Research Article

On a Generalized Hyers-Ulam Stability of Trigonometric Functional Equations

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Let G be an Abelian group, let $\mathbb{C}$ be the field of complex numbers, and let $f, g : G \rightarrow \mathbb{C}$. We consider the generalized Hyers-Ulam stability for a class of trigonometric functional inequalities, $|f(x-y) - f(x)g(y) + g(x)f(y)| \leq \varphi(y)$, $|g(x-y) - g(x)g(y) - f(x)f(y)| \leq \varphi(y)$, where $\varphi : G \rightarrow \mathbb{R}$ is an arbitrary nonnegative function.

1. Introduction

The Hyers-Ulam stability problems of functional equations go back to 1940 when S. M. Ulam proposed a question concerning the approximate homomorphisms from a group to a metric group (see [1]). A partial answer was given by Hyers et al. [2, 3] under the assumption that the target space of the involved mappings is a Banach space. After the result of Hyers, Aoki [4], and Bourgin [5, 6] dealt with this problem, however, there were no other results on this problem until 1978 when Rassias [7] dealt again with the inequality of Aoki [4]. Following the Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [2, 7–21]. The following four functional equations are called *trigonometric functional equations*.

$$f(x+y) - f(x)g(y) - g(x)f(y) = 0 \quad (1.1)$$

$$g(x+y) - g(x)g(y) + f(x)f(y) = 0 \quad (1.2)$$

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \quad (1.3)$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0 \quad (1.4)$$

The four functional equations have been investigated separately. The general solutions and regular solutions of the above equations are introduced [22, 23]. In particular, the last equation (1.4) is most interesting in the sense that (1.4) alone characterizes the two trigonometric functions $f(x) = \cos(ax)$, $g(x) = \sin(ax)$ under some regularities of g , which none of the remaining equations are able to do.

In [19], Székelyhidi developed his idea of using invariant subspaces of functions defined on a group or semigroup to obtain the Hyers-Ulam stability of the trigonometric functional equations (1.1) and (1.2). As results, he obtained the Hyers-Ulam stability when for each fixed y the difference

$$T_1(x) := f(x+y) - f(x)g(y) - g(x)f(y) \quad (1.5)$$

is a bounded function of x and the Hyers-Ulam stability when for each fixed y the difference

$$T_2(x) := g(x+y) - g(x)g(y) + f(x)f(y) \quad (1.6)$$

is a bounded function of x , where f, g are mappings from an Abelian (amenable) group G to the field $\mathbb{C}$ of complex numbers.

In this paper, we complete the parallel Hyers-Ulam stability to that of [19] for the functional equations (1.3) and (1.4). As results, we obtained the Hyers-Ulam stability when for each fixed y the difference

$$T_3(x) := f(x-y) - f(x)g(y) + g(x)f(y) \quad (1.7)$$

is a bounded function of x and the Hyers-Ulam stability when for each fixed y the difference

$$T_4(x) := g(x-y) - g(x)g(y) - f(x)f(y) \quad (1.8)$$

is a bounded function of x .

In fact, the authors [10] obtained weaker versions of the Hyers-Ulam stability for the functional equations (1.3) and (1.4), that is, we proved the Hyers-Ulam stability of (1.3) when the difference

$$T_3(x, y) := f(x-y) - f(x)g(y) + g(x)f(y) \quad (1.9)$$

is uniformly bounded for all x and y , and we proved the Hyers-Ulam stability of (1.4) when the difference

$$T_4(x, y) := g(x-y) - g(x)g(y) - f(x)f(y) \quad (1.10)$$

is uniformly bounded for all x and y .

So, the results in this paper would be generalizations of those in [10]. We refer the reader to [9, 15, 16, 20, 21] for some related Hyers-Ulam stability of functional equations of trigonometric type.

2. Main Theorems

A function a from a semigroup $\langle S, + \rangle$ to the field $\mathbb{C}$ of complex numbers is said to be an *additive function* provided that $a(x+y) = a(x) + a(y)$ and $m : S \rightarrow \mathbb{C}$ is said to be an *exponential function* provided that $m(x+y) = m(x)m(y)$. Throughout this paper, we denote by G an Abelian group, $\mathbb{C}$ the set of complex numbers, and $\varphi : G \rightarrow \mathbb{R}$ a fixed nonnegative function. For the proof of stabilities of (1.3) and (1.4), we need the following.

Lemma 2.1 (see [2]). *Let S be a semigroup. Assume that $f, g : S \rightarrow \mathbb{C}$ satisfy the inequality; for each $y \in S$, there exists a positive constant M_y such that*

$$|f(x+y) - f(x)g(y)| \leq M_y, \quad (2.1)$$

for all $x \in S$, then either f is a bounded function or g is an exponential function.

Proof. Suppose that g is not exponential, then there are $y, z \in S$ such that $g(y+z) \neq g(y)g(z)$. Now we have

$$\begin{aligned} f(x+y+z) - f(x+y)g(z) &= (f(x+y+z) - f(x)g(y+z)) \\ &\quad - g(z)(f(x+y) - f(x)g(y)) + f(x)(g(y+z) - g(y)g(z)), \end{aligned} \quad (2.2)$$

and hence,

$$\begin{aligned} f(x) &= (g(y+z) - g(y)g(z))^{-1} \\ &\quad \times ((f(x+y+z) - f(x+y)g(z)) - (f(x+y+z) - f(x)g(y+z)) \\ &\quad + g(z)(f(x+y) - f(x)g(y))). \end{aligned} \quad (2.3)$$

In view of (2.1), the right hand side of (2.3) is bounded as a function of x . Consequently, f is bounded. $\square$

We discuss the general solutions $f, g : G \rightarrow \mathbb{C}$ of the corresponding trigonometric functional equations

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0, \quad (2.4)$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0. \quad (2.5)$$

Lemma 2.2 (see [22, 23]). *The general solutions (f, g) of the functional equation (2.4) are given by one of the following:*

- (i) $f = 0$ and g is arbitrary,
- (ii) $f(x) = \lambda_1(m(x) - m(-x))$ and $g(x) = \lambda_2 f(x) + (1/2)(m(x) + m(-x))$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, where m is an exponential function,
- (iii) $f(x) = a(x)m(x)$, $g(x) = (1 + \lambda a(x))m(x)$ for some $\lambda \in \mathbb{C}$, where a is an additive function and m is an exponential function satisfying $m^2 \equiv 1$.

Also, the general solutions (g, f) of the functional equation (2.5) are given by one of the following:

- (i) $g(x) = \lambda$ and $f(x) = \pm\sqrt{\lambda - \lambda^2}$ for some $\lambda \in \mathbb{C}$,
- (ii) $g(x) = (1/2)(m(x) + m(-x))$ and $f(x) = (1/2i)(m(x) - m(-x))$, where m is an exponential function.

Proof. The solutions of the functional equation (2.4) are given in [23, p. 217, Theorem 11]. For the functional equation (2.5), combining the result of L. Vietoris [22, p. 177] and that of J. A. Baker [23, p. 220], we obtain that every nonconstant function g satisfying (2.5) has the form

$$g(x) = \frac{1}{2}(m(x) + m(-x)), \quad (2.6)$$

for some exponential function m . Thus, using (2.5), we have

$$f(x) = \frac{1}{2i}(m(x) - m(-x)). \quad (2.7)$$

This completes the proof. $\square$

For the proof of the stability of (1.1), we need the following. Throughout this paper, we denote by φ an arbitrary nonnegative function on G .

Lemma 2.3. *Let $f, g : G \rightarrow \mathbb{C}$ satisfy the inequality*

$$|f(x - y) - f(x)g(y) + g(x)f(y)| \leq \varphi(y), \quad (2.8)$$

for all $x, y \in G$, then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$|\lambda_1 f(x) - \lambda_2 g(x)| \leq M, \quad (2.9)$$

or else

$$f(x - y) - f(x)g(y) + g(x)f(y) = 0, \quad (2.10)$$

for all $x, y \in G$.

Proof. Suppose that the inequality (2.9) holds only when $\lambda_1 = \lambda_2 = 0$. Let

$$k(x, y) = f(x + y) - f(x)g(-y) + g(x)f(-y), \quad (2.11)$$

and choose y_1 satisfying $f(-y_1) \neq 0$. Now it can be easily calculated that

$$g(x) = \lambda_0 f(x) + \lambda_1 f(x + y_1) - \lambda_1 k(x, y_1), \quad (2.12)$$

where $\lambda_0 = g(-y_1)/f(-y_1)$ and $\lambda_1 = -1/f(-y_1)$. By (2.11), we have

$$f(x + (y + z)) = f(x)g(-y - z) - g(x)f(-y - z) + k(x, y + z). \quad (2.13)$$

Also by (2.11) and (2.12), we have

$$\begin{aligned} f((x + y) + z) &= f(x + y)g(-z) - g(x + y)f(-z) + k(x + y, z) \\ &= (f(x)g(-y) - g(x)f(-y) + k(x, y))g(-z) \\ &\quad - (\lambda_0 f(x + y) + \lambda_1 f(x + y + y_1) - \lambda_1 k(x + y, y_1))f(-z) + k(x + y, z) \\ &= (f(x)g(-y) - g(x)f(-y) + k(x, y))g(-z) \\ &\quad - \lambda_0 (f(x)g(-y) - g(x)f(-y) + k(x, y))f(-z) \\ &\quad - \lambda_1 (f(x)g(-y - y_1) - g(x)f(-y - y_1) + k(x, y + y_1))f(-z) \\ &\quad + \lambda_1 k(x + y, y_1)f(-z) + k(x + y, z). \end{aligned} \quad (2.14)$$

From (2.13) and (2.14), we have

$$\begin{aligned} &(g(-y)g(-z) - \lambda_0 g(-y)f(-z) - \lambda_1 g(-y - y_1)f(-z) - g(-y - z))f(x) \\ &\quad + (-g(-y)g(-z) + \lambda_0 f(-y)g(-z) + \lambda_1 f(-y - y_1)f(-z) + f(-y - z))g(x) \\ &= -k(x, y)g(-z) + \lambda_0 k(x, y)f(-z) + \lambda_1 k(x, y + y_1)f(-z) \\ &\quad - \lambda_1 k(x + y, y_1)f(-z) - k(x + y, z) + k(x, y + z). \end{aligned} \quad (2.15)$$

Since $k(x, y)$ is bounded by $\varphi(-y)$, if we fix y, z , the right hand side of (2.15) is bounded by a constant M , where

$$\begin{aligned} M &= \varphi(-y)|g(-z)| + \varphi(-y)|\lambda_0 f(-z)| + \varphi(-y - y_1)|\lambda_1 f(-z)| \\ &\quad + \varphi(-y_1)|\lambda_1 f(-z)| + \varphi(-z) + \varphi(-y - z). \end{aligned} \quad (2.16)$$

So by our assumption, the left hand side of (2.15) vanishes, so does the right hand side. Thus, we have

$$\begin{aligned} &(-\lambda_0 k(x, y) - \lambda_1 k(x, y + y_1) + \lambda_1 k(x + y, y_1))f(-z) + k(x, y)g(-z) \\ &= k(x, y + z) - k(x + y, z). \end{aligned} \quad (2.17)$$

Now by the definition of k , we have

$$\begin{aligned} k(x + y, z) - k(x, y + z) &= f(x + y + z) - f(x + y)g(-z) + g(x + y)f(-z) \\ &\quad - f(x + y + z) + f(x)g(-y - z) - g(x)f(-y - z) \\ &= f(-y - z - x) - f(-y - z)g(x) + g(-y - z)f(x) \\ &\quad - f(-z - x - y) + f(-z)g(x + y) - g(-z)f(x + y) \\ &= k(-y - z, -x) - k(-z, -x - y). \end{aligned} \quad (2.18)$$

Hence, the right hand side of (2.17) is bounded by $\psi(x) + \psi(x+y)$. So if we fix x, y in (2.17), the left hand side of (2.17) is a bounded function of z . Thus, by our assumption, we conclude that $k(x, y) \equiv 0$. This completes the proof. $\square$

In the following theorem, we assume that

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi(-2^k x) < \infty, \quad (2.19)$$

or

$$\Phi_2(x) := \sum_{k=1}^{\infty} 2^k \psi(-2^{-k} x) < \infty. \quad (2.20)$$

For the proof, we discuss the following property.

Lemma 2.4. *Let $m : G \rightarrow \mathbb{C}$ be a bounded exponential function satisfying $m(x) \neq m(-x)$ for some $x \in G$, then there exists $y \in G$ such that*

$$|m(y) - m(-y)| \geq \sqrt{3}. \quad (2.21)$$

Furthermore, the constant $\sqrt{3}$ is the best one.

Proof. Since m is a bounded exponential, there exists $C > 0$ such that $|m(x)|^k = |m(kx)| \leq C$ for all $k \in \mathbb{Z}$ and $x \in G$, which implies $|m(x)| = 1$ for all $x \in G$. Assume that $m(x_0) \neq m(-x_0)$, then we have $m(x_0) \neq \pm 1$, and we may assume that $m(x_0) = e^{i\theta}$, $0 < \theta < \pi$. If $\theta \in [\pi/3, 2\pi/3]$, we have $|m(x_0) - m(-x_0)| = |e^{i\theta} - e^{-i\theta}| \geq \sqrt{3}$. If $\theta \in [0, \pi/3]$, there exists a positive integer k such that $k\theta \in [\pi/3, 2\pi/3]$, and we have $|m(kx_0) - m(-kx_0)| = |e^{ik_0\theta} - e^{-ik_0\theta}| \geq \sqrt{3}$. If $\theta \in [2\pi/3, 5\pi/6]$, then $2\theta \in [4\pi/3, 5\pi/3]$, and we have $|m(2x_0) - m(-2x_0)| = |e^{i2\theta} - e^{-i2\theta}| \geq \sqrt{3}$. Finally, if $\theta \in [5\pi/6, \pi]$, there exists a positive integer k such that $2k\theta \in [-\pi/3, -2\pi/3]$, and we have $|m(2kx_0) - m(-2kx_0)| = |e^{i2k\theta} - e^{-i2k\theta}| \geq \sqrt{3}$. Now define $m : \mathbb{Z} \rightarrow \mathbb{C}$ by $m(k) = e^{ik\pi/3}$. Then we have $|m(3k+1) - m(-3k-1)| = \sqrt{3}$ for all $k \in \mathbb{Z}$. Thus, $\sqrt{3}$ is the biggest one. This completes the proof. $\square$

Theorem 2.5. *Let $f, g : G \rightarrow \mathbb{C}$ satisfy the inequality*

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \leq \psi(y), \quad (2.22)$$

for all $x, y \in G$, then (f, g) satisfies one of the following:

- (i) $f = 0$, g is arbitrary,
- (ii) f and g are bounded functions,
- (iii) $f(x) = \lambda_1(m(x) - m(-x))$ and $g(x) = \lambda_2 f(x) + (1/2)(m(x) + m(-x))$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, where m is an exponential function,
- (iv) there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m such that

$$g(x) = \lambda f(x) + m(x), \quad (2.23)$$

for all $x \in G$, and f satisfies the condition; there exists $d \geq 0$ satisfying

$$|f(x)| \leq \frac{2}{\sqrt{3}}(\psi(-x) + d), \quad (2.24)$$

for all $x \in G$,

(v) there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m satisfying $m^2 \equiv 1$ such that

$$g(x) = \lambda f(x) + m(x), \quad (2.25)$$

for all $x \in G$, and f satisfies one of the following conditions; there exists an additive function $a_1 : G \rightarrow \mathbb{C}$ such that

$$|f(x) - (a_1(x) + f(0))m(x)| \leq \Phi_1(x), \quad (2.26)$$

for all $x \in G$, or there exists an additive function $a_2 : G \rightarrow \mathbb{C}$ such that

$$|f(x) - (a_1(x) + f(0))m(x)| \leq \Phi_2(x), \quad (2.27)$$

for all $x \in G$, where Φ_1 and Φ_2 are the functions given in (2.19) and (2.20).

Proof. In view of Lemma 2.3, we first consider the case when f, g satisfy (2.9). If $f = 0$, g is arbitrary which is the case (i). If f is a nontrivial bounded function, in view of (2.22), g is also bounded which gives the case (ii). If f is unbounded, it follows from (2.9) that $\lambda_2 \neq 0$ and

$$g(x) = \lambda f(x) + m(x), \quad (2.28)$$

for some $\lambda \in \mathbb{C}$ and a bounded function m . Putting (2.28) in (2.22), we have

$$|f(x - y) - f(x)m(y) + m(x)f(y)| \leq \psi(y), \quad (2.29)$$

for all $x, y \in G$. Replacing y by $-y$ and using the triangle inequality, we have, for some $C > 0$,

$$|f(x + y) - f(x)m(-y)| \leq C|f(-y)| + \psi(-y), \quad (2.30)$$

for all $x, y \in G$. By Lemma 2.1, m is an exponential function. If $m = 0$, putting $y = 0$ in (2.29), we have

$$|f(x)| \leq \psi(0). \quad (2.31)$$

Thus, we have $m \neq 0$ since f is unbounded. Since m is a nonzero bounded exponential function, it follows from the equalities

$$m(x) = m(x - y)m(y), \quad x, y \in G \quad (2.32)$$

that $m(0) = 1$ and $m(x) \neq 0$, for all $x \in G$. Putting $x = 0$ in (2.29) and replacing y by $-y$ multiplying $|m(x)|$ in the result, we have

$$|m(x)f(-y) + m(x)f(y) - f(0)m(x)m(-y)| \leq \psi(-y), \quad (2.33)$$

for all $y \in G$. Replacing y by $-y$ in (2.29) and using (2.33), we have

$$|f(x+y) - f(x)m(-y) - m(x)f(y) + f(0)m(x)m(-y)| \leq 2\psi(-y). \quad (2.34)$$

First we consider the case $m(x) \neq m(-x)$ for some $x \in G$. Replacing x by y and y by x in (2.34), we have

$$|f(y+x) - f(y)m(-x) - m(y)f(x) + f(0)m(y)m(-x)| \leq 2\psi(-x), \quad (2.35)$$

for all $x, y \in G$. From (2.34) and (2.35), using the triangle inequality, putting $y = y_0$ such that $|m(y_0) - m(-y_0)| \geq \sqrt{3}$ and dividing $|m(y_0) - m(-y_0)|$, we have

$$|f(x)| \leq \frac{2}{\sqrt{3}}(\psi(-x) + d), \quad (2.36)$$

for all $x \in G$, where $d = \psi(-y_0) + |f(y_0)| + |f(0)|$, which gives (iv). Now we consider the case $m(x) = m(-x)$, for all $x \in G$. Dividing both the sides of (2.34) by $m(x)m(y)$, we have

$$|F(x+y) - F(x) - F(y)| \leq 2\psi(-y), \quad (2.37)$$

for all $x, y \in G$, where $F(x) = f(x)/m(x) - f(0)$. By the well-known results in [4], there exists a unique additive function $a_1(x)$ given by

$$a_1(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x) \quad (2.38)$$

such that

$$|F(x) - a_1(x)| \leq \Phi_1(x) \quad (2.39)$$

if $\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi(-2^k x) < \infty$, and there exists a unique additive function $a_2(x)$ given by

$$a_2(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n} x) \quad (2.40)$$

such that

$$|F(x) - a_2(x)| \leq \Phi_2(x) \quad (2.41)$$

if $\Phi_2(x) := \sum_{k=1}^{\infty} 2^k \psi(-2^{-k}x) < \infty$. Multiplying $|m(x)|$ in both sides of (2.39) and (2.41), we get (v). Now we consider the case when f, g satisfy (2.10). In view of Lemma 2.2, the solutions of (2.10) are given by (i), (iii), or contained in the case (v). This completes the proof. $\square$

Let X be a real normed space, and let $\psi : X \rightarrow \mathbb{R}$ be given by $\psi(x) = \epsilon \|x\|^p$, $p \geq 0$, $p \neq 1$, then ψ satisfies the conditions assumed in Theorem 2.5. In view of (2.19) and (2.20), we have

$$\Phi_1(x) = \frac{2\epsilon \|x\|^p}{2 - 2^p} \quad (2.42)$$

if $0 < p < 1$,

$$\Phi_2(x) = \frac{2\epsilon \|x\|^p}{2^p - 2} \quad (2.43)$$

if $p > 1$. Thus, as a direct consequence of Theorem 2.5, we have the following.

Corollary 2.6. *Let $f, g : X \rightarrow \mathbb{C}$ satisfy the inequality*

$$|f(x - y) - f(x)g(y) + g(x)f(y)| \leq \epsilon \|y\|^p, \quad p \neq 1, \quad p \geq 0, \quad (2.44)$$

for all $x, y \in X$, then (f, g) satisfies one of the following:

- (i) $f = 0$, g is arbitrary,
- (ii) f and g are bounded functions,
- (iii) $f(x) = \lambda_1(m(x) - m(-x))$ and $g(x) = \lambda_2 f(x) + (1/2)(m(x) + m(-x))$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, where m is an exponential function,
- (iv) there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m such that

$$g(x) = \lambda f(x) + m(x), \quad (2.45)$$

for all $x \in X$, and f satisfies the condition; there exists $d \geq 0$ satisfying

$$|f(x)| \leq \frac{2}{\sqrt{3}}(\psi(-x) + d), \quad (2.46)$$

for all $x \in X$,

- (v) there exist $\lambda \in \mathbb{C}$ and a bounded exponential function m satisfying $m^2 \equiv 1$ such that

$$g(x) = \lambda f(x) + m(x), \quad (2.47)$$

for all $x \in X$, and f satisfies one of the following conditions; there exists an additive function $a : X \rightarrow \mathbb{C}$ such that

$$|f(x) - (a(x) + f(0))m(x)| \leq \frac{2\epsilon\|x\|^p}{|2 - 2^p|}, \quad (2.48)$$

for all $x \in X$.

Now we prove the stability of (1.2). For the proof, we need the following.

Lemma 2.7. Let $f, g : G \rightarrow \mathbb{C}$ satisfy the inequality

$$|g(x - y) - g(x)g(y) - f(x)f(y)| \leq \psi(y), \quad (2.49)$$

for all $x, y \in G$, then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$|\lambda_1 f(x) - \lambda_2 g(x)| \leq M, \quad (2.50)$$

or else

$$g(x - y) - g(x)g(y) - f(x)f(y) = 0, \quad (2.51)$$

for all $x, y \in G$.

Proof. Suppose that $\lambda_1 f(x) - \lambda_2 g(x)$ is bounded only when $\lambda_1 = \lambda_2 = 0$, and let

$$l(x, y) = g(x + y) - g(x)g(-y) - f(x)f(-y). \quad (2.52)$$

Since we may assume that f is nonconstant, we can choose y_1 satisfying $f(-y_1) \neq 0$. Now it can be easily get that

$$f(x) = \lambda_0 g(x) + \lambda_1 g(x + y_1) - \lambda_1 l(x, y_1), \quad (2.53)$$

where $\lambda_0 = -g(-y_1)/f(-y_1)$ and $\lambda_1 = 1/f(-y_1)$. From the definition of l and the use of (2.53), we have the following two equations:

$$\begin{aligned} g((x + y) + z) &= g(x + y)g(-z) + f(x + y)f(-z) + l(x + y, z) \\ &= (g(x)g(-y) + f(x)f(-y) + l(x, y))g(-z) \\ &\quad + (\lambda_0 g(x + y) + \lambda_1 g(x + y + y_1) - \lambda_1 l(x + y, y_1))f(-z) + l(x + y, z) \\ &= (g(x)g(-y) + f(x)f(-y) + l(x, y))g(-z) \\ &\quad + \lambda_0 (g(x)g(-y) + f(x)f(-y) + l(x, y))f(-z) \\ &\quad + \lambda_1 (g(x)g(-y - y_1) + f(x)f(-y - y_1) + l(x, y + y_1))f(-z) \\ &\quad - \lambda_1 l(x + y, y_1)f(-z) + l(x + y, z), \end{aligned} \quad (2.54)$$

$$g(x + (y + z)) = g(x)g(-y - z) + f(x)f(-y - z) + l(x, y + z). \quad (2.55)$$

Equating (2.54) and (2.55), we have

$$\begin{aligned}
 & g(x)(g(-y)g(-z) + \lambda_0 g(-y)f(-z) + \lambda_1 g(-y - y_1)f(-z) - g(-y - z)) \\
 & + f(x)(f(-y)g(-z) + \lambda_0 f(-y)f(-z) + \lambda_1 f(-y - y_1)f(-z) - f(-y - z)) \\
 & = -l(x, y)g(-z) - \lambda_0 l(x, y)f(-z) - \lambda_1 l(x, y + y_1)f(-z) \\
 & + \lambda_1 l(x + y, y_1)f(-z) - l(x + y, z) + l(x, y + z).
 \end{aligned} \tag{2.56}$$

In (2.56), when y, z are fixed, the right hand side is bounded, so by our assumption, we have

$$l(x, y)g(-z) + (\lambda_0 l(x, y) + \lambda_1 l(x, y + y_1) - \lambda_1 l(x + y, y_1))f(-z) = l(x, y + z) - l(x + y, z). \tag{2.57}$$

Also we can write

$$\begin{aligned}
 l(x, y + z) - l(x + y, z) &= g(x + y + z) - g(x)g(y + z) - f(x)f(y + z) \\
 &\quad - g(x + y + z) + g(x + y)g(z) + f(x + y)f(z) \\
 &= l(y + z, x) - l(z, x + y) \\
 &\leq \psi(-x) + \psi(-x - y).
 \end{aligned} \tag{2.58}$$

Thus, if we fix x, y in (2.57), the right hand side of (2.57) is bounded. By our assumption, we have $l(x, y) \equiv 0$. This completes the proof. $\square$

Theorem 2.8. Let $f, g : G \rightarrow \mathbb{C}$ satisfy the inequality

$$|g(x - y) - g(x)g(y) - f(x)f(y)| \leq \psi(y), \tag{2.59}$$

for all $x, y \in G$, then (f, g) satisfies one of the following:

- (i) f and g are bounded functions,
- (ii) $g(x) = (1/2)(m(x) + m(-x))$ and $f(x) = (1/2)(m(x) - m(-x))$, where m is an exponential function,
- (iii) $f = \pm i(g - m)$ for a bounded exponential function m , and g satisfies

$$\left| g(x) - \frac{1}{2}(g(0)m(-x) + m(x)) \right| \leq \frac{1}{2}\psi(x), \tag{2.60}$$

for all $x \in G$. In particular if $\psi(0) = 0$, one has $g(0) = 1, f(0) = 0$.

Proof. In view of Lemma 2.7, we first consider the case when f, g satisfy (2.51). If f is bounded, then in view of the inequality (2.59), for each y , $g(x + y) - g(x)g(-y)$, is also bounded. It follows from Lemma 2.1 that g is bounded or a nonzero exponential function. If g is bounded, the case (i) follows. If g is a nonzero exponential function, from (2.59), using the triangle inequality, we have for some $d \geq 0$,

$$|g(x)(g(-y) - g(y))| \leq \psi(y) + d, \tag{2.61}$$

for all $x, y \in G$. Thus, it follows that

$$g(y) = g(-y), \quad (2.62)$$

for all $y \in G$, or else g is bounded, and equality (2.62) implies $g^2 \equiv 1$, which gives the case (i).

If f is unbounded, then in view of (2.59), g is also unbounded, and hence, $\lambda_1 \lambda_2 \neq 0$ and

$$f(x) = \lambda g(x) + r(x), \quad (2.63)$$

for some $\lambda \neq 0$ and a bounded function r . Putting (2.63) in (2.59), replacing y by $-y$, and using the triangle inequality, we have

$$\left| g(x+y) - g(x) \left((\lambda^2 + 1)g(-y) + \lambda r(-y) \right) \right| \leq |(\lambda g(-y) + r(-y))r(x)| + \psi(-y). \quad (2.64)$$

From Lemma 2.1, we have

$$(\lambda^2 + 1)g(y) + \lambda r(y) = m(y), \quad (2.65)$$

for some exponential function m . If $\lambda^2 \neq -1$, we have

$$f(x) = \frac{\lambda m(x) + r(x)}{\lambda^2 + 1}, \quad g(x) = \frac{m(x) - \lambda r(x)}{\lambda^2 + 1}. \quad (2.66)$$

Putting (2.66) in (2.59), multiplying $|\lambda^2 + 1|$ in the result, and using the triangle inequality, we have for some $d \geq 0$,

$$|m(x)(m(-y) - m(y))| \leq |\lambda^2 + 1| \psi(y) + d, \quad (2.67)$$

for all $x, y \in G$. Since m is unbounded, we have

$$m(y) = m(-y), \quad (2.68)$$

for all $y \in G$, which implies $m^2 \equiv 1$, contradicting to the fact that m is unbounded. Thus, it follows that $\lambda^2 = -1$, and we have

$$f = \pm i(g - m), \quad (2.69)$$

where m is a bounded exponential function. Putting (2.69) in (2.59), we have

$$|g(x-y) - g(x)m(y) - g(y)m(x) + m(x)m(y)| \leq \psi(y), \quad (2.70)$$

for all $x, y \in G$. Replacing y by x in (2.70) and dividing the result by $2m(x)$, we have

$$\left| g(x) - \frac{1}{2}(g(0)m(-x) + m(x)) \right| \leq \frac{1}{2}\psi(x), \quad (2.71)$$

for all $x \in G$. From (2.69), (2.71), we get (iii). Now we consider the case when f, g satisfy (2.51). In view of Lemma 2.2, the solutions of (2.51) are contained in (i) or given by (ii). Furthermore, if $\psi(0) = 0$, then putting $x = y = 0$ in (2.70), we have $g(0) = 1$, and from (2.69), we also have $f(0) = 0$. This completes the proof. $\square$

In particular, if $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ is a continuous function and $\psi(x) = \epsilon|x|^p$, $p > 0$, $p \neq 1$, then Theorem 2.8 is reduced as follows.

Corollary 2.9. *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ be a continuous function satisfying (2.59) for $\psi(x) = \epsilon|x|^p$, then (f, g) satisfies one of the following:*

- (i) f and g are bounded functions,
- (ii) $g(x) = \cos(c \cdot x)$ and $f(x) = \sin(c \cdot x)$ for some $c \in \mathbb{C}^n$,
- (iii) there exists $a \in \mathbb{R}^n$ such that

$$\begin{aligned} |f(x) - \sin(a \cdot x)| &\leq \frac{\epsilon}{2}|x|^p, \\ |g(x) - \cos(a \cdot x)| &\leq \frac{\epsilon}{2}|x|^p, \end{aligned} \quad (2.72)$$

for all $x \in \mathbb{R}^n$.

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