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Research Article

Existence of Weak Solutions for Nonlinear Fractional Differential Inclusion with Nonseparated Boundary Conditions

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We discuss the existence of solutions, under the Pettis integrability assumption, for a class of boundary value problems for fractional differential inclusions involving nonlinear nonseparated boundary conditions. Our analysis relies on the Mönch fixed point theorem combined with the technique of measures of weak noncompactness.

1. Introduction

This paper is mainly concerned with the existence results for the following fractional differential inclusion with non-separated boundary conditions:

$$\begin{gathered} {}^{c}D^{\alpha}u(t) \in F(t,u(t)), \quad t \in J := [0,T],\ T > 0, \\ u(0) = \lambda_1 u(T) + \mu_1, \quad u'(0) = \lambda_2 u'(T) + \mu_2, \quad \lambda_1 \neq 1,\ \lambda_2 \neq 1, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative. $F : J \times E \to \mathcal{P}(E)$ is a multivalued map, E is a Banach space with the norm $\|\cdot\|$, and $\mathcal{P}(E)$ is the family of all nonempty subsets of E.

Recently, fractional differential equations have found numerous applications in various fields of physics and engineering [1, 2]. It should be noted that most of the books and papers on fractional calculus are devoted to the solvability of initial value problems for differential equations of fractional order. In contrast, the theory of boundary value problems

for nonlinear fractional differential equations has received attention quite recently and many aspects of this theory need to be explored. For more details and examples, see [3–18] and the references therein.

To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of weak noncompactness, which is an important method for seeking solutions of differential equations. This technique was mainly initiated in the monograph of Banaś and Goebel [19] and subsequently developed and used in many papers; see, for example, Banaś and Sadarangani [20], Guo et al. [21], Krzyśka and Kubiaczyk [22], Lakshmikantham and Leela [23], Mönch's [24], O'Regan [25, 26], Szufla [27, 28], and the references therein.

In 2007, Ouahab [29] investigated the existence of solutions for α-fractional differential inclusions by means of selection theorem together with a fixed point theorem. Very recently, Chang and Nieto [30] established some new existence results for fractional differential inclusions due to fixed point theorem of multivalued maps. Problem (1.1) was discussed for single valued case in the paper [31]; some existence results for single- and multivalued cases for an extension of (1.1) to non-separated integral boundary conditions were obtained in the article [32] and [33]. About other results on fractional differential inclusions, we refer the reader to [34]. As far as we know, there are very few results devoted to weak solutions of nonlinear fractional differential inclusions. Motivated by the above mentioned papers, the purpose of this paper is to establish the existence results for the boundary value problem (1.1) by virtue of the Mönch fixed point theorem combined with the technique of measures of weak noncompactness.

The remainder of this paper is organized as follows. In Section 2, we present some basic definitions and notations about fractional calculus and multivalued maps. In Section 3, we give main results for fractional differential inclusions. In the last section, an example is given to illustrate our main result.

2. Preliminaries and Lemmas

In this section, we introduce notation, definitions, and preliminary facts that will be used in the remainder of this paper. Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^*, and let $(E,\omega) = (E,\sigma(E,E^*))$ denote the space E with its weak topology. Here, let $C(J,E)$ be the Banach space of all continuous functions from J to E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \le t \le T\}, \tag{2.1}$$

and let $L^1(J,E)$ denote the Banach space of functions $y : J \to E$ that are the Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt. \tag{2.2}$$

We let $L^\infty(J,E)$ to be the Banach space of bounded measurable functions $y : J \to E$ equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : \|y(t)\| \le c, \text{ a.e. } t \in J\}. \tag{2.3}$$

Also, $AC^1(J,E)$ will denote the space of functions $y : J \to E$ that are absolutely continuous and whose first derivative, y', is absolutely continuous.

Let $(E, \|\cdot\|)$ be a Banach space, and let $P_{\mathrm{cl}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$, $P_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$, $P_{\mathrm{cp}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$, and $P_{\mathrm{cp},c}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact and convex}\}$. A multivalued map $G : E \to P(E)$ is *convex* (*closed*) valued if $G(x)$ is convex (closed) for all $x \in E$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The mapping G is called *upper semicontinuous* (*u.s.c.*) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E and if for each open set N of E containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. We say that G is *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(E)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). The mapping G has a *fixed point* if there is $x \in E$ such that $x \in G(x)$. The set of fixed points of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G : J \to P_{\mathrm{cl}}(E)$ is said to be *measurable* if for every $y \in E$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\} \tag{2.4}$$

is measurable. For more details on multivalued maps, see the books of Aubin and Cellina [35], Aubin and Frankowska [36], Deimling [37], Hu and Papageorgiou [38], Kisielewicz [39], and Covitz and Nadler [40].

Moreover, for a given set V of functions $v : J \mapsto \mathbb{R}$, let us denote by $V(t) = \{v(t) : v \in V\}$, $t \in J$, and $V(J) = \{v(t) : v \in V, t \in J\}$.

For any $y \in C(J,E)$, let $S_{F,y}$ be the set of selections of F defined by

$$S_{F,y} = \left\{ f \in L^1(J,E) : f(t) \in F(t, y(t)) \text{ a.e. } t \in J \right\}. \tag{2.5}$$

Definition 2.1. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n(t) \to x(t)$ in (E,ω) then $h(x_n(t)) \to h(x(t))$ in (E,ω) for each $t \to J$).

Definition 2.2. A function $F : Q \to P_{\mathrm{cl,cv}}(Q)$ has a weakly sequentially closed graph if for any sequence $(x_n, y_n)_1^\infty \in Q \times Q$, $y_n \in F(x_n)$ for $n \in \{1,2,\ldots\}$ with $x_n(t) \to x(t)$ in (E,ω) for each $t \in J$ and $y_n(t) \to y(t)$ in (E,ω) for each $t \in J$, then $y \in F(x)$.

Definition 2.3 (see [41]). The function $x : J \to E$ is said to be the Pettis integrable on J if and only if there is an element $x_J \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J,E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Lemma 2.4 (see [41]). *If $x(\cdot)$ is Pettis' integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis' integrable.*

Definition 2.5 (see [42]). Let E be a Banach space, Ω_E the set of all bounded subsets of E, and B_1 the unit ball in E. The *De Blasi* measure of weak noncompactness is the map $\beta : \Omega_E \to [0,\infty)$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E \text{ such that } X \subset \epsilon B_1 + \Omega\}. \tag{2.6}$$

Lemma 2.6 (see [42]). *The De Blasi measure of noncompactness satisfies the following properties:*

(a) $S \subset T \Rightarrow \beta(S) \leq \beta(T)$;

(b) $\beta(S) = 0 \Leftrightarrow S$ *is relatively weakly compact;*

(c) $\beta(S \cup T) = \max\{\beta(S), \beta(T)\}$;

(d) $\beta(\overline{S}^{\omega}) = \beta(S)$, *where* $\overline{S}^{\omega}$ *denotes the weak closure of* S;

(e) $\beta(S + T) \leq \beta(S) + \beta(T)$;

(f) $\beta(aS) = |a|\alpha(S)$;

(g) $\beta(\mathrm{conv}(S)) = \beta(S)$;

(h) $\beta(\cup_{|\lambda|\leq h}\lambda S) = h\beta(S)$.

The following result follows directly from the Hahn-Banach theorem.

Lemma 2.7. *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For completeness, we recall the definitions of the Pettis-integral and the Caputo derivative of fractional order.

Definition 2.8 (see [25]). Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^{\alpha}h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \tag{2.7}$$

where the sign "$\int$" denotes the Pettis integral and Γ is the gamma function.

Definition 2.9 (see [3]). For a function $h : J \to E$, the Caputo fractional-order derivative of h is defined by

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds, \quad n-1 < \alpha < n, \tag{2.8}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.10 (see [43]). *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C(J,E)$. Suppose $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{0\} \cup F(V)) \Longrightarrow V \textit{ is relatively weakly compact} \tag{2.9}$$

holds for every subset V of Q, then the operator inclusion $x \in F(x)$ has a solution in Q.

3. Main Results

Let us start by defining what we mean by a solution of problem (1.1).

Definition 3.1. A function $y \in AC^1(J,E)$ is said to be a solution of (1.1), if there exists a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$ for a.e. $t \in J$, such that

$$ {}^cD^\alpha y(t) = v(t) \quad \text{a.e. } t \in J,\ 1 < \alpha \le 2, \tag{3.1}$$

and y satisfies conditions $u(0) = \lambda_1 u(T) + \mu_1,\ u'(0) = \lambda_2 u'(T) + \mu_2,\ \lambda_1 \neq 1,\ \lambda_2 \neq 1$.

To prove the main results, we need the following assumptions:

(H1) $F : J \times E \to P_{\mathrm{cp,cv}}(E)$ has weakly sequentially closed graph;

(H2) for each continuous $x \in C(J,E)$, there exists a scalarly measurable function $v : J \to E$ with $v(t) \in F(t,x(t))$ a.e. on J and v is Pettis integrable on J;

(H3) there exist $p_f \in L^\infty(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le p_f(t)\psi(\|u\|); \tag{3.2}$$

(H4) for each bounded set $D \subset E$, and each $t \in I$, the following inequality holds:

$$\beta(F(t,D)) \le p_f(t) \cdot \beta(D); \tag{3.3}$$

(H5) there exists a constant $R > 0$ such that

$$\frac{R}{g^* + \|p_f\|_{L^\infty}\psi(R)G^*} > 1, \tag{3.4}$$

where g^* and G^* are defined by (3.9).

Theorem 3.2. *Let E be a Banach space. Assume that hypotheses (H1)–(H5) are satisfied. If*

$$\|p_f\|_{L^\infty} G^* < 1, \tag{3.5}$$

then the problem (1.1) *has at least one solution on J.*

Proof. Let $\rho \in C[0,T]$ be a given function; it is obvious that the boundary value problem [18]

$$\begin{gathered} {}^cD^\alpha u(t) = \rho(t), \quad t \in (0,T),\ 1 < \alpha \le 2 \\ u(t) = \lambda_1 u(T) + \mu_1, \quad u'(0) = \lambda_2 u'(T) + \mu_2, \quad \lambda_1 \neq 1,\ \lambda_2 \neq 1 \end{gathered} \tag{3.6}$$

has a unique solution

$$u(t) = \int_0^T G(t,s)\rho(s)ds + g(t), \tag{3.7}$$

where $G(t,s)$ is defined by the formula

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{\lambda_1(T-s)^{\alpha-1}}{(\lambda_1-1)\Gamma(\alpha)} + \dfrac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{\alpha-2}}{(\lambda_2-1)(\lambda_1-1)\Gamma(\alpha-1)}, & \text{if } 0 \le s \le t \le T, \\ -\dfrac{\lambda_1(T-s)^{\alpha-1}}{(\lambda_1-1)\Gamma(\alpha)} + \dfrac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{\alpha-2}}{(\lambda_2-1)(\lambda_1-1)\Gamma(\alpha-1)}, & \text{if } 0 \le t \le s \le T, \end{cases} \tag{3.8}$$

$$g(t) = \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{\lambda_1-1}.$$

From the expression of $G(t,s)$ and $g(t)$, it is obvious that $G(t,s)$ is continuous on $J \times J$ and $g(t)$ is continuous on J. Denote by

$$G^* = \sup\left\{\int_0^T |G(t,s)|ds,\ t \in J\right\}, \qquad g^* = \max_{0\le t\le T}\|g(t)\|. \tag{3.9}$$

We transform the problem (1.1) into fixed point problem by considering the multivalued operator $N : C(J,E) \to P_{\mathrm{cl,cv}}(C(J,E))$ defined by

$$N(x) = \left\{h \in C(J,E) : h(t) = g(t) + \int_0^T G(t,s)v(s)ds, v \in S_{F,x}\right\}, \tag{3.10}$$

and refer to [31] for defining the operator N. Clearly, the fixed points of N are solutions of Problem (1.1). We first show that (3.10) makes sense. To see this, let $x \in C(J,E)$; by (H2) there exists a Pettis' integrable function $v : J \to E$ such that $v(t) \in F(t,x(t))$ for a.e. $t \in J$. Since $G(t,\cdot) \in L^\infty(J)$, then $G(t,\cdot)v(\cdot)$ is Pettis integrable and thus N is well defined.

Let $R > 0$, and consider the set

$$\begin{aligned} D = \Bigg\{ & x \in C(J,E) : \|x\|_\infty \le R,\ \|x(t_1) - x(t_2)\| \le \|g(t_1) - g(t_2)\| \\ & + \|p_f\|_{L^\infty}\psi(R)\int_0^T \|G(t_2,s) - G(t_1,s)\|ds \text{ for } t_1,t_2 \in J\Bigg\}; \end{aligned} \tag{3.11}$$

clearly, the subset D is a closed, convex, bounded, and equicontinuous subset of $C(J,E)$. We shall show that N satisfies the assumptions of Lemma 2.10. The proof will be given in four steps.

Step 1. We will show that the operator $N(x)$ is convex for each $x \in D$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exists Pettis' integrable functions $v_1(t)$, $v_2(t) \in F(t, x(t))$ such that, for all $t \in J$, we have

$$h_i(t) = g(t) + \int_0^T G(t,s)v_i(s)\,ds, \quad i = 1,2. \tag{3.12}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$[dh_1 + (1-d)h_2](t) = g(t) + \int_0^T G(t,s)[dv_1(s) + (1-d)v_2(s)]\,ds. \tag{3.13}$$

Since F has convex values, $(dv_1 + (1-d)v_2)(t) \in F(t,y)$ and we have $dh_1 + (1-d)h_2 \in N(x)$.

Step 2. We will show that the operator N maps D into D.

To see this, take $u \in ND$. Then there exists $x \in D$ with $u \in N(x)$ and there exists a Pettis integrable function $v : J \to E$ with $v(t) \in F(t, x(t))$ for a.e. $t \in J$. Without loss of generality, we assume $u(s) \neq 0$ for all $s \in J$. Then, there exists $\varphi_s \in E^*$ with $\|\varphi_s\| = 1$ and $\varphi_s(u(s)) = \|u(s)\|$. Hence, for each fixed $t \in J$, we have

$$\begin{aligned}
\|u(t)\| &= \varphi_t(u(t)) = \varphi_t\left(g(t) + \int_0^T G(t,s)v(s)ds\right) \\
&\le \varphi_t(g(t)) + \varphi_t\left(\int_0^T G(t,s)v(s)ds\right) \\
&\le \|g(t)\| + \int_0^T \|G(t,s)\|\varphi_t(v(s))\,ds \\
&\le g^* + G^*\psi(\|x\|_\infty)\|p_f\|_{L^\infty}.
\end{aligned} \tag{3.14}$$

Therefore, by (H5), we have

$$\|u\|_\infty \le g^* + \|p_f\|_{L^\infty} G^*\psi(\|R\|_\infty) \le R. \tag{3.15}$$

Next suppose $u \in ND$ and $\tau_1, \tau_2 \in J$, with $\tau_1 < \tau_2$ so that $u(\tau_2) - u(\tau_1) \neq 0$. Then, there exists $\varphi \in E^*$ such that $\|u(\tau_2) - u(\tau_1)\| = \varphi(u(\tau_2) - u(\tau_1))$. Hence,

$$\begin{aligned}
\|u(\tau_2) - u(\tau_1)\| &= \varphi\left(g(t_2) - g(t_1) + \int_0^T [G(\tau_2,s) - G(\tau_1,s)] \cdot v(s)ds\right) \\
&\le \varphi(g(t_2) - g(t_1)) + \varphi\left(\int_0^T [G(\tau_2,s) - G(\tau_1,s)] \cdot v(s)\,ds\right) \\
&\le \|g(t_2) - g(t_1)\| + \int_0^T \|G(\tau_2,s) - G(\tau_1,s)\| \cdot \|v(s)\| ds \\
&\le \|g(t_2) - g(t_1)\| + \psi(R)\|p_f\|_{L^\infty} \int_0^T \|G(\tau_2,s) - G(\tau_1,s)\| ds;
\end{aligned} \tag{3.16}$$

this means that $u \in D$.

Step 3. We will show that the operator N has a weakly sequentially closed graph.

Let $(x_n, y_n)_1^\infty$ be a sequence in $D \times D$ with $x_n(t) \to x(t)$ in (E, ω) for each $t \in J$, $y_n(t) \to y(t)$ in (E, ω) for each $t \in J$, and $y_n \in N(x_n)$ for $n \in \{1, 2, \ldots\}$. We will show that $y \in Nx$. By the relation $y_n \in N(x_n)$, we mean that there exists $v_n \in S_{F,x_n}$ such that

$$y_n(t) = g(t) + \int_0^T G(t,s) v_n(s) ds. \tag{3.17}$$

We must show that there exists $v \in S_{F,x}$ such that, for each $t \in J$,

$$y(t) = g(t) + \int_0^T G(t,s) v(s) ds. \tag{3.18}$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence v_{n_m} such that

$$\begin{gathered} v_{n_m}(\cdot) \longrightarrow v(\cdot) \text{ in } (E, \omega) \text{ as } m \longrightarrow \infty \\ v_{n_m}(t) \in F(t, x_n(t)) \text{ a.e. } t \in J. \end{gathered} \tag{3.19}$$

Since $F(t,\cdot)$ has a weakly sequentially closed graph, $v \in F(t,x)$. The Lebesgue dominated convergence theorem for the Pettis integral then implies that for each $\varphi \in E^*$,

$$\varphi(y_n(t)) = \varphi\left(g(t) + \int_0^T G(t,s) v_n(s) ds\right) \longrightarrow \varphi\left(g(t) + \int_0^T G(t,s) v(s) ds\right); \tag{3.20}$$

that is, $y_n(t) \to Nx(t)$ in (E, w). Repeating this for each $t \in J$ shows $y(t) \in Nx(t)$.

Step 4. The implication (2.9) holds. Now let V be a subset of D such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{\text{conv}}(N(V) \cup \{0\})$ for all $t \in J$. Hence, $NV(t) \subset ND(t)$, $t \in J$, is bounded in $P(E)$.

Since function g is continuous on J, the set $\overline{\{g(t), t \in J\}} \subset E$ is compact, so $\beta(g(t)) = 0$. By assumption (H4) and the properties of the measure β, we have for each $t \in J$

$$\begin{aligned}
\beta(N(V)(t)) &= \beta\left\{g(t) + \int_0^T G(t,s)v(s)ds : v \in S_{F,x},\ x \in V,\ t \in J\right\} \\
&\le \beta\{g(t) : t \in J\} + \beta\left\{\int_0^T G(t,s)v(s)ds : v \in S_{F,x},\ x \in V,\ t \in J\right\} \\
&\le \beta\left\{\int_0^T G(t,s)v(s)ds : v(t) \in F(t,x(t)),\ x \in V,\ t \in J\right\} \\
&\le \int_0^T \|G(t,s)\| \cdot p_f(s) \cdot \beta(V(s))ds \\
&\le \|p_f\|_{L^\infty} \cdot \int_0^T \|G(t,s)\| \cdot \beta(V(s))ds \\
&\le \|p_f\|_{L^\infty} \cdot G^* \cdot \int_0^T \beta(V(s))ds,
\end{aligned} \tag{3.21}$$

which gives

$$\|v\|_\infty \le \|p_f\|_{L^\infty} \cdot \|v\|_\infty \cdot G^*. \tag{3.22}$$

This means that

$$\|v\|_\infty \cdot \left[1 - \|p_f\|_{L^\infty} \cdot G^*\right] \le 0. \tag{3.23}$$

By (3.5) it follows that $\|v\|_\infty = 0$; that is, $v = 0$ for each $t \in J$, and then V is relatively weakly compact in E. In view of Lemma 2.10, we deduce that N has a fixed point which is obviously a solution of Problem (1.1). This completes the proof. □

In the sequel we present an example which illustrates Theorem 3.2.

4. An Example

Example 4.1. We consider the following partial hyperbolic fractional differential inclusion of the form

$$\begin{gathered}
({}^cD^\alpha u_n)(t) \in \frac{1}{7e^{t+13}}(1 + |u_n(t)|), \quad t \in J := [0,T],\ 1 < \alpha \le 2, \\
u(0) = \lambda_1 u(T) + \mu_1, \qquad u'(0) = \lambda_2 u'(T) + \mu_2,
\end{gathered} \tag{4.1}$$

Set $T = 1$, $\lambda_1 = \lambda_2 = -1$, $\mu_1 = \mu_2 = 0$, then $g(t) = 0$. So $g^* = 0$.

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\} \tag{4.2}$$

with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|. \tag{4.3}$$

Set

$$\begin{gathered} u = (u_1, u_2, \ldots, u_n, \ldots), \quad f = (f_1, f_2, \ldots, f_n, \ldots), \\ f_n(t, u_n) = \frac{1}{7e^{t+13}}(1 + |u_n|), \quad t \in J. \end{gathered} \tag{4.4}$$

For each $u_n \in \mathbb{R}$ and $t \in J$, we have

$$|f_n(t, u_n)| \leq \frac{1}{7e^{t+13}}(1 + |u_n|). \tag{4.5}$$

Hence conditions (H1), (H2), and (H3) hold with $p_f(t) = 1/(7e^{t+13})$, $t \in J$, and $\psi(u) = 1 + u$, $u \in [0, \infty)$. For any bounded set $D \subset l^1$, we have

$$\beta(F(t, D)) \leq \frac{1}{7e^{t+13}} \cdot \beta(D), \quad \forall t \in J. \tag{4.6}$$

Hence (H4) is satisfied. From (3.8), we have

$$G(t, s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, & \text{if } 0 \leq s \leq t \leq 1, \\ -\dfrac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{4.7}$$

So, we get

$$\begin{aligned}\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\ &= \int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\right]ds \\ &\quad + \int_t^1 \left[-\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\right]ds \\ &= \frac{4t^\alpha - 2}{4\Gamma(\alpha+1)} + \frac{1-2t}{4\Gamma(\alpha)}.\end{aligned} \tag{4.8}$$

A simple computation gives

$$G^* < \frac{1}{4\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha+1)} := A_\alpha. \tag{4.9}$$

We shall check that condition (3.5) is satisfied. Indeed

$$\|p\|_{L^\infty} G^* < \frac{1}{7e^{13}} A_\alpha < 1, \tag{4.10}$$

which is satisfied for some $\alpha \in (1,2]$, and (H5) is satisfied for $R > A_\alpha/(7e^{13} - A_\alpha)$. Then by Theorem 3.2, the problem (4.1) has at least one solution on J for values of α satisfying (4.10).

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