Research Article

The Group Involutory Matrix of the Combinations of Two Idempotent Matrices

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1. Introduction

Throughout this paper $\mathbb{C}^{n \times n}$ stands for the set of $n \times n$ complex matrices. Let $A \in \mathbb{C}^{n \times n}$. $A$ is said to be idempotent if $A^2 = A$. $A$ is said to be group invertible if there exists an $X \in \mathbb{C}^{n \times n}$ such that

$$AXA = A, \quad XAX = X, \quad AX =XA$$ (1.1)

hold. If such an $X$ exists, then it is unique, denoted by $A_g$, and called the group inverse of $A$. It is well known that the group inverse of a square matrix $A$ exists if and only if $\text{rank}(A^2) = \text{rank}(A)$ (see, e.g., [1] for details). Clearly, not every matrix is group invertible. But the group inverse of every idempotent matrix exists and is this matrix itself.

Recall that a matrix $A$ with the group inverse is said to be group involutory if $A_g = A$. $A$ is the group involutory matrix if and only if it is tripotent, that is, satisfies $A^3 = A$ (see [2]). Thus, for a nonzero idempotent matrix $P$ and a nonzero scalar $a$, $aP$ is a group involutory matrix if and only if either $a = 1$ or $a = -1$.

Recently, some properties of linear combinations of idempotents or projections are widely discussed (see, e.g., [3–12] and the literature mentioned below). In [13], authors...
established a complete solution to the problem of when a linear combination of two different projectors is also a projector. In [14], authors considered the following problem: when a linear combination of nonzero different idempotent matrices is the group involutory matrix. In [15], authors provided the complete list of situations in which a linear combination of two idempotent matrices is the group involutory matrix. In [16], authors discussed the group inverse of \( aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ \) of idempotent matrices \( P \) and \( Q \), where \( a, b, c, d, e, f, g \in \mathbb{C} \) with \( a, b \neq 0 \), deduced its explicit expressions, and some necessary and sufficient conditions for the existence of the group inverse of \( aP + bQ + cPQ \).

In this paper, we will investigate the following problem: when \( aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ \) is group involutory. To this end, we need the results below.

**Lemma 1.1** (see [16, Theorems 2.1 and 2.4]). Let \( P, Q \in \mathbb{C}^{n \times n} \) be two different nonzero idempotent matrices. Suppose \((QP)^2 = (PQ)^2\). Then for any scalars \( a, b, c, d, e, f, g \), where \( a, b \neq 0 \) and \( \theta = a + b + c + d + e + f + g \), \( aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ \) is group invertible, and

(i) if \( \theta \neq 0 \), then

\[
\begin{align*}
(aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ) &= \frac{1}{a} P + \frac{1}{b} Q - \left( \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ - \left( \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP \\
&\quad + \left( \frac{2}{a} + \frac{1}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} \right) PQP + \left( \frac{1}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - af}{ab^2} \right) QPQ \\
&\quad - \left( \frac{2}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} + \frac{cd - af}{ab^2} - \frac{1}{\theta} \right) PQPQ.
\end{align*}
\]

(ii) if \( \theta = 0 \), then

\[
\begin{align*}
(aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ) &= \frac{1}{a} P + \frac{1}{b} Q - \left( \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ - \left( \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP \\
&\quad + \left( \frac{2}{a} + \frac{1}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} \right) PQP + \left( \frac{1}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - af}{ab^2} \right) QPQ \\
&\quad - \left( \frac{2}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} + \frac{cd - af}{ab^2} \right) (PQ)^2.
\end{align*}
\]

**Lemma 1.2** (see [16, Theorem 3.1]). Let \( P, Q \in \mathbb{C}^{n \times n} \) be two different nonzero idempotent matrices. Suppose \((QP)^2 = 0\). Then for any scalars \( a, b, c, d, e, f, \) and \( g \), where \( a, b \neq 0 \), \( aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ \) is group invertible, and

\[
\begin{align*}
(aP + bQ + cPQ + dQP + ePQP + f QPQ + gQPQ) &= \frac{1}{a} P + \frac{1}{b} Q - \left( \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ - \left( \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP
\end{align*}
\]
Let \( P, Q \in \mathbb{C}^{n \times n} \) be two different nonzero idempotent matrices with \((PQ)^2 = (QP)^2\), and let \( A \) be a combination of the form

\[
A = aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ,
\]

where \( a, b, c, d, e, f, g \in \mathbb{C} \) with \( a, b \neq 0 \). Denote \( \theta = a + b + c + d + e + f + g \). Then the following list comprises characteristics of all cases where \( A \) is the group involutory matrix:

(a) the cases denoted by \((a_1) \sim (a_3)\), in which

\[
PQ = QP,
\]

and any of the following sets of additional conditions hold:

(a1) either \( a = 1 \) or \( a = -1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \), and \( Q = PQ \);

(a2) either \( b = 1 \) or \( b = -1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \), and \( P = PQ \);

(a3) either \( a = 1 \) or \( a = -1 \), either \( b = 1 \) or \( b = -1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \) or \( PQ = 0 \).

(b) the cases denoted by \((b_1) \sim (b_6)\), in which

\[
PQ \neq QP, \quad PQP = QQP,
\]

and any of the following sets of additional conditions hold:

(b1) \( a = \pm 1, b = \mp 1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \) or \( PQP = 0 \);

(b2) \( a = b = \pm 1, c = d = \mp 1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \) or \( PQP = 0 \);

(b3) \( a = b = \pm 1, c = \mp 1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \), and \( QP = PQP \);

(b4) \( a = b = \pm 1, c = \mp 1 \), either \( \theta = 1 \) or \( \theta = -1 \) or \( \theta = 0 \), and \( PQ = PQP \);

(b5) \( a = b = \pm 1, c = \mp 1 \), and \( QP = 0 \);

(b6) \( a = b = \pm 1, d = \mp 1 \), and \( PQ = 0 \),

(c) the cases denoted by \((c_1) \sim (c_{18})\), in which

\[
PQP \neq QPQ, \quad PQPQ = QQPQ,
\]
and any of the following sets of additional conditions hold:

\( (c_1) \) \( a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1, \) either \( \theta = 1 \) or \( \theta = -1, \) and \( PQP = PQP; \)
\( (c_2) \) \( a = b = e = \pm 1, c = d = \mp 1, \) either \( \theta = 1 \) or \( \theta = -1, \) and \( PQP = PQP; \)
\( (c_3) \) \( a = b = f = \pm 1, c = d = \mp 1, \) either \( \theta = 1 \) or \( \theta = -1, \) and \( PQP = PQP; \)
\( (c_4) \) \( a = b = f = \pm 1, c = d = \mp 1, \) either \( \theta = 1 \) or \( \theta = -1, \) and \( PQP = PQP; \)
\( (c_5) \) \( a = b = f = \pm 1, c = d = \mp 1, \) either \( \theta = 1 \) or \( \theta = -1, \) and \( PQP = PQP; \)
\( (c_6) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) either \( g = 1 \) or \( g = -1; \)
\( (c_7) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) either \( g = 1 \) or \( g = 3; \)
\( (c_8) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) and \( PQP = 0; \)
\( (c_9) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) and \( PQP = 0; \)
\( (c_{10}) \) \( a = b = f = \pm 1, c = d = \mp 1, \) and \( PQP = 0; \)
\( (c_{11}) \) \( a = \pm 1, b = \mp 1, c + d + 2e \pm cd = \pm 1, c + d + 2f \mp cd = \mp 1, \) and \( PQP = 0; \)
\( (c_{12}) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) and \( PQP = 0; \)
\( (c_{13}) \) \( a = \pm 1, b = \mp 1, 2e + c + d \pm cd = \pm 1, \) and \( PQP = PQP; \)
\( (c_{14}) \) \( a = b = e = f = \pm 1, c = d = \mp 1, \) and \( PQP = PQP; \)
\( (c_{15}) \) \( a = \pm 1, b = \mp 1, 2f + c + d \pm cd = \pm 1, \) and \( PQP = PQP; \)
\( (c_{16}) \) \( a = \pm 1, b = \mp 1, 2f + c + d \mp cd = \mp 1, \) and \( PQP = PQP; \)
\( (c_{17}) \) \( a = \pm 1, b = \mp 1, 2f + c + d \pm cd = \pm 1, \) and \( PQP = 0; \)
\( (c_{18}) \) \( a = \pm 1, b = \mp 1, 2f + c + d \mp cd = \mp 1, \) and \( PQP = 0; \)

Proof. Obviously, the condition (2.2) implies that the group inverse of \( A \) exists and is of the form (1.2) when \( \theta \neq 0 \) or the form (1.3) when \( \theta = 0 \) by Lemma 1.1. So do the conditions (2.2), (2.3), and (2.4). We will straightforwardly show that a matrix \( A \) of the form (2.1) is the group involutory matrix if and only if \( A - A^*_g = 0. \)

(a) Under the condition (2.2), \( A = aP + bQ + \mu PQ, \) where \( \mu = c + d + e + f + g. \)

(i) If \( \theta \neq 0, \) then

\[
A^*_g = \frac{1}{a} P + \frac{1}{b} Q + \left( \frac{1}{\theta} - \frac{1}{a} - \frac{1}{b} \right) PQ. \tag{2.5}
\]

and so

\[
A - A^*_g = \left( a - \frac{1}{a} \right) P + \left( b - \frac{1}{b} \right) Q + \left( \mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b} \right) PQ = 0. \tag{2.6}
\]

Multiplying (2.6) by \( P \) and \( Q, \) respectively, leads to

\[
\left( a - \frac{1}{a} \right) P + \left( b - \frac{1}{b} \right) PQ + \left( \mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b} \right) PQ = 0, \tag{2.7}
\]

\[
\left( a - \frac{1}{a} \right) PQ + \left( b - \frac{1}{b} \right) Q + \left( \mu - \frac{1}{\theta} + \frac{1}{a} + \frac{1}{b} \right) PQ = 0.
\]
Thus, we have three situations: 

\[ (a - \frac{1}{a})P + (b - \frac{1}{b})PQ = (a - \frac{1}{a})PQ + (b - \frac{1}{b})Q. \]  

(2.8)

Multiplying the above equation, respectively, by \( P \) and by \( Q \), we get

\[ (a - \frac{1}{a})(P - PQ) = 0, \quad (b - \frac{1}{b})(Q - PQ) = 0. \]  

(2.9)

Thus, since \( P \neq Q \), we have three situations: \( P = PQ \) and \( b = b^{-1}; a = a^{-1} \) and \( Q = PQ; a = a^{-1} \) and \( b = b^{-1} \).

When \( Q = PQ \) and \( a = a^{-1} \), (2.6) becomes \((\theta - \theta^{-1})Q = 0\) and then \( \theta = \pm 1 \). Therefore, we obtain \((a_1)\) except the situation \( \theta = 0 \). Similarly, when \( b = b^{-1} \) and \( P = PQ \), we have \((a_2)\) except the situation \( \theta = 0 \). When \( a = a^{-1} \) and \( b = b^{-1} \), (2.6) becomes \((\theta - \theta^{-1})PQ = 0\) and then \( \theta = \pm 1 \) or \( PQ = 0 \). Therefore, we obtain \((a_3)\) except the situation \( \theta = 0 \).

(2) If \( \theta = 0 \), then

\[ A_s = \frac{1}{a}P + \frac{1}{b}Q - \left( \frac{1}{a} + \frac{1}{b} \right)PQ, \]  

(2.10)

and then

\[ A - A_s = \left( a - \frac{1}{a} \right)P + \left( b - \frac{1}{b} \right)Q + \left( \mu + \frac{1}{a} + \frac{1}{b} \right)PQ = 0. \]  

(2.11)

Analogous to the process of reaching (2.9) in \((a)(1)\), we have

\[ \left( b - \frac{1}{b} \right)(Q - PQ) = 0, \quad \left( a - \frac{1}{a} \right)(P - PQ) = 0. \]  

(2.12)

Thus, we have three situations: \( P = PQ \) and \( b = b^{-1}; a = a^{-1} \) and \( Q = PQ; a = a^{-1} \) and \( b = b^{-1} \), since \( P \neq Q \). Similar to the argument in \((a)(1)\), substituting them, respectively, into (2.11), we can obtain the situation \( \theta = 0 \), respectively, in \((a_1), (a_2), \) and \((a_3)\).

(b) Under the condition (2.3), \( A = aP + bQ + cPQ + dQP + vPQP \), where \( v = e + f + g \).

(1) If \( \theta \neq 0 \), then

\[ A_s = \frac{1}{a}P + \frac{1}{b}Q - \left( \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right)PQ - \left( \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right)QP \]
\[ + \left( \frac{1}{a} + \frac{1}{b} + \frac{c + d}{ab} + \frac{1}{\theta} \right)PQP, \]  

(2.13)
and so

\[ A - A_x = \left( a - \frac{1}{a} \right) P + \left( b - \frac{1}{b} \right) Q + \left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ \]

\[ + \left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP + \left( v - \frac{1}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{1}{\theta} \right) PQP = 0. \tag{2.14} \]

Multiplying the above equation, respectively, on the two sides by \( P \) yields

\[ 0 = \left( a - \frac{1}{a} \right) P + \left( c + b + \frac{1}{a} + \frac{c}{ab} \right) PQ + \left( v + d - \frac{c}{ab} - \frac{1}{\theta} \right) PQP; \tag{2.15} \]

\[ 0 = \left( a - \frac{1}{a} \right) P + \left( b + d + \frac{1}{a} + \frac{d}{ab} \right) QP + \left( v + c - \frac{d}{ab} - \frac{1}{\theta} \right) PQP. \tag{2.16} \]

Multiplying (2.15) on the left sides by \( Q \) and (2.16) on the right sides by \( Q \), by (2.3), we have

\[ \left( a - \frac{1}{a} \right) QP + \left( b + c + d + v + \frac{1}{a} - \frac{1}{\theta} \right) QPQ = 0, \]

\[ \left( a - \frac{1}{a} \right) PQ + \left( b + c + d + v + \frac{1}{a} - \frac{1}{\theta} \right) PQP = 0, \tag{2.17} \]

and then \((a - a^{-1})(Q P - P Q) = 0\). Since \( Q P \neq P Q, a = a^{-1} \). Similarly, \( b = b^{-1} \).

Substituting \( a = a^{-1} \) inside (2.17) yields \((\theta - \theta^{-1})Q P Q = 0\) and then \( \theta = \theta^{-1} \) or \( Q P Q = 0 \).

We will discuss the remainder for detail as follows:

When \( a = a^{-1}, b = b^{-1} \), (2.14) becomes

\[ 0 = \left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ + \left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP \]

\[ + \left( v - \frac{1}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{1}{\theta} \right) PQP, \tag{2.18} \]

(i) if \( a + b = 0 \), then

\[ c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0, \quad d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0, \tag{2.19} \]

and so it follows from (2.18) that

\[ \left( \theta - \frac{1}{\theta} \right) PQP = \left( v + c + d - \frac{1}{\theta} \right) PQP = 0. \tag{2.20} \]

Therefore, either \( \theta = \theta^{-1} \) or \( PQ P = 0 \) implies that (2.18) holds, namely, (2.14) holds. Thus, we have \((b_1)\) except the situation \( \theta = 0 \).
(ii) if $a = b$, then (2.18) becomes

$$0 = (2c + 2a)PQ + (2d + 2a)QP + \left(2v - \theta - \frac{1}{\theta}\right)PQP.$$  

(2.21)

Multiplying the above equation, respectively, on the right side by $P$ and then on the left side by $Q$, we have

$$0 = (2c + 2a)PQ + \left(v + d - c - \frac{1}{\theta}\right)PQP,$$

(2.22)

$$0 = (2d + 2a)QP + \left(v + c - d - \frac{1}{\theta}\right)PQP.$$  

(2.23)

So if $\theta = \theta^{-1}$, then the two equations above (2.22) and (2.23) become, respectively,

$$(c + a)(PQ - PQP) = 0, \quad (d + a)(QP - PQP) = 0.$$  

(2.24)

Or if $PQP = 0$, then (2.22) and (2.23) become, respectively,

$$(c + a)PQ = 0, \quad (d + a)QP = 0.$$  

(2.25)

Since $PQ \neq QP$, it follows from (2.24) and (2.25) that we have the six situations: $\theta = \theta^{-1}$ and $c = d = -a$; $\theta = \theta^{-1}$, $c = -a$ and $QP = PQP$; $\theta = \theta^{-1}$, $d = -a$, and $QP = PQP$; $c = d = -a$ and $QP = 0$; $d = -a$ and $QP = 0$; $c = d = -a$ and $PQP = 0$. Thus, we have $(b_2) \sim (b_4)$ except the situation $\theta = 0$, and $(b_5)$ and $(b_6)$.

(2) If $\theta = 0$, then

$$A_s = \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ - \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(\frac{1}{a} + \frac{1}{b} + \frac{c + d}{ab}\right)PQP,$$  

(2.26)

and then

$$A - A_s = \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ
+ \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP + \left(v - \frac{1}{a} - \frac{1}{b} - \frac{c + d}{ab}\right)PQP = 0.$$  

(2.27)

Analogous to the process in (b)(1), using (2.27) we can obtain

$$\left(a - \frac{1}{a}\right)QP - \left(a - \frac{1}{a}\right)PQP = 0,$$

$$\left(a - \frac{1}{a}\right)PQ - \left(a - \frac{1}{a}\right)PQP = 0.$$  

(2.28)
Thus, since $PQ \neq QP$, $PQ \neq QP$ and/or $QP \neq PQP$ and then $a = a^{-1}$. Similarly, $b = b^{-1}$. Hence, $a = \pm b$.

(i) If $a = -b$, then

$$c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0,$$

$$d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0,$$

$$\nu - \frac{1}{a} - \frac{1}{b} - \frac{c + d}{ab} = -2(a + b) = 0.$$ (2.29)

Thus, (2.27) holds. Hence we have the situation $\theta = 0$ in $(b_1)$.

(ii) If $a = b$, then (2.27) becomes

$$(c + a)PQ + (d + a)QP + \nu PQP = 0.$$ (2.30)

Multiplying the above equation on the left side, respectively, by $P$ and by $Q$, we have

$$(c + a)(PQ - PQP) = 0, \quad (d + a)(QP - PQ) = 0.$$ (2.31)

Thus, $c = d = -a; c = -a$ and $QP = PQP; d = -a$ and $PQ = PQ$. Hence, we have the situation $\theta = 0$, respectively, in $(b_2), (b_3)$, and $(b_4)$.

(c) Under the condition (2.4),

$$A = aP + bQ + cQP + dQ + ePQP + fQPQ + gPQPQ.$$ (2.32)

(1) If $\theta \neq 0$, then

$$A_\theta = \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ - \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP$$

$$+ \left(\frac{2}{a} + \frac{1}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b}\right)PQP + \left(\frac{1}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - af}{ab^2}\right)QQP$$

$$- \left(\frac{2}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} + \frac{cd - af}{ab^2} - \frac{1}{\theta}\right)PQPQ,$$ (2.33)

and so

$$A - A_\theta = \left(a - \frac{1}{a}\right)P + \left(b - \frac{1}{b}\right)Q + \left(c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP$$

$$+ \left(e - \frac{2}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{cd - be}{a^2b}\right)PQP$$
Thus, a Journal of Applied Mathematics 9

\[ + \left( f - \frac{1}{a} - \frac{2}{b} - \frac{c + d}{ab} - \frac{cd - af}{ab^2} \right) \text{QPQ} \]

\[ + \left( g + \frac{2}{a} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} + \frac{cd - af}{ab^2} - \frac{1}{\theta} \right) \text{QPQ} = 0. \] (2.34)

If \( PQ = 0 \), then \( \text{QPQ} = 0 = \text{QP} \) and so it contradicts (2.4). Thus \( PQ \neq 0 \). Similarly, \( QP \neq 0 \).

Multiplying (2.34) on the left side by \( QP \) yields

\[ \left( a - \frac{1}{a} \right) \text{QP} + \left( b + c + \frac{1}{a} + \frac{c}{ab} \right) \text{QPQ} + \left( d + e + f + g - \frac{c}{ab} - \frac{1}{\theta} \right) \text{QPQ} = 0. \] (2.35)

Multiplying the above equation, respectively, on the left side by \( P \) and on the right side by \( PQ \) yields, by (2.4),

\[ 0 = \left( a - \frac{1}{a} \right) \text{PQP} + \left( \frac{1}{a} - a + \theta - \frac{1}{\theta} \right) \text{PQPQ}, \] (2.36)

\[ 0 = \left( a - \frac{1}{a} \right) \text{QPQ} + \left( \frac{1}{a} - a + \theta - \frac{1}{\theta} \right) \text{PQPQ}. \] (2.37)

Since \( \text{PQP} \neq \text{QPQ} \), \( a = a^{-1} \) by (2.36) and (2.37). Similarly, we can gain \( b = b^{-1} \). Substituting \( a = a^{-1} \) inside (2.36) yields \( \theta = \theta^{-1} \) or \( \text{PQPQ} = 0 \).

(i) Consider the case of \( a = a^{-1} \), \( b = b^{-1} \) and \( \theta = \theta^{-1} \).

Substituting \( a = a^{-1} \), \( b = b^{-1} \), and \( \theta = \theta^{-1} \) inside (2.35) yields

\[ \left( a + b + c + \frac{c}{ab} \right) \left( \text{QPQ} - \text{PQPQ} \right) = 0. \] (2.38)

Similarly, we have

\[ \left( a + b + d + \frac{d}{ab} \right) \left( \text{PQP} - \text{QPQ} \right) = 0. \] (2.39)

If \( \text{PQP} = \text{PQPQ} \), then \( \text{QPQ} \neq \text{PQPQ} \) by the hypothesis \( \text{PQP} \neq \text{QPQ} \) and so \( a + b + c + c/ab = 0 \) by (2.38). Multiplying (2.34) on the right side by \( Q \) yields

\[ \left( a + c + d + 2f - \frac{cd}{a} \right) \left( \text{QPQ} - \text{PQPQ} \right) = 0. \] (2.40)

Thus, \( a + c + d + 2f - cd/a = 0 \) and then (2.14) becomes

\[ \left( a + b + d + \frac{d}{ab} \right) \text{QP} + \left( f - a - 2b - \frac{c + d}{ab} - \frac{cd - af}{a} \right) \text{QPQ} \]

\[ + \left( 2.38 \right) \text{QPQ} = 0. \] (2.41)
Multiplying the above equation on the right side by \( P \) yields

\[
(a + b + d + \frac{d}{ab}) (QP - PQP) = 0. \tag{2.42}
\]

Assume \( PQ = PQP \). Then \( QPQ = PQPQ = PQP = PQ = PQP \) and it contradicts the hypothesis \( PQP \neq QPQ \). Thus, \( a + b + d + d/ab = 0 \).

Similarly, if \( QPQ = PQPQ \), then we can obtain \( a + b + d + d/ab = 0 \) and \( a + b + c + c/ab = 0 \).

Obviously, if \( QPQ 
eq PQPQ \) and \( QPQ 
eq PQQP \), we have \( a + b + d + d/ab = 0 \), \( a + b + c + c/ab = 0 \), \( b + c + d + 2e - cd/b = 0 \), and \( a + c + d + 2f - cd/a = 0 \).

Next, we calculate these scalars. If \( a + b = 0 \), then \( a + b + c + c/ab = 0 \) for any \( c \) and \( a + b + d + d/ab = 0 \) for any \( d \), and so \( c, d, e \) are chosen to satisfy \( b + c + d + 2e - cd/b = 0 \).

Similarly, \( c, d, f \) are chosen to satisfy \( a + c + d + 2f - cd/a = 0 \).

If \( a = b \), then \( c = d = -a \), and \( e = a \) by solving \( b + c + d + 2e - cd/b = 0 \), and \( f = a \) by solving \( a + c + d + 2f - cd/a = 0 \).

Note that \( b + c + d + 2e - cd/b = 0 \) and \( a + c + d + 2f - cd/a = 0 \) imply \( g = \theta - (a + b) \).

Hence, we have \( (c_1) \sim (c_6) \).

(ii) Consider the case of \( a = a^{-1} \), \( b = b^{-1} \), and \( PQPQ = 0 \).

Multiplying (2.34), respectively, on the right side by \( QP \) and on the left side by \( PQ \) yields

\[
\left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQPQ = 0, \tag{2.43}
\]

\[
\left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) PQPQ = 0.
\]

If \( PQ = 0 \), then \( PQP \neq 0 \) and so \( a + b + d + d/ab = 0 \) and (2.34) becomes

\[
0 = \left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ + \left( e - \frac{2}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{cd - be}{a^2b} \right) PQP. \tag{2.44}
\]

Multiplying (2.44) on right side by \( Q \) yields

\[
\left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ = 0. \tag{2.45}
\]

Since \( PQ \neq 0 \), \( a + b + c + c/ab = 0 \) and then (2.44) becomes

\[
\left( 2e + b + c + d - \frac{cd}{b} \right) PQP. \tag{2.46}
\]

Thus, \( 2e + b + c + d - cd/b = 0 \).

If \( PQP = 0 \), then we, similarly, have \( a + b + c + c/ab = 0 \), \( a + b + d + d/ab = 0 \), and \( 2f + a + c + d - cd/a = 0 \).
If \( PQP \neq 0 \) and \( QPQ \neq 0 \), then, multiplying (2.34), on the right side by \( Q \) and on the left side by \( P \) yields \( a + b + c + c/ab = 0 \), and multiplying (2.34) on the right side by \( P \) and on the left side by \( Q \) yields \( a + b + d + d/ab = 0 \). Thus, (2.34) becomes

\[
\left( e - \frac{2}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{cd - be}{a^2b} \right) PQP + \left( f - \frac{1}{a} - \frac{2}{b} - \frac{c + d}{ab} - \frac{cd - af}{ab^2} \right) QPQ = 0. \tag{2.47}
\]

Multiplying the equation above on the right side, respectively, by \( P \) and by \( Q \) yields

\[
2e + b + c + d - \frac{cd}{b} = 0, \quad 2f + a + c + d - \frac{cd}{a} = 0. \tag{2.48}
\]

As the argument above in (i), we have \( (c_7) \sim (c_{12}) \).

(2) If \( \theta = 0 \), then

\[
A_s = \frac{1}{a} P + \frac{1}{b} Q - \left( \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ - \left( \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP
+ \left( \frac{2}{a} + \frac{1}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} \right) PQP + \left( \frac{1}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - af}{ab^2} \right) QPQ \tag{2.49}
\]

and so

\[
A - A_s = \left( a - \frac{1}{a} \right) P + \left( b - \frac{1}{b} \right) Q + \left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ
+ \left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP + \left( e - \frac{2}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{cd - be}{a^2b} \right) PQP
+ \left( f - \frac{1}{a} - \frac{2}{b} - \frac{c + d}{ab} - \frac{cd - af}{ab^2} \right) QPQ
+ \left( g + \frac{2}{a} + \frac{2}{b} + \frac{c + d}{ab} + \frac{cd - be}{a^2b} + \frac{cd - af}{ab^2} \right) PQPQ = 0. \tag{2.50}
\]

Analogous to the process in (c)(1), using (2.50), we can get

\[
\left( a - \frac{1}{a} \right) (PQP - PQPQ) = 0, \tag{2.51}
\]

\[
\left( a - \frac{1}{a} \right) (QPQ - PQPQ) = 0.
\]
Thus, since \(PQP \neq QPQ\), \(PQP \neq QPQ\) and/or \(QPQ \neq PQP\) and then \(a = a^{-1}\). Similarly, \(b = b^{-1}\). Therefore, multiplying (2.50) on the right side by \(Q\) and on the left side by \(P\) yields

\[
\left(a + b + c + \frac{c}{ab}\right)(PQ - PQP) = 0.
\]

Multiplying (2.50) on the right side by \(P\) and on the left side by \(Q\) yields

\[
\left(a + b + d + \frac{d}{ab}\right)(QP - PQP) = 0.
\]

Since \(PQ \neq PQPQ\) and \(QP \neq PQPQ\), \(a + b + c + ab = 0\) and \(a + b + d + ab = 0\). Multiplying (2.50) on the left side, respectively, by \(P\) and by \(Q\) yields

\[
\left(2e + b + c + d - \frac{cd}{b}\right)(PQP - PQPQ) = 0,
\]

\[
\left(2f + a + c + d - \frac{cd}{a}\right)(QPQ - PQPQ) = 0.
\]

Thus, we have \(2e + b + c + d - cd/b = 0\) and \(QPQ = PQPQ\); \(2f + a + c + d - cd/a = 0\) and \(PQP = PQPQ\); \(2e + b + c + d - cd/b = 0\) and \(2f + a + c + d - cd/a = 0\).

Note that \(2e + b + c + d - cd/b = 0\) and \(2f + a + c + d - cd/a = 0\) imply \(g = -(a + b)\) by \(\theta = 0\). As the argument above in (c)(1), we have \((c_{13}) \sim (c_{18})\).

**Remark 2.2.** Clearly, [15, (a) and (b) in Theorem] are the special cases in Theorem 2.1.

**Example 2.3.** Let

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(2.55)
Then they, obviously, are idempotent, and \((PQ)^2 = (QP)^2\) but \(PQP \neq QPQ\). By Theorem 2.1(c),

\[
A = P - Q + 2PQ + 2QP - \frac{7}{2} PQP - \frac{1}{2} QPQ + PQPQ
\]

(2.56)

is the group involutory matrix, namely, \(A = A_g\), since \(2 + 2 + 2 \cdot (-7/2) + 2 \cdot 2 = 1\) and \(2 + 2 + 2 \cdot (-1/2) - 2 \cdot 2 = -1\). By Theorem 2.1(c),

\[
P - Q + P - 2QP + 2PQP - QPQ
\]

(2.57)

is group involutory since \(1 - 2 + 2 \cdot 2 + 1 \cdot (-2) = 1\) and \(1 - 2 + 2 \cdot (-1) - 1 \cdot (-2) = -1\).

Next, we will study the situation \((PQ)^2 = 0\) or \((QP)^2 = 0\).

**Theorem 2.4.** Let \(P, Q \in \mathbb{C}^{n \times n}\) be two different nonzero idempotent matrices, and let \(A\) be a combination of the form

\[
A = aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ,
\]

where \(a, b, c, d, e, f, g \in \mathbb{C}\) with \(a, b \neq 0\). Suppose that

\[
PQPQ \neq 0, \quad QPQ = 0,
\]

(2.59)

and any of the following sets of additional conditions hold:

\((d_1)\) \(a = b = \pm 1, c = d = \mp 1, e = f = \pm 1, g = \mp 1\);

\((d_2)\) \(a = \pm 1, b = \mp 1, 2e + c + d \pm cd = \pm 1, 2f + c + d \mp cd = \mp 1\).

Then \(A\) is the group involutory matrix.

**Proof.** By Lemma 1.2,

\[
A - A_g = \left(\frac{1}{a} - \frac{1}{a}\right)P + \left(\frac{1}{b} - \frac{1}{b}\right)Q + \left(\frac{1}{a} + \frac{1}{b} + \frac{c}{ab}\right)PQ + \left(\frac{1}{a} + \frac{1}{b} + \frac{d}{ab}\right)QP
\]

\[
+ \left(\frac{e}{a} - \frac{2}{a} - \frac{1}{b} - \frac{c + d}{ab} - \frac{cd - be}{a^2 b}\right)PQP
\]

\[
+ \left(\frac{f}{a} - \frac{2}{b} - \frac{c + d}{ab} - \frac{cd - af}{a^2 b}\right)QPP
\]

\[
+ \left(\frac{g}{a} + \frac{2}{b} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2 b} + \frac{cd - af - cf}{a^2 b^2} + \frac{c^2 d}{a^2 b^2}\right)(PQ)^2.
\]

(2.60)
Since \( PQPQ \neq 0 \), multiplying (2.60), respectively, on the right side and on the right side by \( PQPQ \) yields

\[
\left( a - \frac{1}{a} \right) PQPQ = 0, \quad \left( b - \frac{1}{b} \right) PQPQ = 0, \tag{2.61}
\]

and so \( a = a^{-1} \) and \( b = b^{-1} \). Substituting them inside (2.60), we get

\[
0 = \left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQ + \left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP \\
+ \left( e - \frac{2}{a} - \frac{1}{b} - \frac{c + d - cd - be}{a^2 b} \right) PQP \\
+ \left( f - \frac{1}{a} - \frac{2}{b} - \frac{c + d - cd - af}{ab^2} \right) QPQ \\
+ \left( g + \frac{2}{a} + \frac{2}{b} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2 b} + \frac{cd - af - cf}{ab^2} + \frac{c^2 d}{a^2 b^2} \right) PQPQ. \tag{2.62}
\]

Multiplying (2.62) on the left side by \( PQP \) yields

\[
\left( c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} \right) PQPQ = 0, \tag{2.63}
\]

and then

\[
c + \frac{1}{a} + \frac{1}{b} + \frac{c}{ab} = 0. \tag{2.64}
\]

So (2.62) becomes

\[
0 = \left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) QP + \left( e - \frac{2}{a} - \frac{1}{b} - \frac{c + d - cd - be}{a^2 b} \right) PQP \\
+ \left( f - \frac{1}{a} - \frac{2}{b} - \frac{c + d - cd - af}{ab^2} \right) QPQ \\
+ \left( g + \frac{2}{a} + \frac{2}{b} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2 b} + \frac{cd - af - cf}{ab^2} + \frac{c^2 d}{a^2 b^2} \right) PQPQ. \tag{2.65}
\]

Multiplying (2.65) on the left side by \( PQ \) and on the right side by \( P \) yields

\[
\left( d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} \right) PQPQ = 0. \tag{2.66}
\]
Therefore,
\[
d + \frac{1}{a} + \frac{1}{b} + \frac{d}{ab} = 0.
\] (2.67)

Similarly, we can obtain
\[
0 = e - \frac{c + d}{a} - \frac{cd - be}{ab^2},
\]
\[
0 = f - \frac{1}{a} - \frac{c + d}{ab} - \frac{cd - af}{ab^2},
\]
\[
0 = g + \frac{2}{a} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2b} + \frac{cd - af - cf}{ab^2} + \frac{c^2d}{a^2b^2}.
\] (2.68)

By (2.64) and (2.67), we can obtain
\[
\frac{1}{b} + c + d + 2e - \frac{cd}{b} = 0, \quad \frac{1}{a} + c + d + 2f - \frac{cd}{a} = 0.
\] (2.69)

Since \(a = a^{-1}\) and \(b = b^{-1}\), \(a = \pm b\). If \(a = -b\), then (2.64) holds for any \(c\), (2.67) holds for any \(d\), and, for any \(c, d, e, f\) satisfying (2.69) and any \(g\),
\[
g + \frac{2}{a} + \frac{2c + d + g}{ab} + \frac{cd - be - ce}{a^2b} + \frac{cd - af - cf}{ab^2} + \frac{c^2d}{a^2b^2}
\]
\[
= c^2d - 2c - d - (e + f) + \frac{c}{a}(e - f)
\]
\[
= c^2d - 2c - d + (c + d) + \frac{c}{a}\left(1 - \frac{cd}{a}\right) = 0.
\] (2.70)

If \(a = b\), then, by (2.64) - (2.69), \(c = d = -a\) and \(e = f = a\) and so \(g = -a\) from (2.68).

Hence, we have \((d_1)\) and \((d_2)\). \(\square\)

**Example 2.5.** Let
\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & -1 & 1
\end{pmatrix}.
\] (2.71)

Obviously they are idempotent, and \((QP)^2 = 0\) but \((PQ)^2 \neq 0\). By Theorem 2.4(d2),
\[
P - Q + 2PQ - 2QP + \frac{5}{2}PQP - \frac{5}{2}QQP - 2PQP
\] (2.72)

is group involutory since \(2 - 2 + 2 \cdot (5/2) + 2 \cdot (-2) = 1\) and \(2 - 2 + 2 \cdot (-5/2) - 2 \cdot (-2) = -1\).
Similarly, we have the following result.

Theorem 2.6. Let \( P, Q \in \mathbb{C}^{n \times n} \) be two different nonzero idempotent matrices, and let \( A \) be a combination of the form

\[
A = aP + bQ + cPQ + dQP + ePQP + fQPQ + hQPQP,
\]

where \( a, b, c, d, e, f, h \in \mathbb{C} \) with \( a, b \neq 0 \). Suppose that

\[
QPQP \neq 0, \quad PQP = 0,
\]

and any of the following sets of additional conditions hold:

\((e_1)\) \( a = b = \pm 1, \quad c = d = \mp 1, \quad e = f = \pm 1, \quad h = \mp 1; \)

\((e_2)\) \( a = \pm 1, \quad b = \mp 1, \quad 2e + c + d \pm cd = \pm 1, \quad 2f + c + d \mp cd = \mp 1. \)

Then \( A \) is the group involutory matrix.

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