Research Article

General Helices of AW(k)-Type in the Lie Group

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We study curves of AW(k)-type in the Lie group G with a bi-invariant metric. Also, we characterize general helices in terms of AW(k)-type curve in the Lie group G .

1. Introduction

The geometry of curves and surfaces in a 3-dimensional Euclidean space $\mathbb{R}^3$ represented for many years a popular topic in the field of classical differential geometry. One of the important problems of the curve theory is that of Bertrand-Lancret-de Saint Venant saying that a curve in $\mathbb{R}^3$ is of constant slop; namely, its tangent makes a constant angle with a fixed direction if and only if the ratio of torsion τ and curvature κ is a constant. These curves are said to be general helices. If both τ and κ are nonzero constants, the curve is called cylindrical helix. Helix is one of the most fascinating curves in science and nature. Scientists have long held a fascinating, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, α -helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA.

The problem of Bertrand-Lancret-de Saint Venant was generalized for curves in other 3-dimensional manifolds—in particular space forms or Sasakian manifolds. Such a curve has the property that its tangent makes a constant angle with a parallel vector field on the manifold or with a Killing vector field, respectively. For example, a curve $\alpha(s)$ in a 3-dimensional space form is called a general helix if there exists a Killing vector field $V(s)$ with constant length along α and such that the angle between V and α' is a non-zero constant (see [1]). A general helix defined by a parallel vector field was studied in [2]. Moreover, in [3] it is shown that general helices in a 3-dimensional space form are extremal curvatures of a functional involving a linear combination of the curvature, the torsion, and a constant. General helices also called the Lancret curves are used in many applications (e.g., [4–7]).

The notion of AW(k)-type submanifolds was introduced by Arslan and West in [8]. In particular, many works related to curves of AW(k)-type have been done by several authors. For example, in [9, 10] the authors gave curvature conditions and characterizations related to these curves in $\mathbb{R}^n$. Also, in [11] they investigated curves of AW(k) type in a 3-dimensional null cone and gave curvature conditions of these kinds of curves. However, to the author's knowledge, there is no article dedicated to studying the notion of AW(k)-type curves immersed in Lie group.

In this paper, we investigate curvature conditions of curves of AW(k)-type in the Lie group G with a bi-invariant metric. Moreover, we characterize general helices of AW(k)-type in the Lie group G .

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$ and D the Levi-Civita connection of the Lie group G . If $\mathfrak{g}$ denotes the Lie algebra of G , then we know that $\mathfrak{g}$ is isomorphic to $T_e G$, where e is identity of G . If $\langle \cdot, \cdot \rangle$ is a bi-invariant metric on G , then we have

$$\begin{aligned}\langle X, [Y, Z] \rangle &= \langle [X, Y], Z \rangle, \\ D_X Y &= \frac{1}{2}[X, Y]\end{aligned}\tag{2.1}$$

for all $X, Y, Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be a unit speed curve with parameter s and $\{V_1, V_2, \dots, V_n\}$ an orthonormal basis of $\mathfrak{g}$. In this case, we write that any vector fields W and Z along the curve α as $W = \sum_{i=1}^n w_i V_i$ and $Z = \sum_{i=1}^n z_i V_i$, where $w_i : I \rightarrow \mathbb{R}$ and $z_i : I \rightarrow \mathbb{R}$ are smooth functions. Furthermore, the Lie bracket of two vector fields W and Z is given by

$$[W, Z] = \sum_{i=1}^n w_i z_j [V_i, V_j].\tag{2.2}$$

Let $D_{\alpha'} W$ be the covariant derivative of W along the curve α , $V_1 = \alpha'$, and $W' = \sum_{i=1}^n w'_i V_i$, where $w'_i = dw_i/ds$. Then we have

$$D_{\alpha'} W = W' + \frac{1}{2}[V_1, W].\tag{2.3}$$

A curve α is called a Frenet curve of osculating order d if its derivatives $\alpha'(s), \alpha''(s), \alpha'''(s), \dots, \alpha^{(d)}(s)$ are linearly dependent and $\alpha'(s), \alpha''(s), \alpha'''(s), \dots, \alpha^{(d+1)}(s)$ are no longer linearly independent for all s . To each Frenet curve of order d one can associate an orthonormal d -frame $V_1(s), V_2(s), V_3(s), \dots, V_d(s)$ along α (such that $\alpha'(s) = V_1(s)$) called the Frenet frame

and the functions $k_1, k_2, \dots, k_{d-1} : I \rightarrow \mathbb{R}$ said to be the Frenet curvatures, such that the Frenet formulas are defined in the usual way:

$$\begin{aligned}
 D_{V_1} V_1(s) &= k_1(s) V_2(s), \\
 D_{V_1} V_2(s) &= -k_1(s) V_1(s) + k_2(s) V_3(s), \\
 &\vdots \\
 D_{V_1} V_i(s) &= -k_{i-1}(s) V_{i-1}(s) + k_i(s) V_{i+1}(s), \\
 D_{V_1} V_{i+1}(s) &= -k_i(s) V_i(s).
 \end{aligned} \tag{2.4}$$

If $\alpha : I \rightarrow G$ is a Frenet curve of osculating order 3 in G , then we define

$$\bar{k}_2(s) = \frac{1}{2} \langle [V_1, V_2], V_3 \rangle. \tag{2.5}$$

Proposition 2.1. *Let α be a Frenet curve of osculating order 3 in G . Then one has*

$$\begin{aligned}
 [V_1, V_2] &= \langle [V_1, V_2], V_3 \rangle V_3 = 2\bar{k}_2 V_3, \\
 [V_1, V_3] &= \langle [V_1, V_3], V_2 \rangle V_2 = -2\bar{k}_2 V_2, \\
 [V_2, V_3] &= \langle [V_2, V_3], V_1 \rangle V_1 = 2\bar{k}_2 V_1.
 \end{aligned} \tag{2.6}$$

Proof. Let α be a Frenet curve of osculating order 3 with the Frenet frame $\{V_1, V_2, V_3\}$. Since $[V_1, V_2] = a_1 V_1 + a_2 V_2 + a_3 V_3$, taking the inner product with V_1, V_2 , and V_3 , respectively, we have $a_1 = a_2 = 0$ and $\langle [V_1, V_2], V_3 \rangle = a_3$. Thus, we find

$$[V_1, V_2] = \langle [V_1, V_2], V_3 \rangle V_3. \tag{2.7}$$

From (2.5), we get

$$[V_1, V_2] = 2\bar{k}_2 V_3. \tag{2.8}$$

By using the above similar method, we can obtain $[V_1, V_3] = -2\bar{k}_2 V_2$ and $[V_2, V_3] = 2\bar{k}_2 V_1$. $\square$

Remark 2.2. Let G be a 3-dimensional Lie group with a bi-invariant metric. Then it is one of the Lie groups $SO(3)$, S^3 or a commutative group, and the following statements hold (see [6, 12]).

- (i) If G is $SO(3)$, then $\bar{k}_2(s) = 1/2$.
- (ii) If G is $S^3 \cong SU(2)$, then $\bar{k}_2(s) = 1$.
- (iii) If G is a commutative group, then $\bar{k}_2(s) = 0$.

Proposition 2.3. *Let α be a Frenet curve of osculating order 3 in G . Then one has*

$$\begin{aligned}
 \alpha'(s) &= V_1(s), \\
 \alpha''(s) &= k_1(s)V_2(s), \\
 \alpha'''(s) &= -k_1^2(s)V_1(s) + k_1'(s)V_2(s) + k_1(s)\tau_1(s)V_3(s), \\
 \alpha''''(s) &= -3k_1(s)k_1'(s)V_1(s) + \left[k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) \right. \\
 &\quad \left. - k_1(s)\bar{k}_2^2(s) \right] V_2(s) + (2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s))V_3(s),
 \end{aligned} \tag{2.9}$$

where $\tau_1(s) = k_2(s) - \bar{k}_2(s)$.

Proof. Let α be a Frenet curve of osculating order 3 in G . Then we have

$$\alpha''(s) = \frac{d^2\alpha}{ds^2} = V_1'(s) = D_{V_1}V_1(s) - \frac{1}{2}[V_1(s), V_1(s)] = k_1(s)V_2(s). \tag{2.10}$$

This implies that

$$\begin{aligned}
 \alpha'''(s) &= k_1'(s)V_2(s) + k_1(s)V_2'(s) \\
 &= k_1'(s)V_2(s) + k_1(s)\left(D_{V_1}V_2(s) - \frac{1}{2}[V_1(s), V_2(s)]\right) \\
 &= k_1'(s)V_2(s) + k_1(s)\left(-k_1(s)V_1(s) + k_3(s) - \bar{k}_2(s)V_3(s)\right) \\
 &= -k_1^2(s)V_1(s) + k_1'(s)V_2(s) + k_1(s)\left(k_2(s) - \bar{k}_2(s)\right)V_3(s).
 \end{aligned} \tag{2.11}$$

Also, we have the following:

$$\begin{aligned}
 \alpha''''(s) &= -2k_1(s)k_1'(s)V_1(s) + k_1''(s)V_2(s) + \left(k_1(s)k_2(s) - k_1(s)\bar{k}_2(s)\right)'V_3(s) \\
 &\quad - k_1^2(s)V_1'(s) + k_1'(s)V_2'(s) + k_1(s)\left(k_2(s) - \bar{k}_2(s)\right)V_3'(s) \\
 &= -2k_1(s)k_1'(s)V_1(s) + k_1''(s)V_2(s) + \left(k_1(s)k_2(s) - k_1(s)\bar{k}_2(s)\right)'V_3(s) \\
 &\quad - k_1^2(s)\left(D_{V_1}V_1(s) - \frac{1}{2}[V_1(s), V_1(s)]\right) + k_1'(s)\left(D_{V_1}V_2(s) - \frac{1}{2}[V_1(s), V_2(s)]\right) \\
 &\quad + k_1(s)\left(k_2(s) - \bar{k}_2(s)\right)\left(D_{V_1}V_3(s) - \frac{1}{2}[V_1(s), V_3(s)]\right)
 \end{aligned}$$

$$\begin{aligned}
&= -3k_1(s)k_1'(s)V_1(s) + \left[k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) \right. \\
&\quad \left. - k_1(s)\bar{k}_2^2(s) \right] V_2(s) + (2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s))V_3(s).
\end{aligned} \tag{2.12}$$

□

Notation. Let we put

$$\begin{aligned}
N_1(s) &= k(s)V_2(s), \\
N_2(s) &= k_1'(s)V_2(s) + k_1(s)\tau_1(s)V_3(s), \\
N_3(s) &= \left[k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s) \right] V_2(s) \\
&\quad + (2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s))V_3(s).
\end{aligned} \tag{2.13}$$

3. Curves of AW(k)-Type

In this section, we consider the properties of curves of AW(k)-type in the Lie group G .

Definition 3.1 (see, cf. [13]). The Frenet curves of osculating order 3 are

(i) of type weak AW(2) if they satisfy

$$N_3(s) = \langle N_3(s), N_2^*(s) \rangle N_2^*(s), \tag{3.1}$$

(ii) of type weak AW(3) if they satisfy

$$N_3(s) = \langle N_3(s), N_1^*(s) \rangle N_1^*(s), \tag{3.2}$$

where

$$\begin{aligned}
N_1^*(s) &= \frac{N_1(s)}{\|N_1(s)\|}, \\
N_2^*(s) &= \frac{N_2(s) - \langle N_2(s), N_1^*(s) \rangle N_1^*(s)}{\|\langle N_2(s), N_1^*(s) \rangle N_1^*(s)\|}.
\end{aligned} \tag{3.3}$$

Definition 3.2 (see [8]). The Frenet curves of osculating order 3 are

(i) of type AW(1) if they satisfy $N_3(s) = 0$,

(ii) of type AW(2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s) \rangle N_2(s), \tag{3.4}$$

(iii) of type AW(3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s) \rangle N_1(s). \quad (3.5)$$

From the definitions of type AW(k), we can obtain the following propositions.

Proposition 3.3. *Let α be a Frenet curve of osculating order 3. Then α is of weak AW(2)-type if and only if*

$$k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s) = 0. \quad (3.6)$$

Proposition 3.4. *Let α be a Frenet curve of osculating order 3. Then α is of weak AW(3)-type if and only if*

$$2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s) = 0. \quad (3.7)$$

Proposition 3.5. *Let α be a Frenet curve of osculating order 3. Then α is of AW(1)-type if and only if*

$$\begin{aligned} k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s) &= 0, \\ k_1^2(s)\tau_1(s) &= c, \end{aligned} \quad (3.8)$$

where c is a constant.

Proposition 3.6. *Let α be a Frenet curve of osculating order 3. Then α is of type AW(2) if and only if*

$$\begin{aligned} &k_1'(s)(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)) \\ &= k_1(s)\tau_1(s) \left(k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s) \right) = 0. \end{aligned} \quad (3.9)$$

Proposition 3.7. *Let α be a Frenet curve of osculating order 3. Then α is of type AW(3) if and only if*

$$k_1^2(s)\tau_1(s) = c, \quad (3.10)$$

where c is a constant.

4. General Helices of AW(k)-Type

In this section, we study general helices of AW(k)-type in the Lie group G with a bi-invariant metric and characterize these curves.

Definition 4.1 (see [6]). Let $\alpha : I \rightarrow G$ be a parameterized curve. Then α is called a general helix if it makes a constant angle with a left-invariant vector field.

Note that in the definition the left-invariant vector field may be assumed to be with unit length, and if the curve α is parametrized by arc-length s , then we have

$$\langle \alpha'(s), X \rangle = \cos \theta, \quad (4.1)$$

for $X \in \mathfrak{g}$, where θ is a constant.

If G is a commutative group $\mathbb{R}^3$, then Definition 4.1 reduces to the classical definition (see [14]). Since a left-invariant vector field in G is a Killing vector field, Definition 4.1 is similar to the definition given in [1].

Theorem 4.2 (see [6]). *A curve of osculating order 3 in G is a general helix if and only if*

$$\tau_1 = ck_1, \quad (4.2)$$

where c is a constant.

From (4.2), a curve with $k_1 \neq 0$ is a general helix if and only if $(\tau_1/k_1)(s) = \text{constant}$. As a Euclidean sense, if both $k_1(s) \neq 0$ and $\tau_1(s)$ are constants, it is a cylindrical helix. We call such a curve a circular helix.

Theorem 4.3. *Let α be a Frenet curve of osculating order 3. Then $\alpha''(s)$, $\alpha'''(s)$, and $\alpha''''(s)$ are linearly dependent if and only if $\alpha(s)$ is general helix.*

Proof. If $\alpha''(s)$, $\alpha'''(s)$, and $\alpha''''(s)$ are linearly dependent, then the following equation holds:

$$\begin{vmatrix} 0 & k_1 & 0 \\ -k_1^2 & k_1' & k_1\tau_1' \\ -3k_1k_1' & k_1'' - k_1^3 - k_1k_2^2 + 2k_1k_2\bar{k}_2 - k_1\bar{k}_2^2 & 2k_1'\tau_1 + k_1\tau_1' \end{vmatrix} = 0. \quad (4.3)$$

By a direct computation, we have

$$k_1\tau_1' - k_1'\tau_1 = 0; \quad (4.4)$$

it follows that

$$\frac{d}{ds} \left(\frac{\tau_1}{k_1} \right) = 0. \quad (4.5)$$

Thus, $\tau_1/k_1 = \text{constant}$; that is, α is general helix. The converse statement is trivial. $\square$

Theorem 4.4. *Let α be a general helix of osculating order 3. Then α is of weak AW(3)-type if and only if α is a circular helix.*

Proof. From (3.7) and (4.2), we can obtain that $k_1 = \text{constant}$; it follows that $\tau_1 = \text{constant}$. Thus, α is a circular helix. The converse statement is trivial. $\square$

Theorem 4.5. *A general helix of type AW(2) has Frenet curvatures*

$$k_1(s) = \frac{1}{\sqrt{-(1+c^2)s^2 + d_1s + d_2}}, \quad \tau_1(s) = ck_1(s), \quad (4.6)$$

where c , d_1 , and d_2 are constants.

Proof. If α is a general helix of type AW(2), then from (3.9) and (4.2) we have

$$\begin{aligned} & k_1'(s)(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)) \\ &= k_1(s)\tau_1(s)\left(k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s)\right) = 0, \end{aligned} \quad (4.7)$$

$$\frac{\tau_1(s)}{k_1(s)} = c; \quad (4.8)$$

where c is a constant.

Combining (4.7) and (4.8), we have

$$k_1(s)k_1''(s) - 3(k_1'(s))^2 - (1+c^2)k_1^4(s) = 0. \quad (4.9)$$

To solve this differential equation, we take

$$k_1(s) = x. \quad (4.10)$$

Then, (4.9) can be rewritten as the form

$$x \frac{d^2x}{ds^2} - 3\left(\frac{dx}{ds}\right)^2 = (1+c^2)x^4. \quad (4.11)$$

Let us put

$$x = y^p. \quad (4.12)$$

Then (4.11) becomes

$$py^{2p-1} \frac{d^2y}{ds^2} - p(2p+1)y^{2p-2} \left(\frac{dy}{ds}\right)^2 = (1+c^2)y^{4p}. \quad (4.13)$$

If we choose $p = -1/2$, then the above equation is

$$\frac{d^2y}{ds^2} = -2(1+c^2), \quad (4.14)$$

its general solution is given by

$$y = -(1 + c^2)s^2 + d_1s + d_2, \quad (4.15)$$

where d_1 and d_2 are constants.

Thus, we have

$$k_1(s) = \frac{1}{\sqrt{-(1 + c^2)s^2 + d_1s + d_2}}, \quad (4.16)$$

so, the theorem is proved. $\square$

Corollary 4.6. *There exists no a circular helix of osculating order 3 of type AW(2) in G .*

Theorem 4.7. *Let α be a general helix of osculating order 3. Then α is of type AW(3) if and only if α is a circular helix.*

Proof. Suppose that α is a general helix of type AW(3). Combining (3.10) and (4.2) we find $k_1^3(s) = 1$, that is, $k_1(s) = 1$. From this $\tau_1(s) = c$. Thus, α is a circular helix. $\square$

Theorem 4.8. *Let α be a curve of osculating order 3. There exists no a general helix of type AW(1).*

Proof. We assume that α is a general helix of type AW(1). Then from (3.8) and (4.2) we have

$$k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\bar{k}_2(s) - k_1(s)\bar{k}_2^2(s) = 0, \quad (4.17)$$

$$k_1^2(s)\tau_1(s) = c, \quad (4.18)$$

$$\tau_1(s) = ck_1(s). \quad (4.19)$$

From (4.18) and (4.19), we have

$$k_1(s) = 1. \quad (4.20)$$

Thus, (4.17) becomes

$$k_2^2(s) - 2k_2(s)\bar{k}_2(s) + \bar{k}_2^2(s) = -1, \quad (4.21)$$

equivalently to

$$\left(k_2(s) - \bar{k}_2(s)\right)^2 = -1. \quad (4.22)$$

It is impossible, so the theorem is proved. $\square$

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