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Research Article

Three Positive Periodic Solutions to Nonlinear Neutral Functional Differential Equations with Parameters on Variable Time Scales

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Using two successive reductions: B-equivalence of the system on a variable time scale to a system on a time scale and a reduction to an impulsive differential equation and by Leggett-Williams fixed point theorem, we investigate the existence of three positive periodic solutions to the nonlinear neutral functional differential equation on variable time scales with a transition condition between two consecutive parts of the scale $(d/dt)(x(t) + c(t)x(t-\alpha)) = a(t)g(x(t))x(t) - \sum_{j=1}^{n} \lambda_j f_j(t, x(t - v_j(t)))$, $(t,x) \in \mathbb{T}_0(x)$, $\Delta t|_{(t,x)\in\mathcal{S}_{2i}} = \Pi_i^1(t,x) - t$, $\Delta x|_{(t,x)\in\mathcal{S}_{2i}} = \Pi_i^2(t,x) - x$, where $\Pi_i^1(t,x) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t,x))$ and $\Pi_i^2(t,x) = B_i x + J_i(x) + x$, $i = 1,2,\ldots$. λ_j $(j = 1,2,\ldots,n)$ are parameters, $\mathbb{T}_0(x)$ is a variable time scale with (ω,p)-property, $c(t)$, $a(t)$, $v_j(t)$, and $f_j(t,x)$ $(j = 1,2,\ldots,n)$ are ω-periodic functions of t, $B_{i+p} = B_i$, $J_{i+p}(x) = J_i(x)$ uniformly with respect to $i \in \mathbb{Z}$.

1. Introduction

In the last several decades, the theory of dynamic equations on time scales (DETS) has been developed very intensively. For the full description of the equations we refer to the nicely written books [1, 2] and papers [3, 4]. The equations have a very special transition condition for adjoint elements of time scales. To enlarge the field of applications of the DETS, Akhmet and Turan proposed to generalize the transition operator [5], correspondingly to investigate differential equations on variable time scales with transition condition (DETC). In [6], Akhmet and Turan proposed some basic theory of dynamic equations on variable time scales; the method of investigation is by means of two successive reductions: B-equivalence of the system [7–9] on a variable time scale to a system on a time scale and a reduction to an impulsive differential equation [5, 7]. Consequently, these results are very effective to develop methods of investigation of mechanical models with impacts.

Also, neutral differential equations arise in many areas of applied mathematics, and for this reason these equations have received much attention in the last few decades; they are not only an extension of functional differential equations but also provide good models in many fields including biology, mechanics, and economics. In particular, qualitative analysis such as periodicity and stability of solutions of neutral functional differential equations has been studied extensively by many authors. We refer to [10–19] for some recent work on the subject of periodicity and stability of neutral equations. In [20], the authors discussed a class of neutral functional differential equations with impulses and parameters on nonvariable time scales

$$\begin{gathered}(x(t)+c(t)x(t-r_1))^{\Delta}=a(t)g(x(t))x(t)-\sum_{i=1}^{n}\lambda_i f_i(t,x(t-\tau_i(t))),\\ t\neq t_j,\ t\in\mathbb{T},\ j=1,2,\ldots,q,\\ x\left(t_j^-\right)-x\left(t_j^+\right)=I_j(x(t_j)),\qquad t=t_j,\ j=1,2,\ldots,q,\end{gathered}\tag{1.1}$$

where λ_i, $i=1,2,\ldots,n$ are parameters, $\mathbb{T}$ is an ω-periodic nonvariable time scale, $a\in C(\mathbb{T},\mathbb{R}^+)$, $c\in C(\mathbb{T},[0,1))$ and both of them are ω-periodic functions, $\tau_i\in C(\mathbb{T},\mathbb{R})$, $i=1,2,\ldots,n$ are ω-periodic functions, $f_i\in C(\mathbb{T}\times\mathbb{R}^+,\mathbb{R}^+)$, $i=1,2,\ldots,n$ are nondecreasing with respect to their second arguments and ω-periodic with respect to their first arguments, respectively; $g\in C(\mathbb{R},\mathbb{R}^+)$ and there exist two positive constants l,L such that $0<l\leq g(x)\leq L<\infty$ for all $x>0$, $I_j\in C(\mathbb{R},\mathbb{R}^+)$ $(j=1,2,\ldots,q)$ and is bounded, r_1 is a constant.

To the best of authors' knowledge, there has been no paper published on the existence of solutions to neutral functional differential equations on variable time scales. Our main purpose of this paper is by using theory of dynamic equations on variable time scales to investigate the existence of three positive periodic solutions to the nonlinear neutral functional differential equations on variable time scales with a transition condition between two consecutive parts of the scale

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}(x(t)+c(t)x(t-\alpha))=a(t)g(x(t))x(t)-\sum_{j=1}^{n}\lambda_j f_j\big(t,x\big(t-v_j(t)\big)\big),\qquad (t,x)\in\mathbb{T}_0(x),\\ \Delta t|_{(t,x)\in\mathcal{S}_{2i}}=\Pi_i^1(t,x)-t,\\ \Delta x|_{(t,x)\in\mathcal{S}_{2i}}=\Pi_i^2(t,x)-x,\end{gathered}\tag{1.2}$$

where $\Pi_i^1(t,x)=t_{2i+1}+\tau_{2i+1}(\Pi_i^2(t,x))$ and $\Pi_i^2(t,x)=B_ix+J_i(x)+x$, $i=1,2,\ldots$ λ_j $(j=1,2,\ldots,n)$ are parameters, $\mathbb{T}_0(x)$ is a variable time scale with (ω,p)-property, $c(t)$, $a(t)$, $v_j(t)$, and $f_j(t,x)$ $(j=1,2,\ldots,n)$ are ω-periodic functions of t, $B_{i+p}=B_i$, $J_{i+p}(x)=J_i(x)$ uniformly with respect to $i\in\mathbb{Z}$.

For convenience, we introduce the notation

$$\overline{a}=\max_{t\in[0,\omega]}a(t),\qquad \underline{a}=\min_{t\in[0,\omega]}a(t),\qquad \underline{c}=\min_{t\in[0,\omega]}c(t),\qquad \overline{c}=\max_{t\in[0,\omega]}c(t),\qquad r_0=\exp\left\{\int_{\omega}^{0}a(s)\mathrm{d}s\right\}.\tag{1.3}$$

Throughout this paper, we assume the following.

(H_1) $\mathbb{T}_0(x)$ is a variable time scale with (ω, p)-property, $c \in C(\mathbb{R}, [0,1))$, $a \in C(\mathbb{R}, \mathbb{R}^+)$, $v_j \in C(\mathbb{R}, \mathbb{R})$, and $f_j(t,x) \in C(\mathbb{T}_0(x), \mathbb{R}^+)$, $(j = 1, 2, \ldots, n)$ are ω-periodic functions of t, $J_i \in C(\mathbb{R}, \mathbb{R})$, $B_{i+p} = B_i$, $J_{i+p}(x) = J_i(x)$ uniformly with respect to $i \in \mathbb{Z}$, $J_i(0) = 0$, $B_i \geq 0$ for each $i \in \mathbb{Z}$.

(H_2) $f_j(t,x) \in C(\mathbb{T}_0(x), \mathbb{R}^+)$ is nondecreasing with respect to x and $f_j(t,0) = 0$ for each $j \in \{1, 2, \ldots, n\}$.

(H_3) $g \in C(\mathbb{R}, \mathbb{R}^+)$ and there exist two positive constants l, L such that $0 < l \leq g(x) \leq L < \infty$ for all $x > 0$.

(H_4) There exists a number $\delta > 0$ such that $(\underline{c} + \overline{c})/2 \leq \delta < 1$, $0 \leq ((2\delta - \underline{c})(1 - \underline{c}^2))/(1 - (2\delta - \underline{c})^2) < r_0^L\ (1 - r_0^l)/(1 - r_0^L)$.

2. Preliminaries

Let E be a real Banach space and P be a cone in E. A map ρ is said to be a nonnegative continuous concave functional on P if $\rho : P \to [0, \infty)$ is continuous and

$$\rho(tx + (1-t)y) \geq t\rho(x) + (1-t)\rho(y) \qquad \forall x, y \in P \text{ and } t \in [0,1]. \tag{2.1}$$

For numbers β_1, β_4 such that $0 < \beta_1 < \beta_4$ and ρ is a nonnegative continuous concave function on P, we define the following sets: $P_{\beta_1} = \{x \in P : \|x\| < \beta_1\}$, $\overline{P_{\beta_1}} = \{x \in P : \|x\| \leq \beta_1\}$, $P(\rho, \beta_1, \beta_4) = \{x \in P : \beta_1 \leq \rho(x), \|x\| \leq \beta_4\}$.

Now, we state the following Leggett-Williams fixed-point theorem, which is critical to the proof of our main results.

Lemma 2.1 (see [21]). *Let $T : \overline{P_{\beta 4}} \to \overline{P_{\beta 4}}$ be completely continuous and ρ nonnegative continuous concave functional on P such that $\rho(u) \leq \|u\|$ for all $u \in \overline{P_{\beta 4}}$. Suppose that there exist positive constants $\beta_1, \beta_2, \beta_3, \beta_4$ with $0 < \beta_1 < \beta_2 < \beta_3 \leq \beta_4$ such that*

(1) *$\{u \in P(\rho, \beta_2, \beta_3) : \rho(x) > \beta_2\} \neq \phi$ and $\rho(Tu) > \beta_2$ for $u \in P(\rho, \beta_2, \beta_3)$;*

(2) *$\|Tu\| < \beta_1$ for $u \in \overline{P_{\beta 1}}$;*

(3) *$\rho(Tu) > \beta_2$ for $u \in P(\rho, \beta_2, \beta_4)$ with $\|Tu\| > \beta_3$.*

Then T has at least three fixed points u_1, u_2, u_3 satisfying

$$u_1 \in P_{\beta 1}, \quad u_2 \in \{u \in P(\rho, \beta_2, \beta_4) : \rho(u) > \beta_2\}, \quad u_3 \in \overline{P_{\beta 4}} \setminus \left(P(\rho, \beta_2, \beta_4) \cup \overline{P_{\beta 1}}\right). \tag{2.2}$$

Let $\mathbb{T}$ be a periodic time scale, and let $E = \{x \in C(\mathbb{T}, \mathbb{R}) : x(t) = x(t + \omega)\}$ be a Banach space with the norm $\|x\| = \sup_{t \in [0,\omega] \cap \mathbb{T}} \{|x(t)| : x \in E\}$, and let $\Phi : E \to E$ be defined by

$$(\Phi x)(t) = x(t) + c(t)x(t - \tau). \tag{2.3}$$

Lemma 2.2 (see [20]). *If $0 \le c(t) < 1$ and E is a Banach space, then Φ has a bounded inverse Φ^{-1} on E, and for all $x \in E$,*

$$\left(\Phi^{-1}x\right)(t) = \sum_{j\ge 0} \prod_{0\le i\le j-1} (-1)^k c(t-i\tau)x(t-j\tau) \tag{2.4}$$

and $\|\Phi^{-1}x\| \le \|x\|/(1-\overline{c})$.

Definition 2.3 (see [6]). A nonempty closed set $\mathbb{T}_0(x)$ in $\mathbb{R}\times\mathbb{R}^n$ is said to be a variable time scale if for any $x_0 \in \mathbb{R}^n$ the projection of $\mathbb{T}_0(x_0)$ on time axis, that is, the set $\{t \in \mathbb{R} : (t, x_0) \in \mathbb{T}_0(x_0)\}$ is a time scale in Hilger sense.

Fix a sequence $\{t_i\} \subset \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $|t_i| \to \infty$ as $|i| \to \infty$. Denote $\delta_i = t_{2i-1} - t_{2i}$, $\kappa_i = t_{2i} - t_{2i-1}$ and take a sequence of functions $\{\tau_i(x)\} \subset C(\mathbb{R}^n, \mathbb{R})$. Assume that

(C_1) for some positive numbers θ', $\theta \in \mathbb{R}$, $\theta' \le t_{i+1} - t_i \le \theta$;

(C_2) there exists l_0, $0 < 2l_0 < \theta'$ such that $\|\tau_i(x)\| \le l_0$ for all $x \in \mathbb{R}^n$, $i \in \mathbb{Z}$.

Denote

$$l_i := \inf_{x\in\mathbb{R}^n}\{t_i + \tau_i(x)\}, \qquad r_i := \sup_{x\in\mathbb{R}^n}\{t_i + \tau_i(x)\}. \tag{2.5}$$

We set

$$\begin{aligned}
\mathcal{E}_i &= \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t_{2i} + \tau_{2i}(x) < t < t_{2i+1} + \tau_{2i+1}(x)\},\\
\mathcal{S}_i &= \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t = t_i + \tau_i(x)\},\\
\mathfrak{D}_i &= \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : t_{2i-1} + \tau_{2i-1}(x) \le t \le t_{2i} + \tau_{2i}(x)\}.
\end{aligned} \tag{2.6}$$

Following [6], we denote $\mathbb{T}_0(x) := \bigcup_{i=-\infty}^{\infty} \mathfrak{D}_i$ and $\mathbb{T}_c = \bigcup_{i=-\infty}^{\infty}[t_{2i-1}, t_{2i}]$.

A transition operator $\Pi_i : \mathcal{S}_{2i} \to \mathcal{S}_{2i+1}$, for all $i \in \mathbb{Z}$, such that $\Pi_i(t,y) = (\Pi_i^1(t,y), \Pi_i^2(t,y))$ where $\Pi_i^1 : \mathcal{S}_{2i} \to \mathbb{R}$ and $\Pi_i^2 : \mathcal{S}_{2i} \to \mathbb{R}^n$, and

$$\Pi_i^1(t,y) = t_{2i+1} + \tau_{2i+1}\left(\Pi_i^2(t,y)\right), \quad \Pi_i^2(t,y) = I_i(y) + y, \tag{2.7}$$

where $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is a function. One can easily see that $\Pi_i^1(t,y)$ is the time coordinate of $(t^+, y^+) := \Pi_i(t,y)$, the image of $(t,y) \in \mathcal{S}_{2i}$ under the operator Π_i, and $\Pi_i^2(t,y)$ is the space coordinate of the image.

Let $t = \alpha_i$ and $t = \beta_i$ be the moments that the graph of $y = \varphi(t)$ intersects the surface $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively, where the surfaces are defined previously. Then, we set the nonvariable time scale

$$\mathbb{T}_c^{\varphi} := \bigcup_{i=-\infty}^{\infty} [\alpha_i, \beta_i], \tag{2.8}$$

which is the domain of φ, and define the Δ-derivative as in the introduction. That is, for $t = \beta_i$, we have

$$\begin{aligned} \varphi^{\Delta}(\beta_i) &= \frac{\varphi(\alpha_{i+1}) - \varphi(\beta_i)}{\alpha_{i+1} - \beta_i}, \\ \varphi^{\Delta}(t) &= \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{s - t}, \end{aligned} \tag{2.9}$$

for any other $t \in \mathbb{T}_c^{\varphi}$, whenever the limit exists.

Consider the system on variable time scales:

$$\begin{aligned} y' &= A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ \Delta t|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^1(t, y) - t, \\ \Delta y|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^2(t, y) - y, \end{aligned} \tag{2.10}$$

where $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is an $n \times n$ continuous real-valued matrix function, B_i is an $n \times n$ matrix, functions $f(t, y) : \mathbb{T}_0(y) \to \mathbb{R}^n$ and $J_i(y) : \mathbb{R}^n \to \mathbb{R}^n$ are continuous, $\Pi_i^1(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t, y))$, and $\Pi_i^2(t, y) = B_i y + J_i(y) + y$.

For any $\alpha, \beta \in \mathbb{R}$ we define the oriented interval $[\widehat{\alpha, \beta}]$ as

$$\left[\widehat{\alpha, \beta}\right] = \begin{cases} [\alpha, \beta], & \text{if } \alpha \le \beta, \\ [\beta, \alpha], & \text{otherwise.} \end{cases} \tag{2.11}$$

Consider the nonvariable time scale

$$\mathbb{T}_c^0 = \bigcup_{i=-\infty}^{\infty} [l_{2i-1}, r_{2i}], \tag{2.12}$$

where $l_i, r_i,\ i \in \mathbb{Z}$ are defined by (2.5) for the variable time scale $\mathbb{T}_0(y)$, and take a continuation $\widetilde{f} : \mathbb{T}_c^0 \times \mathbb{R}^n \to \mathbb{R}^n$ of $f : \mathbb{T}_0(y) \to \mathbb{R}^n$ which is Lipschitzian with the same Lipschitz constant l; furthermore, if f is a monotone function, a continuation $\widetilde{f}$ can also have the same monotony with f. Set $\mathbb{T}_c := \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$.

(C_3) $\|\tau_i(x) - \tau_i(y)\| + \|J_i(x) - J_i(y)\| + \|f(t, x) - f(t, y)\| \le l\|x - y\|$ for arbitrary $x, y \in \mathbb{R}^n$, where l is a Lipschitz constant.

By (C_3), in [6], one can see the following important lemma.

Lemma 2.4 (see [6]). *Assume (C_3) is satisfied. Then there are mappings $W_i(z) : \mathbb{R}^n \to \mathbb{R}^n,\ i \in \mathbb{Z}$, such that, corresponding to each solution $y(t)$ of* (2.10)*, there is a solution $z(t)$ of the system*

$$\begin{aligned} z' &= A(t)z + \widetilde{f}(t, z), \quad t \ne t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}), \end{aligned} \tag{2.13}$$

such that $y(t) = z(t)$ for all $t \in \mathbb{T}_c$ except possibly on $[\widehat{t_{2i-1}, \alpha_i}]$ and $[\widehat{\beta_i, t_{2i}}]$, where α_i and β_i are the moments that $y(t)$ meets the surfaces $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively.

Furthermore, the functions W_i satisfy the inequality

$$\|W_i(z) - W_i(y)\| \le k(l)l\|z - y\|, \tag{2.14}$$

uniformly with respect to $i \in \mathbb{Z}$ for all $z, y \in \mathbb{R}^n$ such that $\|z\| \le h$ and $\|y\| \le h$; here $k(l_0) = k(l_0, h)$ is a bounded function. Under the sense of Lemma 2.4, we say that systems (2.10) *and* (2.13) *are B-equivalent.*

Proof. Fix $i \in \mathbb{Z}$. Let $z(t)$ be the solution of (2.10) such that $z(t_{2i}) = z$, and assume that α_i and β_i are solutions of $\alpha = t_{2i-1} + \tau_{2i-1}(z(\alpha))$ and $\beta = t_{2i} + \tau_{2i}(z(\beta))$, respectively. Let $z_1(t)$ be the solution of the system

$$z' = A(t)z + \tilde{f}(t, z) \tag{2.15}$$

with the initial condition $z_1(\alpha_{i+1}) = \Pi_i^2(\beta_i, z(\beta_i))$.

We first note that $z_1(\alpha_{i+1}) = (I + B_i)z(\beta_i) + J_i(z(\beta_i))$. Moreover, for $t \in [\widehat{t_{2i}, \beta_i}]$,

$$z(t) = z(t_{2i}) + \int_{t_{2i}}^{t} \left[A(s)z(s) + \tilde{f}(s, z(s))\right] \mathrm{d}s, \tag{2.16}$$

and for $t \in [\widehat{\alpha_{i+1}, t_{2i+1}}]$,

$$\begin{aligned} z_1(t) &= z_1(\alpha_{i+1}) + \int_{\alpha_{i+1}}^{t} \left[A(s)z_1(s) + f(s, z_1(s))\right] \mathrm{d}s \\ &= (I + B_i)z(\beta_i) + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t} \left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right] \mathrm{d}s \\ &= (I + B_i)\left[z(t_{2i}) + \int_{t_{2i}}^{\beta_i} \left[A(s)z(s) + \tilde{f}(s, z(s))\right] \mathrm{d}s\right] + J_i(z(\beta_i)) \\ &\quad + \int_{\alpha_{i+1}}^{t} \left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right] \mathrm{d}s. \end{aligned} \tag{2.17}$$

Thus, we have

$$\begin{aligned} W_i(z) = (I + B_i) &\int_{t_{2i}}^{\beta_i} \left[A(s)z(s) + \tilde{f}(s, z(s))\right] \mathrm{d}s + J_i(z(\beta_i)) \\ &+ \int_{\alpha_{i+1}}^{t_{2i+1}} \left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right] \mathrm{d}s. \end{aligned} \tag{2.18}$$

Substituting (2.18) in (2.13), we see that $W_i(z)$ satisfies the first conclusion of the lemma.

Next, we prove (2.14). Let $\|z(t_{2i})\| \le h$. By employ integrals (2.16) and (2.17), we find that the solutions $z(t)$ and $z_1(t)$ determined above satisfy the inequalities $\|z(t)\| \le H$ and $\|z_1(t)\| \le H$ on $[\widehat{\beta_i, t_{2i}}]$ and $[\widehat{\alpha_{i+1}, t_{2i+1}}]$, where

$$H = \left[M(1+l) + (1+N+l)(h+Ml)e^{Nl+l^2}\right]e^{Nl+l^2}. \tag{2.19}$$

Let $y(t)$ be the solution of (2.10) such that $y(t_{2i}) = y$, and assume that $\overline{\alpha}_i$ and $\overline{\beta}_i$ are solutions of $\overline{\alpha} = t_{2i-1} + \tau_{2i-1}(y(\overline{\alpha}))$ and $\overline{\beta} = t_{2i} + \tau_{2i}(y(\overline{\beta}))$, respectively. Let $y_1(t)$ be the solution of (2.15) with initial condition $y_1(\overline{\alpha}_{i+1}) = \Pi_i^2(\overline{\beta}_i, y(\overline{\beta}_i))$. Without loss of any generality, we assume that $\overline{\beta}_i \ge \beta_i$ and $\overline{\alpha}_{i+1} \le \alpha_{i+1}$. Application of the Gronwall-Bellman lemma shows that, for $t \in [\widehat{\beta_i, t_{2i}}]$,

$$\|z(t) - y(t)\| \le e^{(N+l)l}\|z - y\|. \tag{2.20}$$

The equation

$$y\left(\overline{\beta}_i\right) = y(\beta_i) + \int_{\beta_i}^{\overline{\beta}_i}\left[A(s)y(s) + \tilde{f}(s, y(s))\right]\mathrm{d}s \tag{2.21}$$

gives us

$$\left\|y\left(\overline{\beta}_i\right) - y(\beta_i)\right\| \le (NH + lH + M)\left(\overline{\beta}_i - \beta_i\right). \tag{2.22}$$

Thus, we obtain

$$\left\|z(\beta_i) - y\left(\overline{\beta}_i\right)\right\| \le e^{(N+l)l}\|z - y\| + (NH + lH + M)\left(\overline{\beta}_i - \beta_i\right). \tag{2.23}$$

Now condition (C_3) together with (2.23) leads to

$$\overline{\beta}_i - \beta_i \le \frac{le^{(N+l)l}}{1 - l(NH + lH + M)}\|z - y\|. \tag{2.24}$$

Hence (2.23) becomes

$$\left\|z(\beta_i) - y\left(\overline{\beta}_i\right)\right\| \le \frac{e^{(N+l)l}}{1 - l(NH + lH + M)}\|z - y\|. \tag{2.25}$$

On the other hand,

$$y_1(\alpha_{i+1}) = y_1(\overline{\alpha}_{i+1}) + \int_{\overline{\alpha}_{i+1}}^{\alpha_{i+1}}\left[A(s)y_1(s) + \tilde{f}(s, y_1(s))\right]\mathrm{d}s \tag{2.26}$$

gives us

$$\left\|y_1(\alpha_{i+1}) - y_1(\overline{\alpha}_{i+1})\right\| \leq (NH + lH + M)(\alpha_{i+1} - \overline{\alpha}_{i+1}). \tag{2.27}$$

Using the transition operators and (2.25) we get

$$\left\|z_1(\alpha_{i+1}) - y_1(\overline{\alpha}_{i+1})\right\| \leq \frac{(1 + N + l)e^{(N+l)l}}{1 - l(NH + lH + M)}\left\|z - y\right\|. \tag{2.28}$$

Condition (C_3) and (2.28) imply that

$$\alpha_{i+1} - \overline{\alpha}_{i+1} \leq \frac{l(1 + N + l)e^{(N+l)l}}{1 - l(NH + lH + M)}\left\|z - y\right\|. \tag{2.29}$$

From (2.27)–(2.29) we obtain

$$\left\|z_1(\alpha_{i+1}) - y_1(\overline{\alpha}_{i+1})\right\| \leq H_1 e^{(N+l)l}\left\|z - y\right\|, \tag{2.30}$$

where $H_1 = (1 + N + l)[1 + l(NH + lH + M)]/[1 - l(NH + lH + M)]$. Solutions $z_1(t)$ and $y_1(t)$ on $[\widehat{\alpha_{i+1}, t_{2i+1}}]$ satisfy the inequality

$$\left\|z_1(t) - y_1(t)\right\| \leq H_1 e^{2(N+l)l}\left\|z - y\right\|. \tag{2.31}$$

Now subtracting the expression

$$\begin{aligned} W_i(y) = (I + B_i)\int_{t_{2i}}^{\overline{\beta}_i}\left[A(s)y(s) + \widetilde{f}(s, y(s))\right]\mathrm{d}s + J_i\left(y\left(\overline{\beta}_i\right)\right) \\ + \int_{\overline{\alpha}_{i+1}}^{t_{2i+1}}\left[A(s)y_1(s) + \widetilde{f}(s, y_1(s))\right]\mathrm{d}s \end{aligned} \tag{2.32}$$

from (2.18) and using (2.20), (2.24), (2.29), and (2.31), we conclude that (2.14) holds. The proof is complete. □

A special transformation called ψ-substitution [5], which is change of the independent variable and defined for $t \in \bigcup_{i=-\infty}^{\infty}(t_{2i-1}, t_{2i}]$ as

$$\psi(t) = \begin{cases} t - \displaystyle\sum_{0<t_{2k}<t}\delta_k, & t \geq 0, \\ t + \displaystyle\sum_{t\leq t_{2k}<0}\delta_k, & t < 0, \end{cases} \tag{2.33}$$

where $\delta_k = t_{2k+1} - t_{2k}$. Setting $s_i = \psi(t_{2i})$, we see that this transformation has an inverse given by

$$\psi^{-1}(s) = \begin{cases} s + \sum\limits_{0<s_k<s} \delta_k, & s \geq 0, \\ s - \sum\limits_{s\leq s_k<0} \delta_k, & s < 0. \end{cases} \tag{2.34}$$

Lemma 2.5 (see [5, 6]). *$\psi'(t) = 1$ if $t \in \bigcup_{i=-\infty}^{\infty}(t_{2i-1}, t_{2i}]$.*

Proof. Assume that $t \geq 0$. Then,

$$\begin{aligned} \psi'(t) &= \lim_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\left(t + h - \sum_{0<t_{2k}<t+h} \delta_k\right) - \left(t - \sum_{0<t_{2k}<t} \delta_k\right)\right] = 1. \end{aligned} \tag{2.35}$$

The assertion for $t < 0$ can be proved in the same way. The proof is complete. □

Definition 2.6 (see [5]). The time scale $\mathbb{T}_0$ is said to have an ω-property if there exists a number $\omega \in \mathbb{R}^+$ such that $t + \omega \in \mathbb{T}_0$ whenever $t \in \mathbb{T}_0$.

Definition 2.7 (see [5]). A sequence $\{a_i\} \subset \mathbb{R}$ is said to satisfy an (ω, p)-property if there exist numbers $\omega \in \mathbb{R}^+$ and $p \in \mathbb{N}$ such that $a_{i+p} = a_i + \omega$ for all $i \in \mathbb{Z}$.

Definition 2.8 (see [6]). The variable time scale $\mathbb{T}_0(y)$ is said to satisfy an (ω, p)-property if $(t \pm \omega, y)$ is in $\mathbb{T}_0(y)$ whenever (t, y) is. In this case, there exists $p \in \mathbb{N}$ such that the sequences $\{t_{2i-1}\}$ and $\{t_{2i}\}$ satisfy the (ω, p)-property and $\tau_{i+p}(y) = \tau_i(y)$ for all $i \in \mathbb{Z}$.

Suppose now that (2.10) is ω-periodic; that is, $\mathbb{T}_0(y)$ satisfies the (ω, p)-property, $A(t)$ and $f(t, y)$ are ω-periodic functions of t, and $B_{i+p} = B_i$, $J_{i+p}(y) = J_i(y)$ uniformly with respect to $i \in \mathbb{Z}$.

Lemma 2.9 (see [6]). *If* (2.10) *is ω-periodic, then the sequence $W_i(z)$ is p-periodic uniformly with respect to $z \in \mathbb{R}^n$.*

Proof. Since the variable time scale $\mathbb{T}_0(y)$ satisfies an (ω, p)-property, by (2.18), one can easily see that $W_i(z)$ is p-periodic uniformly with respect to $z \in \mathbb{R}^n$. The proof is complete. □

Lemma 2.10 (see [5]). *If $\mathbb{T}_0$ has an ω-property, then the sequence $\{s_i\}$, $s_i = \psi(t_{2i})$, is $(\tilde{\omega}, p_0)$-periodic with*

$$\tilde{\omega} = \omega - \sum_{0<t_{2k}<\omega} \delta_k = \psi(\omega). \tag{2.36}$$

Proof. In order to prove this lemma, we only need to verify that $s_{i+p_0} = s_i + \tilde{\omega}$ for all i. Assume that $i \geq 0,\ i = np_0 + j$ for some $n \in \mathbb{Z},\ 0 \leq j < p_0$ and $0 < t_0 < \cdots < t_{2(p_0-1)} < \omega$. Then

$$\begin{aligned} s_{i+p_0} &= \psi\left(t_{2(i+p_0)}\right) = t_{2(i+p_0)} - \sum_{0<t_{2k}<t_{2(i+p_0)}} \delta_k \\ &= t_{2i} + \omega - \sum_{0<t_{2k}<t_{2i}} \delta_k - \sum_{t_{2i}\leq t_{2k}<t_{2(i+p_0)}} \delta_k = \psi(t_{2i}) + \omega - \sum_{k=i}^{i+p_0-1} \delta_k \\ &= s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_{k+np_0} = s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_k = s_i + \omega - \sum_{k=0}^{p_0-1} \delta_k \\ &= s_i + \omega - \sum_{0<t_{2k}<\omega} \delta_k = s_i + \tilde{\omega}, \end{aligned} \tag{2.37}$$

where we have used the fact that

$$\begin{aligned} \sum_{k=j}^{j+p_0-1} \delta_k &= \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=p_0}^{j+p_0-1} \delta_k = \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_{k+p_0} \\ &= \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_k = \sum_{k=0}^{p_0-1} \delta_k. \end{aligned} \tag{2.38}$$

All other cases can be verified similarly. The proof is complete. □

Lemma 2.11 (see [5, 6]). *If $\mathbb{T}_0(y)$ satisfies an (ω, p)-property, then $\psi(t+\omega) = \psi(t) + \psi(\omega)$.*

Proof. Assume that $t \geq 0$. By Lemma 2.10, we have

$$\begin{aligned} \psi(t+\omega) &= t + \omega - \sum_{0<t_{2k}<t+\omega} \delta_k = t + \omega - \sum_{0<t_{2k}<\omega} \delta_k - \sum_{\omega\leq t_{2k}<t+\omega} \delta_k \\ &= t - \sum_{\omega\leq t_{2k}<t+\omega} \delta_k + \psi(\omega) = \psi(t) + \psi(\omega). \end{aligned} \tag{2.39}$$

The assertion for $t < 0$ can be proved in the same way. The proof is complete. □

Lemma 2.12 (see [5, 6]). *A function $\phi(t)$ is an ω-periodic function on $\mathbb{T}_c$, if and only if $\phi(\psi^{-1}(s))$ is an $\tilde{\omega}$-periodic function on $\mathbb{R}$, where $\tilde{\omega} = \psi(\omega)$.*

Proof. By Lemma 2.11, $s + \tilde{\omega} = \psi(t+\omega)$. Then the equality

$$\phi\left(\psi^{-1}(s+\tilde{\omega})\right) = \phi(t+\omega) = \phi(t) = \phi\left(\psi^{-1}(s)\right) \tag{2.40}$$

completes the proof. □

For any fixed $x_0 \in \mathbb{R}$, we set

$$PC(\mathbb{T}(x_0)) = \left\{x \in C((t_{2i-1}+\tau_{2i-1}(x_0),\ t_{2i}+\tau_{2i}(x_0)),\mathbb{R}) : \Pi_i^2(t,x) = B_i x + J_i(x) + x,\ i = 1,2,\dots\right\} \tag{2.41}$$

and consider the Banach space

$$E = \{x : x \in PC(\mathbb{T}(x_0)) : x(t) = x(t+\omega)\}. \tag{2.42}$$

Let $\Phi : E \to E$ be defined by

$$(\Phi x)(t) = x(t) + c(t)x(t-\alpha) := y(t). \tag{2.43}$$

Using the inverse transformation of Φ, we can obtain

$$\Pi_i^1\left(t,\Phi^{-1}y\right) = t_{2i+1} + \tau_{2i+1}\left(\Pi_i^2\left(t,\Phi^{-1}y\right)\right), \quad \Pi_i^2\left(t,\Phi^{-1}y\right) = B_i\Phi^{-1}y + J_i\left(\Phi^{-1}y\right) + \Phi^{-1}y. \tag{2.44}$$

From the second equation, it is easy to get

$$\Phi\left(\Pi_i^2\left(t,\Phi^{-1}y\right)\right) = \left(\Phi B_i\Phi^{-1}\right)y + \left(\Phi J_i\Phi^{-1}\right)y + y := \widetilde{\Pi}_i^2(t,y), \tag{2.45}$$

that is, $\Pi_i^2(t,\Phi^{-1}y) = \Phi^{-1}(\widetilde{\Pi}_i^2(t,y))$. Hence

$$\begin{aligned}\Pi_i^1\left(t,\Phi^{-1}y\right) &= t_{2i+1} + \tau_{2i+1}\left(\Phi^{-1}\widetilde{\Pi}_i^2(t,y)\right) = t_{2i+1} + \tau_{2i+1}\Phi^{-1}\left(\widetilde{\Pi}_i^2(t,y)\right)\\ &:= t_{2i+1} + \widetilde{\tau}_{2i+1}\left(\widetilde{\Pi}_i^2(t,y)\right) := \widetilde{\Pi}_i^1(t,y).\end{aligned} \tag{2.46}$$

Therefore, we can obtain the other variable time scale plane $T_0(y)$ by the inverse transformation of Φ and

$$\begin{aligned}\mathcal{E}_i' &= \left\{(t,y) \in \mathbb{R}\times\mathbb{R} : t_{2i} + \widetilde{\tau}_{2i}(y) < t < t_{2i+1} + \widetilde{\tau}_{2i+1}(y)\right\},\\ \mathcal{S}_i' &= \left\{(t,y) \in \mathbb{R}\times\mathbb{R} : t = t_i + \widetilde{\tau}_i(y)\right\},\\ \mathfrak{D}_i' &= \left\{(t,y) \in \mathbb{R}\times\mathbb{R} : t_{2i-1} + \widetilde{\tau}_{2i-1}(y) \le t \le t_{2i} + \widetilde{\tau}_{2i}(y)\right\}.\end{aligned} \tag{2.47}$$

Hence, (1.2) can be changed into the following form:

$$\begin{gathered} y' = a(t)g\left((\Phi^{-1}y)(t)\right)y(t) - a(t)H(y(t)) - \sum_{j=1}^{n}\lambda_j f_j\left(t,(\Phi^{-1}y)(t-v_j(t))\right), \qquad (t,y) \in \mathbb{T}_0(y),\\ \Delta t|_{(t,y)\in\mathcal{S}_{2i}'} = \widetilde{\Pi}_i^1(t,y) - t,\\ \Delta y|_{(t,y)\in\mathcal{S}_{2i}'} = \widetilde{\Pi}_i^2(t,y) - y,\end{gathered} \tag{2.48}$$

where $H(y(t)) = c(t)g((\Phi^{-1}y)(t))(\Phi^{-1}y)(t-\alpha)$.

Define a cone in E by

$$P_0 = \{y(t) \in E : y(t) \geq k\|y\|\}, \tag{2.49}$$

where $k \in (((2\delta - \underline{c})(1 - \underline{c}^2))/(1 - (2\delta - \underline{c})^2), r_0^L(1 - r_0^l)/(1 - r_0^L)]$.

Lemma 2.13 (see [20]). *Suppose that conditions (H_1)–(H_4) hold and $0 \leq c(t) < 1$ and $y \in P_0$, then*

$$\tilde{\alpha}\|y\| \leq \left(\Phi^{-1}y\right)(t) \leq \frac{1}{1-\bar{c}}\|y\|, \tag{2.50}$$

$$l\underline{c}\tilde{\alpha}\|y\| \leq H(y(t)) \leq \frac{L\bar{c}}{1-\bar{c}}\|y\|, \tag{2.51}$$

where $\tilde{\alpha} = (k/(1 - \underline{c}^2)) - (2\delta - \underline{c})/(1 - (2\delta - \underline{c})^2)$.

(H_5) $\|\tau_i(x) - \tau_i(y)\| + \|J_i(x) - J_i(y)\| + \sum_{j=1}^{n} \|f_j(t,x) - f_j(t,y)\| \leq l_0\|x - y\|$ for arbitrary $x, y \in \mathbb{R}^n$, where l_0 is a Lipschitz constant.

In view of (H_5) by Lemma 2.4, it is easy to get the following lemma.

Lemma 2.14. *Assume that (H_5) is satisfied. Then there are mappings $W_i(z) : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}$ such that, corresponding to each solution $y(t)$ of* (2.48)*, there is a solution $z(t)$ of the system*

$$\begin{aligned} z' &= a(t)g\left(\left(\Phi^{-1}z\right)(t)\right)z(t) - a(t)H(z(t)) - \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(t, \left(\Phi^{-1}z\right)(t - v_j(t))\right), \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= \left(\Phi B_i\Phi^{-1}\right)(z(t_{2i})) + \left(\Phi W_i\Phi^{-1}\right)(z(t_{2i})) + z(t_{2i}), \quad t = t_{2i} \end{aligned} \tag{2.52}$$

such that $y(t) = z(t)$ for all $t \in \mathbb{T}_c$ except possibly on $[\widehat{t_{2i-1}, \alpha_i}]$ and $[\widehat{\beta_i, t_{2i}}]$ where α_i and β_i are the moments that $y(t)$ meets the surfaces $\mathcal{S}'_{2i-1}$ and $\mathcal{S}'_{2i}$, respectively.

Furthermore, the functions W_i satisfy the inequality

$$\|W_i(z) - W_i(y)\| \leq k(l_0)l_0\|z - y\|, \tag{2.53}$$

uniformly with respect to $i \in \mathbb{Z}$ for all $z, y \in \mathbb{R}^n$ such that $\|z\| \leq h$ and $\|y\| \leq h$; here $k(l_0) = k(l_0, h)$ is a bounded function. Under the sense of lemma 2.14, we say that systems (2.48) *and* (2.52) *are B-equivalent.*

Proof. For fixed $i \in \mathbb{Z}$. Let $z(t)$ be the solution of (2.48) such that $z(t_{2i}) = z$, and assume that α_i and β_i are solutions of $\alpha = t_{2i-1} + \tilde{\tau}_{2i-1}(z(\alpha))$ and $\beta = t_{2i} + \tilde{\tau}_{2i}(z(\beta))$, respectively. Let $z_1(t)$ be the solution of the system

$$z' = a(t)g\left(\left(\Phi^{-1}z\right)(t)\right)z(t) - a(t)H(z(t)) - \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(t, \left(\Phi^{-1}z\right)(t - v_j(t))\right) \tag{2.54}$$

with the initial condition $z_1(\alpha_{i+1}) = \tilde{\Pi}_i^2(\beta_i, z(\beta_i))$.

We first note that $z_1(\alpha_{i+1}) = (\Phi B_i \Phi^{-1})(z(\beta_i)) + (\Phi J_i \Phi^{-1})(z(\beta_i)) + z(\beta_i)$. Moreover, for $t \in [\widehat{t_{2i}, \beta_i}]$,

$$z(t) = z(t_{2i}) + \int_{2i}^{t} \left[a(s) g\left(\left(\Phi^{-1} z\right)(s)\right) z(s) - a(s) H(z(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z\right)(s - v_j(s))\right) \right] \mathrm{d}s, \tag{2.55}$$

and for $t \in [\widehat{\alpha_{i+1}, t_{2i+1}}]$,

$$\begin{aligned}
z_1(t) &= z_1(\alpha_{i+1}) + \int_{\alpha_{i+1}}^{t} \left[a(s) g\left(\left(\Phi^{-1} z_1\right)(s)\right) z_1(s) - a(s) H(z_1(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z_1\right)(s - v_j(s))\right) \right] \mathrm{d}s \\
&= \left(\Phi B_i \Phi^{-1}\right)(z(\beta_i)) + \left(\Phi J_i \Phi^{-1}\right)(z(\beta_i)) + z(\beta_i) \\
&\quad + \int_{\alpha_{i+1}}^{t} \left[a(s) g\left(\left(\Phi^{-1} z_1\right)(s)\right) z_1(s) - a(s) H(z_1(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z_1\right)(s - v_j(s))\right) \right] \mathrm{d}s \\
&= \left(\Phi (B_i + I) \Phi^{-1}\right) \left[z(t_{2i}) + \int_{t_{2i}}^{\beta_i} \left[a(s) g\left(\left(\Phi^{-1} z\right)(s)\right) z(s) - a(s) H(z(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z\right)(s - v_j(s))\right) \right] \mathrm{d}s \right] \\
&\quad + \left(\Phi J_i \Phi^{-1}\right)(z(\beta_i)) + \int_{\alpha_{i+1}}^{t} \left[a(s) g\left(\left(\Phi^{-1} z_1\right)(s)\right) z_1(s) - a(s) H(z_1(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z_1\right)(s - v_j(s))\right) \right] \mathrm{d}s.
\end{aligned} \tag{2.56}$$

Thus, we set

$$W_i(z) = \left(\Phi (B_i + I) \Phi^{-1}\right) \int_{t_{2i}}^{\beta_i} \left[a(s) g\left(\left(\Phi^{-1} z\right)(s)\right) z(s) - a(s) H(z(s)) - \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(s, \left(\Phi^{-1} z\right)(s - v_j(s))\right) \right] \mathrm{d}s + \left(\Phi J_i \Phi^{-1}\right)(z(\beta_i))$$

$$+\int_{\alpha_{i+1}}^{t_{2i+1}}\Bigg[a(s)g\Big(\Big(\Phi^{-1}z_1\Big)(s)\Big)z_1(s)$$
$$-a(s)H(z_1(s))-\sum_{j=1}^{n}\lambda_j\tilde{f}_j\Big(s,\Big(\Phi^{-1}z_1\Big)(s-v_j(s))\Big)\Bigg]\mathrm{d}s. \tag{2.57}$$

Substituting (2.57) in (2.52), we see that $W_i(z)$ satisfies the first conclusion of the lemma.

The rest of the proof is similar to that of Lemma 2.4, and we can use Gronwall-Bellman lemma to show that $W_i(z)$ satisfies (2.53) and it will be omitted here. This completes the proof. □

Next, we will use ψ-substitution, reducing (2.52) to an impulsive differential equation. Letting $m(s)=z(\psi^{-1}(s))$, we obtain, for $t\neq t_{2i}$,

$$\begin{aligned}\Big(\Phi^{-1}z\Big)(t-v_j(t))&=\Big(\Phi^{-1}z\Big)\Big(\psi^{-1}(s)-v_j\Big(\psi^{-1}(s)\Big)\Big)\\&=\Big(\Phi^{-1}z\Big)\Big(\psi^{-1}\Big(\psi\Big(\psi^{-1}(s)-v_j\Big(\psi^{-1}(s)\Big)\Big)\Big)\Big)\\&=\Big(\Phi^{-1}m\Big)\Big(\psi\Big(\psi^{-1}(s)-v_j\Big(\psi^{-1}(s)\Big)\Big)\Big):=\nu(s),\end{aligned} \tag{2.58}$$

hence

$$m'=a\Big(\psi^{-1}(s)\Big)g\Big(\Big(\Phi^{-1}m\Big)(s)\Big)m(s)-a\Big(\psi^{-1}(s)\Big)H(m(s))-\sum_{j=1}^{n}\lambda_j\tilde{f}_j\Big(\psi^{-1}(s),\nu(s)\Big), \tag{2.59}$$

and for $t=t_{2i}$, we get

$$m(s_i^-)=z(t_{2i+1})=\Big(\Phi B_i\Phi^{-1}\Big)(m(s_i))+\Big(\Phi W_i\Phi^{-1}\Big)(m(s_i))+m(s_i). \tag{2.60}$$

Thus, the second equation in (2.23) leads to

$$\Delta m|_{s=s_i}=\Big(\Phi B_i\Phi^{-1}\Big)(m(s_i))+\Big(\Phi W_i\Phi^{-1}\Big)(m(s_i)), \tag{2.61}$$

where $\Delta m|_{s=s_i}=m(s_i^-)-m(s_i)$. Hence, $m(s)$ is a solution of the impulsive differential equation:

$$\begin{gathered}m'=a\Big(\psi^{-1}(s)\Big)g\Big(\Big(\Phi^{-1}m\Big)(s)\Big)m(s)-a\Big(\psi^{-1}(s)\Big)H(m(s))-\sum_{j=1}^{n}\lambda_j\tilde{f}_j\Big(\psi^{-1}(s),\nu(s)\Big),\qquad s\neq s_i,\\ \Delta m|_{s=s_i}=\Big(\Phi B_i\Phi^{-1}\Big)(m(s_i))+\Big(\Phi W_i\Phi^{-1}\Big)(m(s_i)),\quad s=s_i\ .\end{gathered} \tag{2.62}$$

In the following, we set

$$\overline{PC} = \left\{ m : m|_{(s_i, s_{i+1})} \in C((s_i, s_{i+1}), \mathbb{R}), m(s_i^+) = m(s_i),\ i = 1, 2, \ldots \right\} \tag{2.63}$$

and consider the Banach space

$$\overline{E} = \left\{ m : m \in \overline{PC}, m(s) = m(s + \tilde{\omega}) \right\} \tag{2.64}$$

with the norm $\|m\| = \sup_{s\in[0,\tilde{\omega}]}\{|m(s)| : m \in \overline{E}\}$, where $\tilde{\omega} = \varphi(\omega)$. Define a cone in $\overline{E}$ by

$$P = \left\{ m(s) \in \overline{E} : m(s) \geq k\|m\| \right\}, \tag{2.65}$$

where $k \in ((2\delta - \underline{c})(1 - \underline{c}^2)/(1 - (2\delta - \underline{c})^2), r_0^L(1 - r_0^l)/(1 - r_0^L)]$.

Let the operator $\Psi : P \to \overline{E}$ be defined by

$$\begin{aligned}(\Psi m)(s) = &\int_s^{s+\tilde{\omega}} G(s,\theta)\left\{ a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\varphi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta \\ &+ \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right),\end{aligned} \tag{2.66}$$

where

$$G(s,\theta) = \frac{e^{\int_\theta^s a(\varphi^{-1}(r))g((\Phi^{-1}m)(r))\mathrm{d}r}}{1 - e^{\int_{\tilde{\omega}}^0 a(\varphi^{-1}(r))g((\Phi^{-1}m)(r))\mathrm{d}r}}, \qquad \theta \in [s, s+\tilde{\omega}]. \tag{2.67}$$

By the assumptions, we have

$$\frac{r_0^L}{1 - r_0^L} \leq G(s,\theta) \leq \frac{1}{1 - r_0^l}. \tag{2.68}$$

Lemma 2.15. *m is an $\tilde{\omega}$-periodic solution of* (2.62) *if and only if m is a fixed point of the operator Ψ.*

Proof. If $m(s)$ is an $\tilde{\omega}$-periodic solution of (2.62), for any $s \in \mathbb{R}$, there exists $i \in \mathbb{Z}$ such that s_i is the first impulsive point after s. Hence, for $\theta \in [s, s_i]$, we have

$$\begin{aligned}m(\theta) = &e^{\int_s^\theta a(\varphi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \\ &- \int_s^\theta e^{\int_r^\theta a(\varphi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau}\left\{ a\left(\varphi^{-1}(r)\right)H(m(r)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\varphi^{-1}(r), v(r)\right)\right\}\mathrm{d}r,\end{aligned} \tag{2.69}$$

then

$$m(s_i) = e^{\int_s^{s_i} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s) - \int_s^{s_i} e^{\int_r^{s_i} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r. \tag{2.70}$$

Again, for $\theta \in (s_i, s_{i+1}]$, then

$$\begin{aligned}
m(s) &= e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s_i^+) \\
&\quad - \int_{s_i}^{\theta} e^{\int_r^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \\
&= e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ \left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i)) + m(s_i) \right\} \\
&\quad - \int_{s_i}^{\theta} e^{\int_r^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \\
&= e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i)) \right) \\
&\quad - \int_{s_i}^{\theta} e^{\int_r^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \\
&\quad + e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s_i),
\end{aligned}$$

$$\begin{aligned}
&e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s_i) \\
&\quad = e^{\int_{s_i}^{\theta} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ e^{\int_s^{s_i} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s) \right. \\
&\qquad\qquad - \int_s^{s_i} e^{\int_r^{s_i} a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \\
&\qquad\qquad \left. \times \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \right\}
\end{aligned}$$

$$= e^{\int_s^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s)$$
$$- \int_s^{s_i} e^{\int_r^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r. \tag{2.71}$$

So we can obtain

$$\begin{aligned} m(s) &= e^{\int_s^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s) \\ &\quad - \int_s^\theta e^{\int_r^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \\ &\quad + e^{\int_{s_i}^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left[\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i)) \right]. \end{aligned} \tag{2.72}$$

Repeating the above process for $\theta \in [s, s+\tilde{\omega}]$, we obtain

$$\begin{aligned} m(s) &= e^{\int_s^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} m(s) \\ &\quad - \int_s^\theta e^{\int_r^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left\{ a\left(\psi^{-1}(r)\right) H(m(r)) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(r), \nu(r)\right) \right\} \mathrm{d}r \\ &\quad + \sum_{i: s_i \in [s,\theta)} e^{\int_{s_i}^\theta a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} \left[\left(\Phi^{-1} B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i)) \right]. \end{aligned} \tag{2.73}$$

Noticing that $m(s) = m(s+\tilde{\omega})$ and $e^{\int_{s+\tilde{\omega}}^s a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau} = e^{\int_{\tilde{\omega}}^0 a(\psi^{-1}(\tau))g((\Phi^{-1}m)(\tau))\mathrm{d}\tau}$, we find that m is a fixed point of Ψ.

Let m be a fixed point of Ψ. If $s \neq s_i, i \in \mathbb{Z}$, we have

$$\begin{aligned} m'(s) &= G(s, s+\tilde{\omega}) \left\{ a\left(\psi^{-1}(s+\tilde{\omega})\right) H(m(s+\tilde{\omega})) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(s+\tilde{\omega}), \nu(s+\tilde{\omega})\right) \right\} \\ &\quad - G(s,s) \left\{ a\left(\psi^{-1}(s)\right) H(m(s)) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(s), \nu(s)\right) \right\} \\ &\quad + a\left(\psi^{-1}(s)\right) g\left(\left(\Phi^{-1}m\right)(s)\right) m(s). \end{aligned} \tag{2.74}$$

By Lemmas 2.11 and 2.12, we have $\psi(t+\omega) = \psi(t) + \psi(\omega) = \psi(t) + \tilde{\omega}$, so it is easy to have $t + \omega = \psi^{-1}(\psi(t) + \tilde{\omega}) = \psi^{-1}(s+\tilde{\omega})$ and $\nu(s+\tilde{\omega}) = \nu(s)$. Therefore, we can obtain

$$m'(s) = a\left(\psi^{-1}(s)\right) g\left(\left(\Phi^{-1}m\right)(s)\right) m(s) - \left\{ a\left(\psi^{-1}(s)\right) H(m(s)) + \sum_{j=1}^n \lambda_j \tilde{f}_j\left(\psi^{-1}(s), \nu(s)\right) \right\}. \tag{2.75}$$

If $s = s_i,\ i \in \mathbb{Z}$, we can get

$$\begin{aligned}
m(s_i^+) - m(s_i^-) &= \sum_{j:s_j\in[s_i^+,s_i^++\tilde{\omega})} G(s_i,s_j)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_j)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_j))\right) \\
&\quad - \sum_{j:s_j\in[s_i^-,s_i^-+\tilde{\omega})} G(s_i,s_j)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_j)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_j))\right) \\
&= G(s_i,s_i+\tilde{\omega})\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_j)+\tilde{\omega}) + \left(\Phi W_i\Phi^{-1}\right)(m(s_j)+\tilde{\omega})\right) \\
&\quad - G(s_i,s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_j)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_j))\right) \\
&= -\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_j)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_j))\right).
\end{aligned} \tag{2.76}$$

Therefore, m is $\tilde{\omega}$-periodic solution of (2.62). The proof is complete. □

Lemma 2.16. *Assume that (H_1)–(H_5) hold, then $\Psi(P) \subset P$, and $\Psi : P \to P$ is compact and continuous.*

Proof. By the definition of P, for $m \in P$, we have

$$\begin{aligned}
(\Psi m)(s+\tilde{\omega}) &= \int_{s+\tilde{\omega}}^{s+2\tilde{\omega}} G(s+\tilde{\omega},\theta)\left\{a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\varphi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta \\
&\quad + \sum_{i:s_i\in[s+\tilde{\omega},s+2\tilde{\omega}]} G(s+\tilde{\omega},s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \\
&= \int_{s}^{s+\tilde{\omega}} G(s+\tilde{\omega},\theta+\tilde{\omega})\left\{a\left(\varphi^{-1}(\theta+\tilde{\omega})\right)H(m(\theta+\tilde{\omega}))\right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\varphi^{-1}(\theta+\tilde{\omega}),\nu(\theta+\tilde{\omega})\right)\right\}\mathrm{d}\theta \\
&\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s+\tilde{\omega},s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \\
&= \int_{s}^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\varphi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta \\
&\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \\
&= (\Psi m)(s).
\end{aligned} \tag{2.77}$$

Thus, $(\Psi m)(s+\tilde{\omega}) = (\Psi m)(s),\ s \in \mathbb{R}$. So in view of (2.66), (2.68), for $m \in P,\ s \in [0,\tilde{\omega}]$, we have

$$\begin{aligned}
(\Psi m)(s) &= \int_s^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\varphi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta \\
&\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \\
&\geq \frac{r_0^L}{1-r_0^L}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\varphi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta\right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right)\right\} \\
&\geq k\frac{1}{1-r_0^l}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\varphi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\varphi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta\right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right)\right\} \\
&\geq k\|\Psi m\|.
\end{aligned} \tag{2.78}$$

Therefore, $\Psi m \subset P$. Next, we will show that Ψ is continuous and compact. Firstly, we will consider the continuity of Ψ. Let $m_n \in P$ and $\|m_n - m\| \to 0$ as $n \to +\infty$, then $m \in P$ and $|m_n(s) - m(s)| \to 0$ as $n \to +\infty$ for any $s \in [0,\tilde{\omega}]$. By the continuity of $f_j\ (j = 1,2,\ldots,n),\ g,\ \Phi,\ \Phi^{-1},\ W_i\ (i = 1,2,\ldots,p)$, for any $s \in [0,\tilde{\omega}]$ and $\varepsilon > 0$, we have

$$|H(m_n(s)) - H(m(s))| \leq \frac{1-r_0^l}{3\tilde{\omega}\overline{a}}\varepsilon, \tag{2.79}$$

and denote $\nu_n(s) := (\Phi^{-1}m_n)(\varphi(\varphi^{-1}(s) - v_j(\varphi^{-1}(s))))$, and it is easy to see that $\|m_n - m\| \to 0$ as $n \to +\infty$ implies $\|\nu_n - \nu\| \to 0$ as $n \to +\infty$, thus

$$\begin{aligned}
&\left|\tilde{f}_j(\varphi^{-1}(s),\nu_n(s)) - \tilde{f}_j(\varphi^{-1}(s),\nu(s))\right| \leq \frac{1-r_0^l}{3\tilde{\omega}\lambda_j n}\varepsilon, \quad j = 1,2,\ldots,n, \\
&\left|\left(\left(\Phi B_i\Phi^{-1}\right)(m_n(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m_n(s_i))\right)\right. \\
&\left. -\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right)\right| \leq \frac{1-r_0^l}{3\tilde{\omega}p}\varepsilon,
\end{aligned} \tag{2.80}$$

where n is sufficiently large. For $s \in [0, \tilde{\omega}]$, we have

$$\begin{aligned}
\|\Psi m_n - \Psi m\| = \sup_{s\in[0,\tilde{\omega}]} \Bigg\{ \Bigg| & \int_s^{s+\tilde{\omega}} G(s,\theta) \left\{ a\left(\psi^{-1}(\theta)\right) H(m_n(\theta)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v_n(\theta)\right) \right\} \mathrm{d}\theta \\
& + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m_n(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m_n(s_i))\right) \\
& - \int_s^{s+\tilde{\omega}} G(s,\theta) \left\{ a\left(\psi^{-1}(\theta)\right) H(m(\theta)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right) \right\} \mathrm{d}\theta \\
& - \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right) \Bigg| \Bigg\} \\
= \sup_{s\in[0,\tilde{\omega}]} \Bigg\{ \Bigg| & \int_s^{s+\tilde{\omega}} G(s,\theta) \Bigg\{ a\left(\psi^{-1}(s)\right)(H(m_n(s)) - H(m(s))) \\
& + \sum_{j=1}^{n} \lambda_j \left(\tilde{f}_j\left(\psi^{-1}(\theta), v_n(\theta)\right) - \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right) \Bigg\} \mathrm{d}\theta \\
& + \sum_{i:s_i\in[s,s+\tilde{\omega})} G(s,s_i)\Bigg(\left(\left(\Phi B_i \Phi^{-1}\right)(m_n(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m_n(s_i))\right) \\
& - \left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right)\Bigg) \Bigg| \Bigg\} \\
\le \frac{\tilde{\omega}}{1-r_0^l} & \left\{ \overline{a} \frac{1-r_0^l}{3\tilde{\omega}\overline{a}} \varepsilon + \sum_{j=1}^{n} \lambda_j \frac{1-r_0^l}{3\tilde{\omega}\lambda_j n} \varepsilon + p \frac{1-r_0^l}{3\tilde{\omega} p} \varepsilon \right\} = \varepsilon.
\end{aligned} \tag{2.81}$$

Therefore, Ψ is continuous on P.

Next, we prove that Ψ is a compact operator. Let $S \subset P$ be an arbitrary bounded set in P, then there exists a number $L_0 > 0$ such that $\|m\| < L_0$ for any $m \in S$. We prove that $\overline{\Psi S}$ is compact. In fact, from (H_1), one has $W_i(0) = 0$, $i = 1, 2, \ldots$; by (2.53), it is easy to see that for any $z \in \mathbb{R}$, one has the following:

$$\|W_i(z)\| \le k(l_0) l_0 \|z\| + \|W_i(0)\| = k(l_0) l_0 \|z\|, \quad i = 1, 2, \ldots. \tag{2.82}$$

So for any $\{m_n\}_{n\in\mathbb{N}} \subset S$ and $s \in [0, \tilde{\omega}]$, we have

$$\begin{aligned}
\|\Psi m_n\| = \sup_{s\in[0,\tilde{\omega}]} \Bigg\{ \Bigg| & \int_s^{s+\tilde{\omega}} (s,\theta) \left\{ a\left(\psi^{-1}(\theta)\right) H(m_n(\theta)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v_n(\theta)\right) \right\} \mathrm{d}\theta \\
& + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m_n(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m_n(s_i))\right) \Bigg| \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\tilde{\omega}}{1-r_0^l}\left\{\overline{a}\frac{L\overline{c}}{1-\overline{c}}+\sum_{j=1}^{n}\max_{\nu\in\left[-L_0',L_0'\right]}\max_{s\in[0,\tilde{\omega}]}\lambda_j\tilde{f}_j\left(\psi^{-1}(s),\nu(s)\right)\right.\\
&\qquad\left.+L_0\sum_{i=1}^{p}(B_i+k(l_0)l_0)\right\}:=K,\\
&\|(\Psi m_n)'\|=\sup_{s\in[0,\tilde{\omega}]}\left\{\left|a\left(\psi^{-1}(s)\right)g\left(\left(\Phi^{-1}m_n\right)(s)\right)m_n(s)\right.\right.\\
&\qquad\left.\left.-\left\{a\left(\psi^{-1}(s)\right)H(m_n(s))+\sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\psi^{-1}(s),\nu_n(s)\right)\right\}\right|\right\}\\
&\leq \overline{a}L_0\|\Psi m_n\|+\overline{a}\frac{L\overline{c}}{1-\overline{c}}+\sum_{j=1}^{n}\max_{\nu\in\left[-L_0',L_0'\right]}\max_{s\in[0,\tilde{\omega}]}\lambda_j\tilde{f}_j\left(\psi^{-1}(s),\nu(s)\right)\\
&\leq \overline{a}L_0K+\overline{a}\frac{L\overline{c}}{1-\overline{c}}+\sum_{j=1}^{n}\max_{\nu\in\left[-L_0',L_0'\right]}\max_{s\in[0,\tilde{\omega}]}\lambda_j\tilde{f}_j\left(\psi^{-1}(s),\nu(s)\right):=Q,
\end{aligned}
\tag{2.83}
$$

where $L_0'=L_0/(1-\overline{c})$, which implies that $\{\Psi m_n\}_{n\in\mathbb{N}}$ and $\{(\Psi m_n)'\}_{n\in\mathbb{N}}$ are uniformly bounded on $[0,\tilde{\omega}]$. Therefore, there exists a subsequence of $\{\Psi m_n\}_{n\in\mathbb{N}}$ which converges uniformly on $[0,\tilde{\omega}]$; namely, $\overline{\Psi S}$ is compact. The proof is complete. □

3. Main Results

Our main results of this paper are as follows.

Theorem 3.1. *Assume that (H_1)–(H_5) hold, $0\leq c(t)<1$, for a sufficiently small Lipschitz constant l_0; suppose that the following conditions hold:*

(H_6) $\tilde{\alpha}_0=(1-\overline{c})(1-r_0^l)-\tilde{\omega}\overline{a}L\overline{c}-(1-\overline{c})\sum_{i=1}^{p}(B_i+k(l_0)l_0)>0$.

(H_7) *There exist positive constants β_1,β_2, and β_4 with $0<\beta_1<\beta_2<\beta_4$ such that*

$$
\frac{\sup_{s\in[0,\tilde{\omega}]}f_j\left(\psi^{-1}(s),\beta_1/(1-\overline{c})\right)}{\left(\beta_1/(1-\overline{c})\right)\tilde{\alpha}_0}<\frac{\sup_{s\in[0,\tilde{\omega}]}f_j\left(\psi^{-1}(s),\beta_4/(1-\overline{c})\right)}{\left(\beta_4/(1-\overline{c})\right)\tilde{\alpha}_0}<\frac{\inf_{s\in[0,\tilde{\omega}]}f_j\left(\psi^{-1}(s),\tilde{\alpha}\beta_2\right)}{\tilde{\alpha}\beta_2\tilde{\beta}_0},
\tag{3.1}
$$

where $\tilde{\beta}_0=((1-r_0^L)/\tilde{\alpha}r_0^L)-\tilde{\omega}\underline{a}l\underline{c}$ and $\tilde{\omega}=\psi(\omega)$.

Then for all $j=1,2,\ldots,n$, $\lambda_j\in(\lambda_{j_1},\lambda_{j_2}]$, (1.2) has at least three positive ω-periodic solutions, where

$$
\lambda_{j_1}=\frac{\tilde{\alpha}\beta_2\tilde{\beta}_0}{\tilde{\omega}n\inf_{s\in[0,\tilde{\omega}]}f_j\left(t,\tilde{\alpha}\beta_2\right)},\qquad \lambda_{j_2}=\frac{\left(\beta_4/(1-\overline{c})\right)\tilde{\alpha}_0}{\tilde{\omega}n\sup_{s\in[0,\tilde{\omega}]}f_j\left(t,\beta_4/(1-\overline{c})\right)},\qquad j=1,2,\ldots,n.
\tag{3.2}
$$

Proof. First of all, since $0 < \tilde{\alpha} < 1/(1-\overline{c})$ and $0 < r_0 < 1$, we have $\tilde{\alpha}_0 > 0$, so

$$\begin{aligned}\tilde{\beta}_0 &= \frac{1-r_0^L}{\tilde{\alpha} r_0^L} - \tilde{\omega}\underline{a}l\underline{c} > \frac{1-r_0^L}{\tilde{\alpha} r_0^L} - \tilde{\omega}\overline{a}l\overline{c} > \frac{1-r_0^L}{\tilde{\alpha}} - (1-\overline{c})\left(1-r_0^l\right) \\ &\quad + (1-\overline{c})\sum_{i=1}^{p}(B_i + k(l_0)l_0) \\ &> (1-\overline{c})\left(r_0^l - r_0^L\right) + (1-\overline{c})\sum_{i=1}^{p}(B_i + k(l_0)l_0) > 0.\end{aligned} \tag{3.3}$$

Furthermore, $0 < \lambda_{j_1} < \lambda_{j_2}$ in view of (3.1).

Now, define for each $\lambda_j \in (\lambda_{j_1}, \lambda_{j_2}]$ and $m \in P$ a mapping $\Psi : P \to P$ by

$$\begin{aligned}(\Psi m)(s) &= \int_s^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta \\ &\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right),\end{aligned} \tag{3.4}$$

and a function $\rho : P \to [0,\infty)$ by

$$\rho(m) = \min_{s\in[0,\tilde{\omega}]} m(s). \tag{3.5}$$

For $m \in \overline{P_{\beta_4}}$, by Lemma 2.13, we have

$$0 < \left(\Phi^{-1} m\right)(s) < \frac{\beta_4}{1-\overline{c}}. \tag{3.6}$$

It follows from (2.51), (2.68), (3.6), and (H_2), for all $j = 1,2,\ldots,n$, $\lambda_j \in (\lambda_{j1}, \lambda_{j2}]$ and $m \in \overline{P_{\beta_4}}$ that

$$\begin{aligned}(\Psi m)(s) &= \int_s^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta \\ &\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right) \\ &\le \frac{1}{1-r_0^l}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta\right. \\ &\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right)\right\} \\ &\le \frac{1}{1-r_0^l}\left\{\overline{a}\tilde{\omega}\frac{L\overline{c}}{1-\overline{c}}\beta_4 + \tilde{\omega}\sum_{j=1}^{n}\lambda_{j2}\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j\left(\psi^{-1}(s), \frac{\beta_4}{1-\overline{c}}\right) + \beta_4\sum_{i=1}^{p}(b_i + k(l_0)l_0)\right\}\end{aligned}$$

$$= \frac{1}{1-r_0^l}\left\{\overline{a}\tilde{\omega}\frac{L\overline{c}}{1-\overline{c}}\beta_4 + \sum_{j=1}^{n}\frac{(\beta_4/(1-\overline{c}))\alpha_0}{\tilde{\omega}n\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j(\psi^{-1}(s),\beta_4/(1-\overline{c}))}\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j\left(\psi^{-1}(s),\frac{\beta_4}{1-\overline{c}}\right)\right.$$
$$\left.+\beta_4\sum_{i=1}^{p}(B_i+k(l_0)l_0)\right\} \le \beta_4. \tag{3.7}$$

By Lemma 2.16, we know that Ψ is completely continuous on $\overline{P_{\beta_4}}$.

We now assert that the condition (2) of Lemma 2.1 holds. Indeed, if $m \in \overline{P_{\beta_1}}$, then similar to above argument, by (3.1), we have

$$\begin{aligned}
(\Psi m)(s) &= \int_s^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\psi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta \\
&\quad + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \\
&\le \frac{1}{1-r_0^l}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\psi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta\right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right)\right\} \\
&\le \frac{1}{1-r_0^l}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j\tilde{f}_j\left(\psi^{-1}(\theta),\nu(\theta)\right)\right\}\mathrm{d}\theta\right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right)\right\} \\
&\le \frac{1}{1-r_0^l}\left\{\overline{a}\tilde{\omega}\frac{L\overline{c}}{1-\overline{c}}\beta_1 + \tilde{\omega}\sum_{j=1}^{n}\lambda_{j2}\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j\left(\psi^{-1}(s),\frac{\beta_1}{1-\overline{c}}\right) + \beta_1\sum_{i=1}^{p}(B_i+k(l_0)\ l_0)\right\} \\
&= \frac{1}{1-r_0^l}\left\{\overline{a}\tilde{\omega}\frac{L\overline{c}}{1-\overline{c}}\beta_1 + \sum_{j=1}^{n}\frac{(\beta_4/(1-\overline{c}))\alpha_0}{\tilde{\omega}n\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j(\psi^{-1}(s),\beta_4/(1-\overline{c}))}\sup_{s\in[0,\tilde{\omega}]}\tilde{f}_j\left(\psi^{-1}(s),\frac{\beta_1}{1-\overline{c}}\right)\right. \\
&\qquad \left. + \beta_1\sum_{i=1}^{p}(B_i+k(l_0)l_0)\right\} \\
&< \frac{1}{1-r_0^l}\left\{\overline{a}\tilde{\omega}\frac{L\overline{c}}{1-\overline{c}}\beta_1 + \frac{\beta_4}{1-\overline{c}}\alpha_0\frac{\beta_1}{\beta_4} + \beta_1\sum_{i=1}^{p}(B_i+k(l_0)l_0)\right\} = \beta_1.
\end{aligned} \tag{3.8}$$

Hence, $\|\Psi m\| < \beta_1$ holds.

Choose a positive constant β_3 such that $0 < \beta_2 < k\beta_3 < \beta_3 \le \beta_4$. Next, we show that the condition (1) of Lemma 2.1 holds. Obviously, ρ is a concave continuous function on P with $\rho(m) \le \|m\|$ for $m \in \overline{P_{\beta_4}}$. We notice that if $m(s) = (2/5)\beta_2 + (3/5)\beta_3$ for $s \in [0,\tilde{\omega}]$, then $m \in \{m \in P(\rho,\beta_2,\beta_3) : \rho(m) > \beta_2\}$ which implies $\{m \in P(\rho,\beta_2,\beta_3) : \rho(m) > \beta_2\} \neq \emptyset$. For $m \in P(\rho,\beta_2,\beta_3)$, we have

$$\beta_2 \le \rho(m) = \min_{s\in[0,\tilde{\omega}]} m(s) \le \|m\| \le \beta_3, \tag{3.9}$$

which implies, from (2.50), that

$$\left(\Phi^{-1}m\right)(s) \ge \tilde{\alpha}\|m\| \ge \alpha\beta_2. \tag{3.10}$$

And it is also clear that $\Phi(x)$ is nondecreasing for $x > 0$ and $B_i, W_i \in C(\mathbb{R},\mathbb{R}^+)$, and we can easily have $\Phi B_i\Phi^{-1},\ \Phi W_i\Phi^{-1} \in C(\mathbb{R},\mathbb{R}^+)$. Hence

$$\begin{aligned}
\rho(\Psi m) &= \min_{s\in[0,\tilde{\omega}]} (\Psi m)(s) \\
&= \min_{s\in[0,\tilde{\omega}]} \left\{ \int_s^{s+\tilde{\omega}} G(s,\theta) \left\{ a\left(\psi^{-1}(\theta)\right) H(m(\theta)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right) \right\} \mathrm{d}\theta \right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \right\} \\
&\ge \frac{r_0^L}{1-r_0^L} \min_{s\in[0,\tilde{\omega}]} \left\{ \int_s^{s+\tilde{\omega}} \left\{ a\left(\psi^{-1}(\theta)\right) H(m(\theta)) + \sum_{j=1}^{n} \lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), \tilde{\alpha}\beta_2\right) \right\} \mathrm{d}\theta \right. \\
&\qquad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]} \left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \right\} \\
&> \frac{r_0^L}{1-r_0^L} \left\{ \tilde{\omega}\underline{a}\underline{l}\underline{c}\tilde{\alpha}\beta_2 + \tilde{\omega}\sum_{j=1}^{n} \lambda_{j_1} \inf_{s\in[0,\tilde{\omega}]} \tilde{f}_j\left(\psi^{-1}(s), \tilde{\alpha}\beta_2\right) \right. \\
&\qquad \left. + \min_{\beta_2\le m\le\beta_3} \sum_{i=1}^{p} \left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \right\} \\
&\ge \frac{r_0^L}{1-r_0^L} \left\{ \tilde{\omega}\underline{a}\underline{l}\underline{c}\tilde{\alpha}\beta_2 + \tilde{\omega}\sum_{j=1}^{n} \frac{\tilde{\alpha}\beta_2\tilde{\beta}_0}{\tilde{\omega}n \inf_{s\in[0,\tilde{\omega}]} \tilde{f}_j\left(\psi^{-1}(s), \tilde{\alpha}\beta_2\right)} \inf_{s\in[0,\tilde{\omega}]} \tilde{f}_j\left(\psi^{-1}(s), \tilde{\alpha}\beta_2\right) \right. \\
&\qquad \left. + \min_{\beta_2\le m\le\beta_3} \sum_{i=1}^{p} \left(\left(\Phi B_i\Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i\Phi^{-1}\right)(m(s_i))\right) \right\} > \beta_2
\end{aligned} \tag{3.11}$$

for all $m \in P(\rho,\beta_2,\beta_3)$.

Finally, we prove that the condition (3) of Lemma 2.1 holds. Let $m \in P(\rho, \beta_2, \beta_4)$ and $\|\Psi m\| > \beta_3$, then $\rho(\Psi m) > \beta_2$. We notice that (3.4) implies that

$$\|\Psi m\| \le \frac{1}{1-r_0^l}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), \tilde{\alpha}\beta_2\right)\right\}\mathrm{d}\theta \right.$$
$$\left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right)\right\}. \tag{3.12}$$

Thus

$$\begin{aligned}
\rho(\Psi m) &= \min_{s\in[0,\tilde{\omega}]}\left\{\int_s^{s+\tilde{\omega}} G(s,\theta)\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta \right.\\
&\quad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]} G(s,s_i)\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right)\right\}\\
&\ge \frac{r_0^L}{1-r_0^L}\min_{s\in[0,\tilde{\omega}]}\left\{\int_s^{s+\tilde{\omega}}\left\{a\left(\psi^{-1}(\theta)\right)H(m(\theta)) + \sum_{j=1}^{n}\lambda_j \tilde{f}_j\left(\psi^{-1}(\theta), v(\theta)\right)\right\}\mathrm{d}\theta \right.\\
&\quad \left. + \sum_{i:s_i\in[s,s+\tilde{\omega}]}\left(\left(\Phi B_i \Phi^{-1}\right)(m(s_i)) + \left(\Phi W_i \Phi^{-1}\right)(m(s_i))\right)\right\}\\
&\ge \frac{r_0^L\left(1-r_0^l\right)}{1-r_0^l}\|\Psi m\| \ge k\beta_3 > \beta_2.
\end{aligned} \tag{3.13}$$

To sum up, all the hypotheses of Lemma 2.1 are satisfied. Hence Ψ has at least three positive fixed points. That is, (1.2) has at least three positive ω-periodic solutions. This completes the proof. □

Corollary 3.2. *Suppose (H_1)–(H_6) hold. If*

$$\lim_{x\to\infty}\frac{\sup_{s\in[0,\tilde{\omega}]} f_j\left(\psi^{-1}(s), x\right)}{x} = 0, \tag{3.14}$$

$$\lim_{x\to 0}\frac{\sup_{s\in[0,\tilde{\omega}]} f_j\left(\psi^{-1}(s), x\right)}{x} = 0, \tag{3.15}$$

where $\tilde{\omega} = \psi(\omega)$; then (1.2) *has at least three positive ω-periodic solutions.*

Proof. In view of (3.14), we can choose $\beta_4 > \beta_2 > 0$ such that the second inequality in (3.1) holds, and in view of (3.15), we can choose $\beta_1 \in (0, \beta_2)$ such that the first inequality in (3.1) holds. Therefore, the conclusion of Theorem 3.1 holds. This completes the proof. □

4. An Example

Let us consider the variable time scale $\mathbb{T}_0(x)$ constructed by $t_i = i$, $\tau_i(x) = (-1)^i l_0 \sin x$, where $|x| > 1$ for all $t \in \mathbb{T}_0(x)$, $0 < l_0 < (1/2)$ and consider π-periodic system:

$$\begin{aligned}
&\left(x(t) + \frac{1}{9}|\cos t| x(t-\alpha)\right)' = \frac{1}{\pi}|\sin t|\left(\frac{1}{3} + \frac{1}{3}e^{-x}\right)x(t) - \frac{l_0}{3n}\sum_{j=1}^{n}\lambda_j x^{1/2}(t)\ \ln\left(x\left(t - e^{(1/j)|\sin t|}\right) + 1\right), \\
&\qquad\qquad (t,x) \in \mathbb{T}_0(x), \\
&x^+ = 0.03\left(\frac{2}{3}\right)^i x + 0.02 l_0 \sin x + x, \\
&t^+ = 2i + 1 - l_0 \sin x,
\end{aligned} \tag{4.1}$$

where α is a constant, λ_j, $j = 1, 2, \ldots, n$ are nonnegative parameters. In this case, $c(t) = (1/9)|\cos t|$, $a(t) = 1/\pi$, $g(x(t)) = (1/3) + (1/3)e^{-x}$, $B_i = 0.03(2/3)^i$, $J_i(x) = 0.02 l_0 \sin x$, $v_j(t) = e^{(1/j)|\sin t|}$ and $f_j(t, x(t - v_j(t))) = (l_0/3n)x^{1/2}(t)\ln(x(t - v_j(t)) + 1)$, $j = 1, 2, \ldots, n$, $i \in \mathbb{N}$. Obviously, (H_1)–(H_3) are satisfied, and it is easy to see that (3.14) and (3.15) hold.

By the formula of φ-substitution and $\delta_k = 1$, one can find

$$\tilde{\omega} = \varphi(\omega) = \omega - \sum_{0<2k<\omega} \delta_k = \pi - 1. \tag{4.2}$$

Clearly, $L = (2/3)$, $l = (1/3)$, $0 \le c(t) \le (1/9) < 1$, $r_0 = e^{-(\pi-1)(1/\pi)} \approx 0.5058$ and $\overline{c} = 1/9$, $\underline{c} = 0$, so we can find $\delta = (\underline{c} + \overline{c})/2 = 1/18$, and it is easy to check that $((2\delta - \underline{c})(1 - \underline{c}^2))/(1 - (2\delta - \underline{c})^2) = 9/80 \approx 0.1125 < (r_0^L(1 - r_0^l))/(1 - r_0^L) = 0.3532$. Thus, (H_4) holds. Furthermore, we also have

$$\begin{aligned}
\left\|\frac{\partial f_j}{\partial x}\right\| &= \frac{l_0}{3n}\left\|\frac{\ln(x+1)}{2x^{1/2}} + \frac{x^{1/2}}{x+1}\right\| \le \frac{l_0}{3n}\left(\left\|\frac{\ln(x+1)^{1/2}}{x^{1/2}}\right\| + \left\|\frac{x^{1/2}}{x+1}\right\|\right) \\
&\le \frac{l_0}{3n}\left(\left\|\frac{(x+1)^{1/2}}{x^{1/2}}\right\| + \left\|\frac{x^{1/2}}{x+1}\right\|\right) = \frac{l_0}{3n}\left(\left\|\left(1 + \frac{1}{x}\right)^{1/2}\right\| + \left\|\frac{x^{1/2}}{x+1}\right\|\right) \\
&< \frac{l_0}{3n}(2+1) = \frac{l_0}{n}.
\end{aligned} \tag{4.3}$$

Hence, for any x_1, $x_2 \in \mathbb{R}$, one can get

$$\sum_{j=1}^{n}\left|f_j(t, x_1) - f_j(t, x_2)\right| \le \left|\frac{\partial f_j}{\partial x}\right||x_1 - x_2| < n \cdot \frac{l_0}{n}|x_1 - x_2| = l_0|x_1 - x_2|. \tag{4.4}$$

So (H_5) is satisfied. For a sufficiently small l_0, one can also have

$$\tilde{\alpha}_0 = (1-\bar{c})\left(1-r_0^l\right) - \tilde{\omega}\bar{a}L\bar{c} - (1-\bar{c})\sum_{i=1}^{p}(B_i + k(l_0)l_0) = 0.0768 - \frac{8}{9}\sum_{i=1}^{p}k(l_0)l_0, \tag{4.5}$$

since $k(l_0)$ is a bounded function; for a sufficiently small l_0, one can have $\tilde{\alpha}_0 > 0$ such that (H_6) holds. Therefore, according to Corollary 3.2, (4.1) has at least three positive π-periodic solutions.

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