Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 580158, 18 pages
doi:10.1155/2012/580158

Research Article

Algorithms for a System of General Variational Inequalities in Banach Spaces

Jin-Hua Zhu,[1] Shih-Sen Chang,[2] and Min Liu[1]

[1] *Department of Mathematics, Yibin University, Sichuan, Yibin 644007, China*
[2] *College of Statistics and Mathematics, Yunnan University of Finance and Economics, Yunnan, Kunming 650221, China*

Correspondence should be addressed to Shih-Sen Chang, changss@yahoo.cn

Received 21 December 2011; Accepted 6 February 2012

Academic Editor: Zhenyu Huang

The purpose of this paper is using Korpelevich's extragradient method to study the existence problem of solutions and approximation solvability problem for a class of systems of finite family of general nonlinear variational inequality in Banach spaces, which includes many kinds of variational inequality problems as special cases. Under suitable conditions, some existence theorems and approximation solvability theorems are proved. The results presented in the paper improve and extend some recent results.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*.

In this paper, we are concerned a finite family of a general system of nonlinear variational inequalities in Banach spaces, which involves finding $(x_1^*, x_2^*, \ldots, x_n^*) \in C \times C \times \cdots \times C$ such that

$$\begin{gathered}
\langle \lambda_1 A_1 x_2^* + x_1^* - x_2^*, j(x - x_1^*) \rangle \geq 0, \quad \forall x \in C, \\
\langle \lambda_2 A_2 x_3^* + x_2^* - x_3^*, j(x - x_2^*) \rangle \geq 0, \quad \forall x \in C, \\
\langle \lambda_3 A_3 x_4^* + x_3^* - x_4^*, j(x - x_3^*) \rangle \geq 0, \quad \forall x \in C, \\
\vdots \\
\langle \lambda_{N-1} A_{N-1} x_N^* + x_{N-1}^* - x_N^*, j(x - x_{N-1}^*) \rangle \geq 0, \quad \forall x \in C, \\
\langle \lambda_N A_N x_1^* + x_N^* - x_1^*, j(x - x_N^*) \rangle \geq 0, \quad \forall x \in C,
\end{gathered} \tag{1.1}$$

where $\{A_i : C \to E,\ i = 1, 2, \ldots, N\}$ is a finite family of nonlinear mappings and λ_i $(i = 1, 2, \ldots, N)$ are positive real numbers.

As special cases of the problem (1.1), we have the following.

(I) If E is a real Hilbert space and $N = 2$, then (1.1) reduces to

$$\begin{aligned} &\langle \lambda_1 A_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \lambda_2 A_2 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.2}$$

which was considered by Ceng et al. [1]. In particular, if $A_1 = A_2 = A$, then the problem (1.2) reduces to finding $(x_1^*, x_2^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda_1 A x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \lambda_2 A x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.3}$$

which is defined by Verma [2]. Furthermore, if $x_1^* = x_2^*$, then (1.3) reduces to the following variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.4}$$

This problem is a fundamental problem in variational analysis and, in particular, in optimization theory. Many algorithms for solving this problem are projection algorithms that employ projections onto the feasible set C of the VI or onto some related set, in order to iteratively reach a solution. In particular, Korpelevich's extragradient method which was introduced by Korpelevich [3] in 1976 generates a sequence $\{x_n\}$ via the recursion

$$\begin{aligned} y_n &= P_C[x_n - \lambda A x_n], \\ x_{n+1} &= P_C[x_n - \lambda A y_n], \quad n \geq 0, \end{aligned} \tag{1.5}$$

where P_C is the metric projection from $\mathbb{R}^n$ onto C, $A : C \to H$ is a monotone operator, and λ is a constant. Korpelevich [3] proved that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, \mathrm{A})$. Note that the setting of the space is Euclid space $\mathbb{R}^n$.

The literature on the VI is vast, and Korpelevich's extragradient method has received great attention by many authors, who improved it in various ways. See, for example, [4–16] and references therein.

(II) If E is still a real Banach space and $N = 1$, then the problem (1.1) reduces to finding $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in \mathrm{C}, \tag{1.6}$$

which was considered by Aoyama et al. [17]. Note that this problem is connected with the fixed point problem for nonlinear mapping, the problem of finding a zero point of a nonlinear operator, and so on. It is clear that problem (1.6) extends problem (1.4) from Hilbert spaces to Banach spaces.

In order to find a solution for problem (1.6), Aoyama et al. [17] introduced the following iterative scheme for an accretive operator A in a Banach space E:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\Pi_C(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.7}$$

where Π_C is a sunny nonexpansive retraction from E to C. Then they proved a weak convergence theorem in a Banach space. For related works, please see [18] and the references therein.

It is an interesting problem of constructing some algorithms with strong convergence for solving problem (1.1) which contains problem (1.6) as a special case.

Our aim in this paper is to construct two algorithms for solving problem (1.1). For this purpose, we first prove that the system of variational inequalities (1.1) is equivalent to a fixed point problem of some nonexpansive mapping. Finally, we prove the strong convergence of the proposed methods which solve problem (1.1).

2. Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \ \|f\| = \|x\|^{q-1} \right\} \tag{2.1}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{n \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the previous limit is attained uniformly for $x, y \in U$. The norm of E is said to be Fréchet differentiable if, for each $x \in U$, the previous limit is attained uniformly for all $y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \ \|x\| = 1, \ \|y\| = \tau \right\}, \tag{2.4}$$

where $\rho : [0, \infty) \to [0, \infty)$ is function. It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0}(\rho(\tau)/\tau) = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is

said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. Note the following.

(1) E is a uniformly smooth Banach space if and only if J is single valued and uniformly continuous on any bounded subset of E.

(2) All Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolev spaces $W_m^p (p \geq 2)$ are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

In our paper, we focus on a 2-uniformly smooth Banach space with the smooth constant K.

Let E be a real Banach space, C a nonempty closed convex subset of E, $T : C \to C$ a mapping, and $F(T)$ the set of fixed points of T.

Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

A bounded linear operator $F : C \in E$ is called strongly positive if there exists a constant $\gamma > 0$ with the property

$$\langle F(x), j(x) \rangle \geq \gamma \|x\|^2, \quad \forall x \in C. \tag{2.6}$$

A mapping $A : C \to E$ is said to be accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \tag{2.7}$$

for all $x, y \in C$, where J is the duality mapping.

A mapping A of C into E is said to be α-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2, \tag{2.8}$$

for all $x, y \in C$.

A mapping A of C into E is said to be α-inverse-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|Ax - Ay\|^2, \tag{2.9}$$

for all $x, y \in C$.

Remark 2.1. Evidently, the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping, which was studied by so many authors; see, for instance, [6, 19, 20].

Let D be a subset of C, and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x) \tag{2.10}$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. Then following lemma concerns the sunny nonexpansive retraction.

Lemma 2.2 (see [21]). *Let C be a closed convex subset of a smooth Banach space E, let D be a nonempty subset of C, and let Π be a retraction from C onto D. Then Π is sunny and nonexpansive if and only if*

$$\langle u - \Pi(u), j(y - \Pi(u)) \rangle \leq 0, \tag{2.11}$$

for all $u \in C$ and $y \in D$.

Remark 2.3. (1) It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from E onto C.

(2) Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with the set $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.

In what follows, we need the following lemmas for proof of our main results.

Lemma 2.4 (see [22]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{2.12}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\Sigma_{n=1}^{\infty} \gamma_n = \infty$,

(b) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.5 (see [23]). *Let X be a Banach space, $\{x_n\}, \{y_n\}$ be two bounded sequences in X and $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{2.13}$$

Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n$, for all $n \geq 1$ and

$$\limsup_{n\to\infty} \{ \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \} \leq 0, \tag{2.14}$$

then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.6 (see [24]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.15}$$

Lemma 2.7 (see [25]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E, and let G be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Gx_n \to 0$, then x is a fixed point of G.*

Lemma 2.8 (see [26]). *Let C be a nonempty closed convex subset of a real Banach space E. Assume that the mapping $F : C \to E$ is accretive and weakly continuous along segments (i.e., $F(x+ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, j(x-x^*)\rangle \ge 0, \quad x \in C, \tag{2.16}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, j(x-x^*)\rangle \ge 0, \quad x \in C. \tag{2.17}$$

Lemma 2.9. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive. For given $(x_1^*, x_2^*, \ldots, x_n^*) \in C \times C \times \cdots \times C$, where $x^* = x_1^*$, $x_i^* = \Pi_C(I-\lambda_i A_i)x_{i+1}^*$, $i \in \{i, 2, \ldots, N-1\}$, $x_N^* = \Pi_C(I-\lambda_N A_N)x_1^*$, then $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution of the problem (1.1) if and only if x^* is a fixed point of the mapping Q defined by*

$$Q(x) = \Pi_C(I-\lambda_1 A_1)\Pi_C(I-\lambda_2 A_2)\cdots\Pi_C(I-\lambda_N A_N)(x), \tag{2.18}$$

where λ_i $(i = 1,2,\ldots,N)$ are real numbers.

Proof. We can rewrite (1.1) as

$$\begin{gathered}
\langle x_1^* - (x_2^* - \lambda_1 A_1 x_2^*), j(x - x_1^*)\rangle \ge 0, \quad \forall x \in C, \\
\langle x_2^* - (x_3^* - \lambda_2 A_2 x_3^*), j(x - x_2^*)\rangle \ge 0, \quad \forall x \in C, \\
\langle x_3^* - (x_4^* - \lambda_3 A_3 x_4^*), j(x - x_3^*)\rangle \ge 0, \quad \forall x \in C, \\
\vdots \\
\langle x_{N-1}^* - (x_N^* - \lambda_{N-1} A_{N-1} x_N^*), j(x - x_{N-1}^*)\rangle \ge 0, \quad \forall x \in C, \\
\langle x_N^* - (x_1^* - \lambda_N A_N x_1^*), j(x - x_N^*)\rangle \ge 0, \quad \forall x \in C.
\end{gathered} \tag{2.19}$$

By Lemma 2.2, we can check (2.19) is equivalent to

$$\begin{aligned}
x_1^* &= \Pi_C(I-\lambda_1 A_1)x_2^*,\\
x_2^* &= \Pi_C(I-\lambda_2 A_2)x_3^*,\\
&\vdots\\
x_{N-1}^* &= \Pi_C(I-\lambda_{N-1} A_{N-1})x_N^*,\\
x_N^* &= \Pi_C(I-\lambda_N A_N)x_1^*.\\
&\Longleftrightarrow\\
Q(x^*) &= \Pi_C(I-\lambda_1 A_1)\Pi_C(I-\lambda_2 A_2)\cdots\Pi_C(I-\lambda_N A_N)(x^*) = x^*.
\end{aligned} \tag{2.20}$$

This completes the proof. □

Throughout this paper, the set of fixed points of the mapping Q is denoted by Ω.

Lemma 2.10. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive. Let Q be defined as Lemma 2.9. If $0 \le \lambda_i \le \gamma_i/K^2$, then $Q : C \to C$ is nonexpansive.*

Proof. First, we show that for all $i \in \{i,2,\ldots,N\}$, the mapping $\Pi_C(I-\lambda_i A_i)$ is nonexpansive. Indeed, for all $x, y \in C$, from the condition $\lambda_i \in [0,\gamma_i/K^2]$ and Lemma 2.6, we have

$$\begin{aligned}
\left\|\Pi_C(I-\lambda_i A_i)x - \Pi_C(I-\lambda_i A_i)y\right\|^2 &\le \left\|(I-\lambda_i A_i)x - (I-\lambda_i A_i)y\right\|^2\\
&= \left\|(x-y) - \lambda_i(A_i x - A_i y)\right\|^2\\
&\le \|x-y\|^2 - 2\lambda_i\langle A_i x - A_i y, j(x-y)\rangle\\
&\quad + 2K^2\lambda_i^2\|A_i x - A_i y\|^2\\
&\le \|x-y\|^2 - 2\lambda_i\gamma_i\|A_i x - A_i y\|^2 + 2K^2\lambda_i^2\|A_i x - A_i y\|^2\\
&= \|x-y\|^2 + 2\lambda_i\left(K^2\lambda_i - \gamma_i\right)\|A_i x - A_i y\|^2\\
&\le \|x-y\|^2,
\end{aligned} \tag{2.21}$$

which implies for all $i \in \{1,2,\ldots,N\}$, the mapping $\Pi_C(I-\lambda_i A_i)$ is nonexpansive, so is the mapping Q. □

3. Main Results

In this section, we introduce our algorithms and show the strong convergence theorems.

Algorithm 3.1. Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E to C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive. Let $B : C \to E$ be a strongly positive bounded linear operator with coefficient $\alpha > 0$ and $F : C \to E$ be a strongly positive bounded linear operator with coefficient $\rho \in (0,\alpha)$. For any $t \in (0,1)$, define a net $\{x_t\}$ as follows:

$$\begin{aligned} x_t &= \Pi_C(tF + (I - tB))y_t, \\ y_t &= \Pi_C(I - \lambda_1 A_1)\Pi_C(I - \lambda_2 A_2)\cdots\Pi_C(I - \lambda_N A_N)x_t, \end{aligned} \tag{3.1}$$

where, for any i, $\lambda_i \in (0,\ \gamma_i/K^2)$ is a real number.

Remark 3.2. We notice that the net $\{x_t\}$ defined by (3.1) is well defined. In fact, we can define a self-mapping $W_t : C \to C$ as follows:

$$W_t x := \Pi_C(tF + (I - tB))\Pi_C(I - \lambda_1 A_1)\Pi_C(I - \lambda_2 A_2)\cdots\Pi_C(I - \lambda_N A_N)x, \quad \forall x \in C. \tag{3.2}$$

From Lemma 2.10, we know that if, for any i, $\lambda_i \in (0,\gamma_i/K^2)$, the mapping $\Pi_C(I - \lambda_1 A_1)\Pi_C(I - \lambda_2 A_2)\cdots\Pi_C(I - \lambda_N A_N) = Q$ is nonexpansive and $\|I - tB\| \le 1 - t\alpha$. Then, for any $x, y \in C$, we have

$$\begin{aligned} \|W_t x - W_t y\| &= \|\Pi_C(tF + (I - tB))Q(x) - \Pi_C(tF + (I - tB))Q(y)\| \\ &\le \|((tF + (I - tB))Q)x - ((tF + (I - tB))Q)y\| \\ &= \|t(Fx - Fy) + (I - tB)(Qx - Qy)\| \\ &\le t\rho\|x - y\| + \|I - tB\|\|Qx - Qy\| \\ &\le t\rho\|x - y\| + (1 - t\alpha)\|x - y\| \\ &= (1 - (\alpha - \rho)t)\|x - y\|. \end{aligned} \tag{3.3}$$

This shows that the mapping W_t is contraction. By Banach contractive mapping principle, we immediately deduce that the net (3.1) is well defined.

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E to C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive. Let $B : C \to E$ be a strongly positive bounded linear operator with coefficient $\alpha > 0$, and let $F : C \to E$ be a strongly positive bounded linear operator with coefficient $\rho \in (0,\alpha)$. Assume that $\Omega \neq \emptyset$ and $\lambda_i \in (0,\ \gamma_i/K^2)$. Then the*

net $\{x_t\}$ *generated by the implicit method* (3.1) *converges in norm, as* $t \to 0^+$ *to the unique solution* $\tilde{x}$ *of VI*

$$\tilde{x} \in \Omega, \quad \langle (B-F)\tilde{x}, j(z-\tilde{x}) \rangle \geq 0, \quad z \in \Omega. \tag{3.4}$$

Proof. We divide the proof of Theorem 3.3 into four steps.

(I) Next we prove that the net $\{x_t\}$ is bounded.

Take that $x^* \in \Omega$, we have

$$\begin{aligned}
\|x_t - x^*\| &= \|\Pi_C(tF + (I - tB))y_t - x^*\| \\
&= \|\Pi_C(tF + (I - tB))y_t - \Pi_C x^*\| \\
&\leq \|(tF + (I - tB))y_t - x^*\| \\
&= \|t(F(y_t) - F(x^*)) + (I - tB)(y_t - x^*) + tF(x^*) - tB(x^*)\| \\
&\leq t\|F(y_t) - F(x^*)\| + \|I - tB\|\|y_t - x^*\| + t\|F(x^*) - B(x^*)\| \\
&\leq t\rho\|y_t - x^*\| + (1 - t\alpha)\|y_t - x^*\| + t\|F(x^*) - B(x^*)\| \\
&= (1 - (\alpha - \rho)t)\|y_t - x^*\| + t\|F(x^*) - B(x^*)\| \\
&= (1 - (\alpha - \rho)t)\|Q(x_t) - Q(x^*)\| + t\|F(x^*) - B(x^*)\| \\
&\leq (1 - (\alpha - \rho)t)\|x_t - x^*\| + t\|F(x^*) - B(x^*)\|.
\end{aligned} \tag{3.5}$$

It follows that

$$\|x_t - x^*\| \leq \frac{\|F(x^*) - B(x^*)\|}{\alpha - \rho}. \tag{3.6}$$

Therefore, $\{x_t\}$ is bounded. Hence, $\{y_t\}, \{By_t\}, \{A_i x_t\}$, and $\{F(y_t)\}$ are also bounded. We observe that

$$\begin{aligned}
\|x_t - y_t\| &= \|\Pi_C(tF + (I - tB))y_t - \Pi_C y_t\| \\
&\leq \|(tF + (I - tB))y_t - y_t\| \\
&= t\|F(y_t) - B(y_t)\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.7}$$

From Lemma 2.10, we know that $Q : C \to C$ is nonexpansive. Thus, we have

$$\|y_t - Q(y_t)\| = \|Q(x_t) - Q(y_t)\| \leq \|x_t - y_t\| \longrightarrow 0. \tag{3.8}$$

Therefore,

$$\lim_{t\to 0}\|x_t - Q(x_t)\| = 0. \tag{3.9}$$

(II) $\{x_t\}$ is relatively norm-compact as $t \to 0^+$.

Let $\{t_n\} \subset (0,1)$ be any subsequence such that $t_n \to 0^+$ as $n \to \infty$. Then, there exists a positive integer n_0 such that $0 < t_n < 1/2$, for all $n \geq n_0$. Let $x_n := x_{t_n}$. It follows from (3.9) that

$$\|x_n - Q(x_n)\| \longrightarrow 0. \tag{3.10}$$

We can rewrite (3.1) as

$$x_t = \Pi_C(tF + (I - tB))y_t - (tF + (I - tB))y_t + (tF + (I - tB))y_t. \tag{3.11}$$

For any $x^* \in \Omega \subset C$, by Lemma 2.2, we have

$$\begin{aligned}&\langle (tF + (I - tB))y_t - x_t, j(x^* - x_t)\rangle \\ &\quad = \langle (tF + (I - tB))y_t - \Pi_C(tF + (I - tB))y_t, j(x^* - \Pi_C(tF + (I - tB))y_t)\rangle \leq 0.\end{aligned} \tag{3.12}$$

With this fact, we derive that

$$\begin{aligned}\|x_t - x^*\|^2 &= \langle x^* - x_t, j(x^* - x_t)\rangle \\ &= \langle x^* - (tF + (I - tB))y_t, j(x^* - x_t)\rangle \\ &\quad + \langle (tF + (I - tB))y_t - \Pi_C(tF + (I - tB))y_t, j(x^* - x_t)\rangle \\ &\leq \langle (tF + (I - tB))(x^* - y_t), j(x^* - x_t)\rangle + t\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle \\ &\leq (1 - t(\alpha - \rho))\|x^* - y_t\|\|x^* - x_t\| + t\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle \\ &= (1 - t(\alpha - \rho))\|Q(x^*) - Q(x_t)\|\|x^* - x_t\| + t\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle \\ &\leq (1 - t(\alpha - \rho))\|x^* - x_t\|\|x^* - x_t\| + t\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle \\ &\leq (1 - t(\alpha - \rho))\|x^* - x_t\|^2 + t\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle.\end{aligned} \tag{3.13}$$

It turns out that

$$\|x_t - x^*\|^2 \leq \frac{1}{\alpha - \rho}\langle B(x^*) - F(x^*), j(x^* - x_t)\rangle, \quad x^* \in \Omega. \tag{3.14}$$

In particular,

$$\|x_n - x^*\|^2 \leq \frac{1}{\alpha - \rho}\langle B(x^*) - F(x^*), j(x^* - x_n)\rangle, \quad x^* \in \Omega. \tag{$**$}$$

Since $\{x_n\}$ is bounded, without loss of generality, $x_n \rightharpoonup \tilde{x} \in C$ can be assumed. Noticing (3.10), we can use Lemma 2.7 to get $\tilde{x} \in \Omega = F(Q)$. Therefore, we can substitute $\tilde{x}$ for x^* in $(**)$ to get

$$\|x_n - \tilde{x}\|^2 \leq \frac{1}{\alpha - \rho}\langle B(\tilde{x}) - F(\tilde{x}), j(\tilde{x} - x_n)\rangle. \tag{3.15}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0^+$.

(III) Now, we prove that $\tilde{x}$ solves the variational inequality (3.4). From (3.1), we have

$$\begin{aligned}
x_t &= \Pi_C(tF + (I - tB))y_t - (tF + (I - tB))y_t + (tF + (I - tB))y_t \\
\Longrightarrow x_t &= \Pi_C(tF + (I - tB))y_t - (tF + (I - tB))y_t - (tF + (I - tB))(x_t - y_t) \\
&\quad + tF(x_t) + (I - tB)(x_t) \\
\Longrightarrow F(x_t) - B(x_t) &= \frac{1}{t}\left[(tF + (I - tB))y_t - \Pi_C(tF + (I - tB))y_t - (tF + (I - tB))(y_t - x_t)\right].
\end{aligned} \tag{3.16}$$

For any $z \in \Omega$, we obtain

$$\begin{aligned}
\langle F(x_t) - B(x_t), j(z - x_t)\rangle &= \frac{1}{t}\langle (tF + (I - tB))y_t - \Pi_C(tF + (I - tB))y_t, j(z - x_t)\rangle \\
&\quad - \frac{1}{t}\langle (tF + (I - tB))(y_t - x_t), j(z - x_t)\rangle \\
&\leq -\frac{1}{t}\langle (tF + (I - tB))(y_t - x_t), j(z - x_t)\rangle \\
&= -\frac{1}{t}\langle y_t - x_t, j(z - x_t)\rangle + \langle (B - F)(y_t - x_t), j(z - x_t)\rangle.
\end{aligned} \tag{3.17}$$

Now we prove that $\langle y_t - x_t, j(z - x_t)\rangle \geq 0$. In fact, we can write $y_t = Q(x_t)$. At the same time, we note that $z = Q(z)$, so

$$\langle y_t - x_t, j(z - x_t)\rangle = \langle z - Q(z) - (x_t - Q(x_t)), j(z - x_t)\rangle. \tag{3.18}$$

Since $I - Q$ is accretive (this is due to the nonexpansivity of Q), we can deduce immediately that

$$\langle y_t - x_t, j(z - x_t)\rangle = \langle z - Q(z) - (x_t - Q(x_t)), j(z - x_t)\rangle \geq 0. \tag{3.19}$$

Therefore,

$$\langle F(x_t) - B(x_t), j(z - x_t)\rangle \leq \langle (B - F)(y_t - x_t), j(z - x_t)\rangle. \tag{3.20}$$

Since B, F is strongly positive, we have

$$\begin{aligned} 0 &\leq (\alpha - \rho)\|z - x_t\|^2 \leq \langle (B - F)(z - x_t), j(z - x_t)\rangle \\ &= \langle (F(x_t) - B(x_t)) - (F(z) - B(z)), j(z - x_t)\rangle. \end{aligned} \tag{3.21}$$

It follows that

$$\langle F(z) - B(z), j(z - x_t)\rangle \leq \langle F(x_t) - B(x_t), j(z - x_t)\rangle. \tag{3.22}$$

Combining (3.20) and (3.22), we get

$$\langle F(z) - B(z), j(z - x_t)\rangle \leq \langle (B - F)(y_t - x_t), j(z - x_t)\rangle. \tag{3.23}$$

Now replacing t in (3.23) with t_n and letting $n \to \infty$, noticing that $x_{t_n} - y_{t_n} \to 0$, we obtain

$$\langle F(z) - B(z), j(z - \tilde{x})\rangle \leq 0, \quad z \in \Omega, \tag{3.24}$$

which is equivalent to its dual variational inequality (see Lemma 2.8)

$$\langle (B - F)\tilde{x}, j(z - \tilde{x})\rangle \geq 0, \quad z \in \Omega, \tag{3.25}$$

that is, $\tilde{x} \in \Omega$ is a solution of (3.4).

(IV) Now we show that the solution set of (3.4) is singleton.

As a matter of fact, we assume that $x^* \in \Omega$ is also a solution of (3.4) Then, we have

$$\langle (B - F)x^*, j(\tilde{x} - x^*)\rangle \geq 0. \tag{3.26}$$

From (3.25), we have

$$\langle (B - F)\tilde{x}, j(x^* - \tilde{x})\rangle \geq 0. \tag{3.27}$$

So,

$$\begin{aligned} &\langle (B - F)x^*, j(\tilde{x} - x^*)\rangle + \langle (B - F)\tilde{x}, j(x^* - \tilde{x})\rangle \geq 0 \\ &\Longrightarrow \langle (B - F)(\tilde{x} - x^*), j(x^* - \tilde{x})\rangle \geq 0 \\ &\Longrightarrow \langle (B - F)(x^* - \tilde{x}), j(x^* - \tilde{x})\rangle \leq 0 \\ &\Longrightarrow (\alpha - \rho)\|x^* - \tilde{x}\|^2 \leq 0. \end{aligned} \tag{3.28}$$

Therefore, $x^* = \tilde{x}$. In summary, we have shown that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

Next, we introduce our explicit method which is the discretization of the implicit method (3.1).

Algorithm 3.4. Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E to C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive. Let $B : C \to E$ be a strongly positive bounded linear operator with coefficient $\alpha > 0$, and let $F : C \to E$ be a strongly positive bounded linear operator with coefficient $\rho \in (0,\alpha)$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\Pi_C(\alpha_n F + (I-\alpha_n B))\Pi_C(I-\lambda_1 A_1)\Pi_C(I-\lambda_2 A_2)\cdots\Pi_C(I-\lambda_N A_N)x_n, \quad n \geq 0, \tag{3.29}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and, for any i, $\lambda_i \in (0,\gamma_i/K^2)$ is a real number.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E, and let Π_C be a sunny nonexpansive retraction from E to C. Let $\{A_i : C \to E,\ i = 1,2,\ldots,N\}$ be a finite family of γ_i-inverse-strongly accretive.Let $B : C \to E$ be a strongly positive bounded linear operator with coefficient $\alpha > 0$, and let $F : C \to E$ be a strongly positive bounded linear operator with coefficient $\rho \in (0,\alpha)$. Assume that $\Omega \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be generated iteratively by* (3.29). *Suppose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n \leq 1$.

Then $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$ which solves the variational inequality (3.4).

Proof. Set $y_n = \Pi_C(I-\lambda_1 A_1)\Pi_C(I-\lambda_2 A_2)\cdots\Pi_C(I-\lambda_N A_N)x_n$ for all $n \geq 0$. Then $x_{n+1} = \beta_n x_n + (1-\beta_n)\Pi_C(\alpha_n F + (I-\alpha_n B))y_n$ for all $n \geq 0$. Pick up $x^* \in \Omega$.

From Lemma 2.10, we have

$$\|y_n - x^*\| = \|Q(x_n) - Q(x^*)\| \leq \|x_n - x^*\|. \tag{3.30}$$

Hence, it follows that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\beta_n x_n + (1-\beta_n)\Pi_C(\alpha_n F + (I-\alpha_n B))y_n - x^*\| \\ &= \|\beta_n(x_n - x^*) + (1-\beta_n)(\Pi_C(\alpha_n F + (I-\alpha_n B))y_n - x^*)\|\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\left\|\Pi_C(\alpha_n F + (I-\alpha_n B))y_n - \Pi_C x^*\right\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\left\|(\alpha_n F + (I-\alpha_n B))y_n - x^*\right\| \\
&= \beta_n\|x_n - x^*\| + (1-\beta_n)\left\|(\alpha_n F + (I-\alpha_n B))(y_n - x^*) + \alpha_n(F(x^*) - B(x^*))\right\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)(\alpha_n\rho + (1-\alpha_n\alpha))\left\|y_n - x^*\right\| + (1-\beta_n)\alpha_n\|F(x^*) - B(x^*)\| \\
&\leq (1-\alpha_n(1-\beta_n)(\alpha-\rho))\|x_n - x^*\| + \alpha_n(1-\beta_n)(\alpha-\rho)\frac{\|F(x^*) - B(x^*)\|}{\alpha-\rho}.
\end{aligned}
\tag{3.31}
$$

By induction, we deduce that

$$
\|x_{n+1} - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|F(x^*) - B(x^*)\|}{\alpha-\rho}\right\}. \tag{3.32}
$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{A_i x_i\}$ $(i = 1, 2, \ldots, N)$, $\{y_n\}$, $\{By_n\}$, and $\{F(y_n)\}$ are also bounded. We observe that

$$
\left\|y_{n+1} - y_n\right\| = \|Q(x_{n+1}) - Q(x_n)\| \leq \|x_{n+1} - x_n\|. \tag{3.33}
$$

Set $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all $n \geq 0$. Then $z_n = \Pi_C(\alpha_n F + (I-\alpha_n B))y_n$. It follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \left\|\Pi_C(\alpha_{n+1}F + (I-\alpha_{n+1}B))y_{n+1} - \Pi_C(\alpha_n F + (I-\alpha_n B))y_n\right\| \\
&\leq \left\|(\alpha_{n+1}F + (I-\alpha_{n+1}B))y_{n+1} - (\alpha_n F + (I-\alpha_n B))y_n\right\| \\
&= \left\|y_{n+1} - y_n + \alpha_{n+1}(F(y_{n+1}) - B(y_{n+1})) - \alpha_n(F(y_n) - B(y_n))\right\| \\
&\leq \left\|y_{n+1} - y_n\right\| + \alpha_{n+1}\left\|F(y_{n+1}) - B(y_{n+1})\right\| - \alpha_n\left\|F(y_n) - B(y_n)\right\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\left\|F(y_{n+1}) - B(y_{n+1})\right\| - \alpha_n\left\|F(y_n) - B(y_n)\right\|.
\end{aligned}
\tag{3.34}
$$

This implies that

$$
\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.35}
$$

Hence, by Lemma 2.5, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently,

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.36}
$$

At the same time, we note that

$$\begin{aligned}\|z_n - y_n\| &= \|\Pi_C(\alpha_n F + (I - \alpha_n B))y_n - y_n\| \\ &= \|\Pi_C(\alpha_n F + (I - \alpha_n B))y_n - \Pi_C y_n\| \\ &\le \|(\alpha_n F + (I - \alpha_n B))y_n - y_n\| \\ &= \alpha_n \|F(y_n) - B(y_n)\| \\ &\longrightarrow 0.\end{aligned} \tag{3.37}$$

It follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.38}$$

From Lemma 2.10, we know that $Q : C \to C$ is nonexpansive. Thus, we have

$$\|y_n - Q(y_n)\| = \|Q(x_n) - Q(y_n)\| \le \|x_n - y_n\| \longrightarrow 0. \tag{3.39}$$

Thus, $\lim_{n\to\infty}\|x_n - Q(x_n)\| = 0$. We note that

$$\begin{aligned}\|z_n - Q(z_n)\| &\le \|z_n - x_n\| + \|x_n - Q(x_n)\| + \|Q(x_n) - Q(z_n)\| \\ &\le 2\|z_n - x_n\| + \|x_n - Q(x_n)\| \\ &= 2\|\Pi_C(\alpha_n F + (I - \alpha_n B))y_n - \Pi_C x_n\| + \|x_n - Q(x_n)\| \\ &\le 2(\|y_n - x_n\| + \alpha_n \|F(y_n) - B(y_n)\|) + \|x_n - Q(x_n)\| \\ &\longrightarrow 0.\end{aligned} \tag{3.40}$$

Next, we show that

$$\limsup_{n\to\infty} \langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle \le 0, \tag{3.41}$$

where $\tilde{x} \in \Omega$ is the unique solution of VI(3.4).

To see this, we take a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that

$$\lim_{n\to\infty} \langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle = \lim_{n_j\to\infty} \left\langle F(\tilde{x}) - B(\tilde{x}), j\left(z_{n_j} - \tilde{x}\right)\right\rangle. \tag{3.42}$$

We may also assume that $z_{n_j} \rightharpoonup z$. Note that $z \in \Omega$ in virtue of Lemma 2.7 and (3.40). It follows from the variational inequality (3.4) that

$$\begin{aligned}\lim_{n\to\infty} \langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle &= \lim_{n_j\to\infty} \left\langle F(\tilde{x}) - B(\tilde{x}), j\left(z_{n_j} - \tilde{x}\right)\right\rangle \\ &= \langle F(\tilde{x}) - B(\tilde{x}), j(z - \tilde{x})\rangle \le 0.\end{aligned} \tag{3.43}$$

Since $z_n = \Pi_C(\alpha_n F + (I - \alpha_n B))y_n$, according to Lemma 2.2, we have

$$\langle (\alpha_n F + (I - \alpha_n B))y_n - \Pi_C(\alpha_n F + (I - \alpha_n B))y_n, j(\tilde{x} - z_n)\rangle \leq 0. \tag{3.44}$$

From (3.44), we have

$$\begin{aligned}
\|z_n - \tilde{x}\|^2 &= \langle \Pi_C(\alpha_n F + (\mathrm{I} - \alpha_n B))y_n - \tilde{x}, j(z_n - \tilde{x})\rangle \\
&= \langle \Pi_C(\alpha_n F + (I - \alpha_n B))y_n - (\alpha_n F + (I - \alpha_n B))y_n, j(z_n - \tilde{x})\rangle \\
&\quad + \langle (\alpha_n F + (I - \alpha_n B))y_n - \tilde{x}, j(z_n - \tilde{x})\rangle \\
&\leq \langle (\alpha_n F + (I - \alpha_n B))y_n - \tilde{x}, j(z_n - \tilde{x})\rangle \\
&= \langle (\alpha_n F + (I - \alpha_n B))(y_n - \tilde{x}), j(z_n - \tilde{x})\rangle + \alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle \\
&\leq (1 - \alpha_n(\alpha - \rho))\|y_n - \tilde{x}\|\|z_n - \tilde{x}\| + \alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle \\
&\leq \frac{(1 - \alpha_n(\alpha - \rho))^2}{2}\|y_n - \tilde{x}\|^2 + \frac{1}{2}\|z_n - \tilde{x}\|^2 + \alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle.
\end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned}
\|z_n - \tilde{x}\|^2 &\leq (1 - \alpha_n(\alpha - \rho))\|y_n - \tilde{x}\|^2 + 2\alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle, \\
&\leq (1 - \alpha_n(\alpha - \rho))\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle.
\end{aligned} \tag{3.46}$$

Finally, we prove $x_n \to \tilde{x}$. From $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ and (3.46), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \beta_n\|x_n - \tilde{x}\|^2 + (1 - \beta_n)\|z_n - \tilde{x}\|^2 \\
&\leq \beta_n\|x_n - \tilde{x}\|^2 + (1 - \beta_n)\Big((1 - \alpha_n(\alpha - \rho))\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle\Big) \\
&= (1 - \alpha_n(1 - \beta_n)(\alpha - \rho))\|x_n - \tilde{x}\|^2 + \alpha_n(1 - \beta_n)(\alpha - \rho) \\
&\quad \times \left\{\frac{2}{\alpha - \rho}\langle F(\tilde{x}) - B(\tilde{x}), j(z_n - \tilde{x})\rangle\right\}.
\end{aligned} \tag{3.47}$$

We can apply Lemma 2.4 to the relation (3.47) and conclude that $x_n \to \tilde{x}$. This completes the proof. □

Acknowledgments

The authors would like to express their thanks to the referees and the editor for their helpful suggestion and comments. This work was supported by the Scientific Reserch Fund of Sichuan Provincial Education Department (09ZB102,11ZB146) and Yunnan University of Finance and Economics.

References

[1] L.-C. Ceng, C.-y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[2] R. U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[3] G. M. Korpelevich, "An extragradient method for finding saddle points and for other problems," *Ekonomika i Matematicheskie Metody*, vol. 12, no. 4, pp. 747–756, 1976.

[4] Y. Censor, A. Gibali, and S. Reich, "Two extensions of Korplevich's extragradient method for solving the variational inequality problem in Euclidean space," Tech. Rep., 2010.

[5] L.-C. Zeng and J.-C. Yao, "Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 10, no. 5, pp. 1293–1303, 2006.

[6] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 3, pp. 341–350, 2005.

[7] N. Nadezhkina and W. Takahashi, "Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 128, no. 1, pp. 191–201, 2006.

[8] Y. Yao and M. A. Noor, "On viscosity iterative methods for variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 776–787, 2007.

[9] Y. Yao, M. A. Noor, R. Chen, and Y.-C. Liou, "Strong convergence of three-step relaxed hybrid steepest-descent methods for variational inequalities," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 175–183, 2008.

[10] R. U. Verma, "Projection methods, algorithms, and a new system of nonlinear variational inequalities," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 1025–1031, 2001.

[11] R. U. Verma, "General convergence analysis for two-step projection methods and applications to variational problems," *Applied Mathematics Letters*, vol. 18, no. 11, pp. 1286–1292, 2005.

[12] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An extragradient method for fixed point problems and variational inequality problems," *Journal of Inequalities and Applications*, Article ID 38752, 12 pages, 2007.

[13] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.

[14] Y. Censor, A. Gibali, and S. Reich, "The subgradient extragradient method for solving variational inequalities in Hilbert space," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 318–335, 2011.

[15] Z. Huang and M. A. Noor, "An explicit projection method for a system of nonlinear variational inequalities with different (γ,r)-cocoercive mappings," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 356–361, 2007.

[16] Z. Huang and M. A. Noor, "Some new unified iteration schemes with errors for nonexpansive mappings and variational inequalities," *Applied Mathematics and Computation*, vol. 194, no. 1, pp. 135–142, 2007.

[17] K. Aoyama, H. Iiduka, and W. Takahashi, "Weak convergence of an iterative sequence for accretive operators in Banach spaces," *Fixed Point Theory and Applications*, Article ID 35390, 13 pages, 2006.

[18] S. Kamimura and W. Takahashi, "Weak and strong convergence of solutions to accretive operator inclusions and applications," *Set-Valued Analysis*, vol. 8, no. 4, pp. 361–374, 2000.

[19] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[20] H. Iiduka, W. Takahashi, and M. Toyoda, "Approximation of solutions of variational inequalities for monotone mappings," *Panamerican Mathematical Journal*, vol. 14, no. 2, pp. 49–61, 2004.

[21] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[22] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[23] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[24] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[25] K. Goebel and W. A. Kirk, "Topics in metric fixed point theory," in *Cambridge Studies in Advanced Mathematics*, vol. 28, Cambridge University Press, Cambridge, UK, 1990.

[26] Y. Yao, Y.-C. Liou, S. M. Kang, and Y. Yu, "Algorithms with strong convergence for a system of nonlinear variational inequalities in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 17, pp. 6024–6034, 2011.