{1}{4}(\lambda_n-\lambda_1)(\mu_n-\mu_1)-[\langle C(A)x,x\rangle\langle C(B)x,x\rangle]^{1/2}, \\ |G(A,B;x)| &\le \frac{1}{4}(\lambda_n-\lambda_1)(\mu_n-\mu_1)-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}\right)x,x\right\rangle\right|\left|\left\langle\left(B-\frac{\mu_1+\mu_n}{2}\right)x,x\right\rangle\right|,\end{aligned} \tag{4.15}$$

for any $x\in\mathbb{C}^n$, $\|x\|=1$.

If $\lambda_1\lambda_n>0$, $\mu_1\mu_n>0$, then

$$\begin{aligned}|G(A,B;x)| &\le \frac{(\lambda_n-\lambda_1)(\mu_n-\mu_1)}{4\left[(\lambda_n\lambda_1)(\mu_n\mu_1)\right]^{1/2}}|\langle Ax,x\rangle\langle Bx,x\rangle|, \\ |G(A,B;x)| &\le \left\{\left(|\lambda_1+\lambda_n|-2(\lambda_1\lambda_n)^{1/2}\right)\left(|\mu_1+\mu_n|-2(\mu_1\mu_n)^{1/2}\right)\right\}^{1/2}[|\langle Ax,x\rangle\langle Bx,x\rangle|]^{1/2},\end{aligned} \tag{4.16}$$

for any $x\in\mathbb{C}^n$, $\|x\|=1$.

Proof. The proof follows by Lemmas 3.1, 3.2, 3.3, and 3.4 on choosing $u=Ax$, $v=Bx$, and $e=x$, $\beta=\lambda_n$, $\alpha=\lambda_1$, $\delta=\mu_n$, and $\gamma=\mu_1$, $x\in\mathbb{C}^n$, $\|x\|=1$, respectively. □

Corollary 4.4. *Let A be a Hermitian matrices, and $C(A) \geq 0$ is defined as above, then*

$$|G(A;x)| \leq \frac{1}{4}(\lambda_n - \lambda_1)^2 - \langle C(A)x, x\rangle, \tag{4.17}$$

$$|G(A;x)| \leq \frac{1}{4}(\lambda_n - \lambda_1)^2 - \left|\left\langle \left(A - \frac{\lambda_1 + \lambda_n}{2} I\right)x, x\right\rangle\right|^2, \tag{4.18}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

If $\lambda_1\lambda_n > 0$, then

$$|G(A;x)| \leq \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)}|\langle Ax, x\rangle|^2, \tag{4.19}$$

$$|G(A;x)| \leq \left(|\lambda_1 + \lambda_n| - 2\sqrt{\lambda_1\lambda_n}\right)|\langle Ax, x\rangle|, \tag{4.20}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

Proof. The proof follows by Theorem 4.3 on choosing $A = B$, respectively. □

Corollary 4.5. *Let A be a Hermitian matrices and $C(A^{-1}) \geq 0$ is defined as above, then one has the following.*

If $\lambda_1\lambda_n > 0$, then

$$\left|G\left(A^{-1};x\right)\right| \leq \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)^2} - \left\langle C\left(A^{-1}\right)x, x\right\rangle, \tag{4.21}$$

$$\left|G\left(A^{-1};x\right)\right| \leq \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)^2} - \left|\left\langle \left(A^{-1} - \frac{\lambda_1 + \lambda_n}{2\lambda_n\lambda_1} I\right)x, x\right\rangle\right|^2, \tag{4.22}$$

$$\left|G\left(A^{-1};x\right)\right| \leq \frac{(\lambda_n - \lambda_1)^2}{4\lambda_n\lambda_1}\left|\left\langle A^{-1}x, x\right\rangle\right|^2, \tag{4.23}$$

$$\left|G\left(A^{-1};x\right)\right| \leq \left(\frac{|\lambda_1 + \lambda_n|}{\lambda_1\lambda_n} - 2\frac{1}{\sqrt{\lambda_1\lambda_n}}\right)\left|\left\langle A^{-1}x, x\right\rangle\right|, \tag{4.24}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

If $\lambda_1\lambda_n < 0$, then

$$\left|G\left(A^{-1};x\right)\right| \leq \frac{(\lambda_k - \lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2} - \left\langle C\left(A^{-1}\right)x, x\right\rangle, \tag{4.25}$$

where $C(A^{-1}) = ((1/\lambda_{k+1})I - A^{-1})(A^{-1} - (1/\lambda_k)I)$, *and*

$$\left|G\left(A^{-1};x\right)\right| \le \frac{(\lambda_k - \lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2} - \left|\left\langle\left(A^{-1} - \frac{\lambda_{k+1}+\lambda_k}{2\lambda_k\lambda_{k+1}}I\right)x,x\right\rangle\right|^2, \tag{4.26}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

Proof. The proof follows by Corollary 4.4 by replacing A with A^{-1}, respectively. □

Theorem 4.6. *Let A be an $n\times n$ invertible Hermitian matrix with real eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then one has the following.*

If $\lambda_1\lambda_n > 0$, then

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)} - \sqrt{\langle C(A)x,x\rangle\langle C(A^{-1})x,x\rangle}, \tag{4.27}$$

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)} - \left|\left\langle\left(A - \frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1} - \frac{\lambda_1+\lambda_n}{2\lambda_n\lambda_1}I\right)x,x\right\rangle\right|, \tag{4.28}$$

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n\lambda_1)^2}\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle, \tag{4.29}$$

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \frac{\left(\sqrt{|\lambda_n|} - \sqrt{|\lambda_1|}\right)^2}{\sqrt{\lambda_n\lambda_1}}\sqrt{\langle Ax,x\rangle\langle A^{-1}x,x\rangle}, \tag{4.30}$$

for any $x \in \mathbb{C}^n$, $\|x\| = 1$.

If $\lambda_1\lambda_n < 0$, then

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \frac{(\lambda_n - \lambda_1)(\lambda_{k+1} - \lambda_k)}{4|\lambda_k\lambda_{k+1}|} - \sqrt{\langle C(A)x,x\rangle\langle C(A^{-1})x,x\rangle}, \tag{4.31}$$

where $C(A^{-1}) = ((1/\lambda_{k+1})I - A^{-1})(A^{-1} - (1/\lambda_k)I)$,

$$\begin{aligned}\left|x^*Axx^*A^{-1}x - 1\right| \le{}& \frac{(\lambda_n - \lambda_1)(\lambda_{k+1} - \lambda_k)}{4|\lambda_k\lambda_{k+1}|} \\ &- \left|\left\langle\left(A - \frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1} - \frac{\lambda_k+\lambda_{k+1}}{2\lambda_1\lambda_{k+1}}I\right)x,x\right\rangle\right|.\end{aligned} \tag{4.32}$$

Proof. Considering

$$\left|x^*Axx^*A^{-1}x - 1\right|^2 \le G(A;x)G\left(A^{-1};x\right). \tag{4.33}$$

When $\lambda_1\lambda_n > 0$, from (4.17) and (4.21), we get

$$\left|x^*Axx^*A^{-1}x-1\right|^2 \le \left\{\frac{1}{4}(\lambda_n-\lambda_1)^2-\langle C(A)x,x\rangle\right\}\left\{\frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-\left\langle C\left(A^{-1}\right)x,x\right\rangle\right\}. \tag{4.34}$$

From $(a^2-b^2)(c^2-d^2) \le (ac-bd)^2$, we have

$$\left|x^*Axx^*A^{-1}x-1\right|^2 \le \left\{\frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)}-\sqrt{\left[\langle C(A)x,x\rangle\langle C(A^{-1})x,x\rangle\right]}\right\}^2, \tag{4.35}$$

then, the conclusion (4.27) holds.

Similarly, from (4.18) and (4.22), we get

$$\begin{aligned}\left|x^*Axx^*A^{-1}x-1\right|^2 &\le \left\{\frac{1}{4}(\lambda_n-\lambda_1)^2-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|^2\right\}\\ &\quad\times\left\{\frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-\left|\left\langle\left(A^{-1}-\frac{\lambda_1+\lambda_n}{2\lambda_n\lambda_1}I\right)x,x\right\rangle\right|^2\right\}\\ &\le \left\{\frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)}-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1}-\frac{\lambda_1+\lambda_n}{2\lambda_n\lambda_1}I\right)x,x\right\rangle\right|\right\}^2,\end{aligned} \tag{4.36}$$

then, the conclusion (4.28) holds.

From (4.19) and (4.23), we get

$$\left|x^*Axx^*A^{-1}x-1\right|^2 \le \frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)}|\langle Ax,x\rangle|^2\frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)}\left|\left\langle A^{-1}x,x\right\rangle\right|^2, \tag{4.37}$$

then, the conclusion (4.29) holds.

From (4.20) and (4.24), we get

$$\left|x^*Axx^*A^{-1}x-1\right|^2 \le \left(|\lambda_1+\lambda_n|-2\sqrt{\lambda_1\lambda_n}\right)|\langle Ax,x\rangle|\left(\frac{|\lambda_1+\lambda_n|}{\lambda_1\lambda_n}-2\frac{1}{\sqrt{\lambda_1\lambda_n}}\right)\left|\left\langle A^{-1}x,x\right\rangle\right|, \tag{4.38}$$

then, the conclusion (4.30) holds.

When $\lambda_1\lambda_n < 0$, from (4.17) and (4.25), we get

$$\left|x^*Axx^*A^{-1}x-1\right|^2 \le \left\{\frac{1}{4}(\lambda_n-\lambda_1)^2-\langle C(A)x,x\rangle\right\}\left\{\frac{(\lambda_k-\lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2}-\left\langle C\left(A^{-1}\right)x,x\right\rangle\right\}, \tag{4.39}$$

then, the conclusion (4.31) holds.

From (4.18) and (4.26), we get

$$\begin{aligned}\left|x^*Axx^*A^{-1}x-1\right|^2 &\le \left\{\frac{1}{4}(\lambda_n-\lambda_1)^2-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|^2\right\}\\ &\quad\times\left\{\frac{(\lambda_k-\lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2}-\left|\left\langle\left(A^{-1}-\frac{\lambda_{k+1}+\lambda_k}{2\lambda_k\lambda_{k+1}}I\right)x,x\right\rangle\right|^2\right\}\\ &\le\left\{\frac{(\lambda_n-\lambda_1)(\lambda_{k+1}-\lambda_k)}{4|\lambda_k\lambda_{k+1}|}\right.\\ &\quad\left.-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|\left|\left\langle\left(A^{-1}-\frac{\lambda_k+\lambda_{k+1}}{2\lambda_1\lambda_{k+1}}I\right)x,x\right\rangle\right|\right\}^2,\end{aligned} \tag{4.40}$$

then, the conclusion (4.32) holds. □

Corollary 4.7. *With the notations above, for any $x\in\mathbb{C}^n$, $\|x\|=1$, one lets*

$$\begin{aligned}|G_1(A;x)| &= \frac{1}{4}(\lambda_n-\lambda_1)^2-\langle C(A)x,x\rangle,\\ |G_2(A;x)| &= \frac{1}{4}(\lambda_n-\lambda_1)^2-\left|\left\langle\left(A-\frac{\lambda_1+\lambda_n}{2}I\right)x,x\right\rangle\right|^2.\end{aligned} \tag{4.41}$$

If $\lambda_1\lambda_n>0$, one lets

$$\begin{aligned}|G_3(A;x)| &= \frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)}|\langle Ax,x\rangle|^2,\\ |G_4(A;x)| &= \left(|\lambda_1+\lambda_n|-2\sqrt{\lambda_1\lambda_n}\right)|\langle Ax,x\rangle|.\end{aligned} \tag{4.42}$$

If $\lambda_1\lambda_n>0$

$$\begin{aligned}\left|G^1\left(A^{-1};x\right)\right| &= \frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-\left\langle C\left(A^{-1}\right)x,x\right\rangle,\\ \left|G^2\left(A^{-1};x\right)\right| &= \frac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-\left|\left\langle\left(A^{-1}-\frac{\lambda_1+\lambda_n}{2\lambda_n\lambda_1}I\right)x,x\right\rangle\right|^2,\\ \left|G^3\left(A^{-1};x\right)\right| &= \frac{(\lambda_n-\lambda_1)^2}{4\lambda_n\lambda_1}\left|\left\langle A^{-1}x,x\right\rangle\right|^2,\\ \left|G^4\left(A^{-1};x\right)\right| &= \left(\frac{|\lambda_1+\lambda_n|}{\lambda_1\lambda_n}-2\frac{1}{\sqrt{\lambda_1\lambda_n}}\right)\left|\left\langle A^{-1}x,x\right\rangle\right|.\end{aligned} \tag{4.43}$$

If $\lambda_1\lambda_n < 0$, *then*

$$\left|G^5\left(A^{-1};x\right)\right| = \frac{(\lambda_k-\lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2} - \left\langle C\left(A^{-1}\right)x,x\right\rangle, \tag{4.44}$$

where $C(A^{-1}) = ((1/\lambda_{k+1})I - A^{-1})(A^{-1} - (1/\lambda_k)I)$, *and*

$$\left|G^6\left(A^{-1};x\right)\right| = \frac{(\lambda_k-\lambda_{k+1})^2}{4(\lambda_k\lambda_{k+1})^2} - \left|\left\langle\left(A^{-1} - \frac{\lambda_{k+1}+\lambda_k}{2\lambda_k\lambda_{k+1}}I\right)x,x\right\rangle\right|^2. \tag{4.45}$$

Then, one has the following.

If $\lambda_1\lambda_n > 0$,

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \sqrt{G^*(A;x)G^{\dot{*}}(A^{-1};x)}, \tag{4.46}$$

where

$$\begin{gathered} G^*(A;x) = \min\{G_1(A;x),G_2(A;x),G_3(A;x),G_4(A;x)\}, \\ G^*(A;x) = \min\left\{G^1\left(A^{-11};x\right),G^2\left(A^{-1};x\right),G^3\left(A^{-1};x\right),G^4\left(A^{-1};x\right)\right\}. \end{gathered} \tag{4.47}$$

If $\lambda_1\lambda_n < 0$

$$\left|x^*Axx^*A^{-1}x - 1\right| \le \sqrt{G^\circ(A;x)G^\bullet(A^{-1};x)}, \tag{4.48}$$

where

$$\begin{gathered} G^\circ(A;x) = \min\{G_1(A;x),G_2(A;x)\}, \\ G^\bullet(A;x) = \min\left\{G^5\left(A^{-1};x\right),G^6\left(A^{-1};x\right)\right\}. \end{gathered} \tag{4.49}$$

Proof. The proof follows from that the conclusions in Corollaries 4.4 and 4.5 are independent. □

Remark 4.8. It is easy to see that if $\lambda_1 > 0$, $\lambda_n > 0$, our result coincides with the inequality of operator versions in [6]. So we conclude that our results give an improvement of the Kantorovich inequality [6] that applies to all invertible Hermite matrices.

5. Conclusion

In this paper, we introduce some new Kantorovich-type inequalities for the invertible Hermitian matrices. Inequalities (4.27) and (4.31) are the same as [4], but our proof is simple. In Theorem 4.6, if $\lambda_1 > 0$, $\lambda_n > 0$, the results are similar to the well-known Kantorovich-type inequalities for operators in [6]. Moreover, for any invertible Hermitian matrix, there exists a similar inequality.

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