Research Article

Periodic Solutions of a Cohen-Grossberg-Type BAM Neural Networks with Distributed Delays and Impulses

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A class of Cohen-Grossberg-type BAM neural networks with distributed delays and impulses are investigated in this paper. Sufficient conditions to guarantee the uniqueness and global exponential stability of the periodic solutions of such networks are established by using suitable Lyapunov function, the properties of $M$-matrix, and some suitable mathematical transformation. The results in this paper improve the earlier publications.

1. Introduction

The research of neural networks with delays involves not only the dynamic analysis of equilibrium point but also that of periodic oscillatory solution. The dynamic behavior of periodic oscillatory solution is very important in learning theory due to the fact that learning usually requires repetition [1, 2].

Cohen and Grossberg proposed the Cohen-Grossberg neural networks (CGNNs) in 1983 [3]. Kosko proposed bi directional associative memory neural networks (BAMNNs) in 1988 [4]. Some important results for periodic solutions of delayed CGNNs have been obtained in [5–10]. Xiang and Cao proposed a class of Cohen-Grossberg BAM neural networks (CGBAMNNs) with distributed delays in 2007 [11]; in addition, many evolutionary processes are characterized by abrupt changes at certain time; these changes are called to be impulsive phenomena, which are included in many neural networks such as Hopfield neural networks, BAM neural networks, CGNNs, and CGBAMNNs and can affect dynamical
behaviors of the systems just as time delays. The results for periodic solutions of CGBAMNNs with or without impulses are obtained in [11–15].

The objective of this paper is to study the existence and global exponential stability of periodic solutions of CGBAMNNs with distributed delays by using suitable Lyapunov function, the properties of $M$-matrix, and some suitable mathematical transformation. Comparing with the results in [13, 14], improved results are successively obtained, the conditions for the existence and globally exponential stability of the periodic solution of such system without impulses have nothing to do with inputs of the neurons and amplification functions; and we also point that CGBAMNNs model is a special case of CGNNs model, many results of CGBAMNNs can be directly obtained from the results of CGNNs.

The rest of this paper is organized as follows. Preliminaries are given in Section 2. Sufficient conditions which guarantee the uniqueness and global exponential stability of periodic solutions for CGBAMNNs with distributed delays and impulses are given in Section 3. Two examples are given in Section 4 to demonstrate the main results.

2. Preliminaries

Consider the following periodic CGNNs model with distributed delays and impulses:

$$
\dot{x}_i(t) = -a_i(x_i(t)) \left[ b_i(t, x_i(t)) - \sum_{j=1}^n p_{ij}(t) f_j(\rho_j x_j(t)) \right. \\
\left. - \sum_{j=1}^n u_{ij}(t) \int_0^{+\infty} k_{ij}(s) f_j(\rho_j x_j(t-s)) ds - I_i(t) \right], \quad t > 0, \quad t \neq t_k,
\Delta x_i(t_k) = -\gamma_{ik} x_i(t_k), \quad t = t_k, \quad k \in \mathbb{Z}^+,
$$

where $1 \leq i \leq n, t > 0$, and $\mathbb{Z}^+ = \{1, 2, \ldots\}$. $x_i(t)$ denotes the state variable of the $i$th neuron, $f_j(\cdot)$ denotes the signal function of the $j$th neuron at time $t$; $I_i$ denotes input of the $i$th neuron at time $t$; $a_i(\cdot)$ represents amplification function; $b_i(t, \cdot)$ is appropriately behaved function; $p_{ij}(t)$ and $u_{ij}(t)$ are connection weights of the neural networks at time $t$; respectively, $\rho_j$ is positive constant, which corresponds to the neuronal gain associated with the neuronal activations and $k_{ij}$ corresponds to the delay kernel function; $p_{ij}(t)$ and $u_{ij}(t)$ are continuously periodic functions on $[0, +\infty)$ with common period $T > 0$.

$\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$; $t_k$ is called impulsive moment and satisfies $0 < t_1 < t_2 < \cdots$, $\lim_{k \to +\infty} t_k = +\infty$; $x_i(t^+)$ and $x_i(t^-)$ denote the left-hand and right-hand limits at $t_k$; respectively, we always assume $x_i(t_k^+) = x_i(t_k^-)$ and $x_i(t_k^+) = x_i(t_k^-)$, $k \in \mathbb{Z}^+$.

For system (2.1), we assume the following.

(H$_1$) The amplification function $a_i(\cdot)$ is continuous, and there exist constants $\underline{a}_i, \overline{a}_i$ such that $0 < \underline{a}_i \leq a_i(x_i(t)) \leq \overline{a}_i$ for $1 \leq i \leq n$.

(H$_2$) The behaved function $b_i(t, \cdot)$ is $T$-periodic about the first argument; there exists continuous $T$-periodic function $a_i(t)$ such that

$$
\frac{b_i(t, x) - b_i(t, y)}{x - y} \geq a_i(t) > 0,
$$

for all $x \neq y, \ 1 \leq j \leq n$. 

(H₃) For activation function \( f_j(\cdot) \), there exists positive constant \( L_j \) such that

\[
L_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|
\]  

for all \( x \neq y, 1 \leq j \leq n \).

(H₄) The kernel function \( k_{ij}(s) \) is nonnegative continuous function on \([0, +\infty)\) and satisfies

\[
\int_{0}^{+\infty} se^{ls} k_{ij}(s) ds < +\infty, \\
K_{ij}(\lambda) = \int_{0}^{+\infty} e^{\lambda s} k_{ij}(s) ds
\]

is differentiable function for \( \lambda \in [0, r_{ij}) \), \( 0 < r_{ij} < +\infty \), \( K_{ij}(0) = 1 \) and \( \lim_{\lambda \to r_{ij}} K_{ij}(\lambda) = +\infty \).

(H₅) There exists positive integer \( k_0 \) such that \( t_{k+k_0} = t_k + T \) and \( y_{i(k+k_0)} = y_{ik} \) hold.

**Remark 2.1.** A typical example of kernel function is given by \( k_{ij}(s) = (s^r / r! r_{ij}^{r+1} e^{-r_{ij}s} \) for \( s \in [0, +\infty) \), where \( r_{ij} \in (0, +\infty) \), \( r \in \{0, 1, \ldots, n\} \). These kernel functions are called as the gamma memory filter [16] and satisfy condition (H₄).

For any continuous function \( S(t) \) on \([0, T]\), \( \underline{S} \) and \( \overline{S} \) denote \( \min_{t \in [0, T]} |S(t)| \) and \( \max_{t \in [0, T]} |S(t)| \), respectively.

For any \( x(t) = (x_1(t), x_2(t), \ldots, x_k(t))^T \in \mathbb{R}^k \), \( t > 0 \), define \( \|x(t)\| = \sum_{i=1}^{k} |x_i(t)| \), and for any \( \varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_k(s))^T \in \mathbb{R}^k \), \( s \in (-\infty, 0] \), define \( \|\varphi(s)\| = \sup_{s \in (-\infty, 0]} \sum_{i=1}^{k} |\varphi_i(s)| \).

Denote

\[
\text{PC}\left((-\infty, 0], \mathbb{R}^k \right) = \{\varphi : [-\infty, 0] \to \mathbb{R}^k \mid \varphi(s) \text{ is bounded and continuous for all but at most a finite number of points } s \in (-\infty, 0], \text{ and at these points } s, \varphi(s^+), \varphi(s^-) \text{ exist and } \varphi(s^-) = \varphi(s) \}.
\]

Then \( \text{PC}((-\infty, 0], \mathbb{R}^k) \) is a Banach space with respect to \( \| \cdot \| \).

The initial conditions of system (2.1) are given by

\[
x_i(s) = \varphi_i(s), \quad -\infty \leq s \leq 0, \quad 1 \leq i \leq n,
\]

where \( \varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s)) \in \text{PC}((-\infty, 0], \mathbb{R}^n) \).

Let \( x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \ldots, x_n(t, \varphi))^T \) denote any solution of the system (2.1) with initial value \( \varphi \in \text{PC}((-\infty, 0], \mathbb{R}^n) \).
Definition 2.2. A solution $x(t, \varphi)$ of system (2.1) is said to be globally exponentially stable, if there exist two constants $\lambda > 0$, $M > 0$ such that

$$\|x(t, \varphi) - x(t, \varphi_0)\| \leq M\|\varphi - \varphi_0\|e^{-\lambda t}, \quad t > 0,$$

(2.7)

for any solutions $x(t, \varphi)$ of system (2.1).

Definition 2.3. A real matrix $A = (a_{ij})_{n \times n}$ is said to be a nonsingular $M$-matrix if $a_{ij} \leq 0$ ($i, j = 1, 2, \ldots, n, i \neq j$), and all successive principle minors of $A$ are positive.

Lemma 2.4 (see [17]). A matrix with nonpositive off-diagonal elements $A = (a_{ij})_{n \times n}$ is a nonsingular $M$-matrix if and only there exists a vector $p = (p_i)_{1 \times n} > 0$ such that $p^TA > 0$ or $Ap$ holds.

Lemma 2.5. Under assumptions $(H_1)$–$(H_3)$, system (2.1) has a $T$-periodic solution which is globally exponentially stable, if the following conditions hold.

$(H_6)$ $\mathcal{M} = A - C$ is a nonsingular $M$-matrix, where

$$A = \text{diag}(a_1, a_2, \ldots, a_n), \quad C = (c_{ij})_{n \times n}, \quad c_{ij} = (\bar{p}_{ij} + u_{ij})_{\rho_i, L_i}.$$

(2.8)

$(H_7)$ $a_i((1 - \gamma_k)s) \geq |1 - \gamma_k|a_i(s)$, for all $s \in \mathbb{R}, i = 1, 2, \ldots, n$.

Proof. Let $x(t, \varphi_1)$ and $x(t, \varphi_2)$ be two solutions of system (2.1) with initial value $\varphi_1 = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $\varphi_2 = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{P}C(\mathbb{R}, \mathbb{R}^n)$, respectively.

Let

$$F_i(\theta) = \mu_i \left( a_i - \frac{\theta}{a_i} \right) - \sum_{j=1}^{n} \mu_j \left( \bar{p}_{ji} + u_{ji}K_{ji}(\theta) \right)_{\rho_j, L_j}, \quad i = 1, 2, \ldots, n.$$

(2.9)

Since $\mathcal{M}$ is a nonsingular $M$-matrix according to condition $(H_6)$, $\mathcal{M}^T$ is also a nonsingular $M$-matrix; we know from Lemma 2.4 that there exists a vector $p = (\mu_1, \mu_2, \ldots, \mu_n)^T$ such that $\mathcal{M}^T p > 0$; that is,

$$\mu_i a_i - \sum_{j=1}^{n} \mu_j (\bar{p}_{ji} + u_{ji})_{\rho_j, L_j} > 0,$$

(2.10)

for $1 \leq i \leq n$, which indicates that $F_i(0) > 0$. Since $F_i(\theta)$ is continuous and differential on $[0, r_{ji})$ and $\lim_{\theta \to r_{ji}} F_i(\theta) = -\infty$ according to condition $(H_4)$, $F_i'(\theta) < 0$ for $\theta \in [0, u_{ji})$. There exist constants $\theta_i$ such that $F_i(\theta_i) = 0$ for $i = 1, 2, \ldots, n$. So we can choose

$$0 < \lambda \leq \min\{\theta_1, \theta_2, \ldots, \theta_n\},$$

(2.11)

such that

$$F_i(\lambda) \geq 0.$$
Define

\[ X_i(t) = |x_i(t, \varphi_2) - x_i(t, \varphi_1)|. \] (2.13)

Now we define a Lyapunov function \( V(t) \) by

\[
V(t) = \sum_{i=1}^{n} \mu_i \left\{ V_i(t) + \sum_{j=1}^{n} \bar{u}_{ij} L_j \int_{0}^{t} k_{ij}(s) \int_{t-s}^{t} X_j(\mu) e^{\lambda(s+\mu)} d\mu \, ds \right\},
\]

in which

\[
V_i(t) = e^{\lambda t} \text{sign}(x_i(t, \varphi_2) - x_i(t, \varphi_1)) \int_{x_i(t, \varphi_1)}^{x_i(t, \varphi_2)} \frac{1}{a_i(s)} \, ds.
\] (2.15)

for \( i = 1, 2, \ldots, n \).

When \( t \neq t_k, \ k \in \mathbb{Z}^+ \), calculating the upper right derivative of \( V(t) \) along solution of (2.1), similar to proof of Theorem 3.1 in [10], corresponding to case in which \( r \to 1, v_{ij}(t) = 0 \) in [10], we obtain from (2.12)–(2.15) that

\[
D^+ V(t)|_{(2.1)} \leq -e^{\lambda t} \left\{ \sum_{i=1}^{n} \mu_i \left( \frac{a_i}{a_i} - 1 \right) X_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \left( \bar{p}_{ij} + \bar{u}_{ij} K_{ij}(\lambda) \right) \rho_j L_j X_j(t) \right\}
\]

\[
= -e^{\lambda t} \left\{ \sum_{i=1}^{n} \mu_i \left( \frac{a_i}{a_i} - 1 \right) X_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \left( \bar{p}_{ij} + \bar{u}_{ij} K_{ij}(\lambda) \right) \rho_j L_j X_j(t) \right\}
\]

\[
= -e^{\lambda t} \sum_{i=1}^{n} F_i(\lambda) X_i(t) \leq 0.
\] (2.16)

When \( t = t_k, \ k \in \mathbb{Z}^+ \), we have

\[
V_i(t_k^+) = e^{\lambda t_0} \text{sign}(x_i(t_k^+, \varphi_2) - x_i(t_k^+, \varphi_1)) \int_{x_i(t_k^+, \varphi_1)}^{x_i(t_k^+, \varphi_2)} \frac{1}{a_i(s)} \, ds
\]

\[
= e^{\lambda t} \text{sign}(1 - \gamma_i)(x_i(t_k, \varphi_2) - x_i(t_k, \varphi_1)) \int_{(1-\gamma_i)x_i(t_k, \varphi_1)}^{(1-\gamma_i)x_i(t_k, \varphi_2)} \frac{1}{a_i(s)} \, ds,
\] (2.17)

\[
= e^{\lambda t} \text{sign}(x_i(t_k, \varphi_2) - x_i(t_k, \varphi_1)) \int_{x_i(t_k, \varphi_1)}^{x_i(t_k, \varphi_2)} \frac{|1 - \gamma_i|}{a_i((1 - \gamma_i)s)} \, ds,
\]
which, together with (H2), leads to

\[ V_i(t_k) - V_i(t_k^+) = e^{\lambda t} \text{sign}(x_i(t_k, \varphi_2) - x_i(t_k, \varphi_1)) \int_{x_i(t_k, \varphi_1)}^{x_i(t_k, \varphi_2)} \left( \frac{1}{a_i(s)} - \frac{|1 - \gamma_k|}{a_i((1 - \gamma_k)s)} \right) ds, \]

\[ \geq e^{\lambda t} \text{sign}(x_i(t_k, \varphi_2) - x_i(t_k, \varphi_1)) \int_{x_i(t_k, \varphi_1)}^{x_i(t_k, \varphi_2)} \frac{a_i((1 - \gamma_k)s) - |1 - \gamma_k|a_i(s)}{a_i(s)a_i((1 - \gamma_k)s)} ds \geq 0, \]

that is,

\[ V_i(t_k^+) \leq V_i(t_k). \tag{2.19} \]

It follows that

\[ V(t_k^+) = \sum_{i=1}^{n} \mu_i \left\{ V_i(t^+) + \sum_{j=1}^{n} \bar{u}_{ij} L_j \rho_j \int_{0}^{\infty} k_{ij}(s) \int_{t-s}^{t^+} X_j(\mu)e^{\lambda(s+\mu)} d\mu ds \right\} \]

\[ \leq \sum_{i=1}^{n} \mu_i \left\{ V_i(t) + \sum_{j=1}^{n} \bar{u}_{ij} L_j \rho_j \int_{0}^{\infty} k_{ij}(s) \int_{t-s}^{t} X_j(\mu)e^{\lambda(s+\mu)} d\mu ds \right\} = V(t_k). \tag{2.20} \]

Then we have

\[ V(t) \leq V(0). \tag{2.21} \]

On the other hand, from (2.14), we have

\[ V(t) \geq m_0 e^{\lambda t} \sum_{i=1}^{n} |x_i(t, \varphi_2) - x_i(t, \varphi_1)|, \quad V(0) \leq M_0 \sup_{t \in (-\infty, 0]} \sum_{i=1}^{n} |\varphi_i(t) - \zeta_i(t)|, \tag{2.22} \]

in which

\[ m_0 = \min_{1 \leq i \leq n} \left( \frac{\mu_i}{\bar{a}_i} \right), \quad M_0 = \max \left\{ M_1, M_2 \right\}, \quad M_1 = \max_{1 \leq i \leq n} \left( \frac{\mu_i}{\bar{a}_i} \right), \]

\[ M_2 = \sum_{j=1}^{n} \mu_j \max_{1 \leq i \leq n} (\bar{u}_{ij} \rho_j L_i) \int_{0}^{\infty} se^{\lambda s} \max_{1 \leq i \leq n} k_{ij}(s) ds. \tag{2.23} \]

Hence, from (2.21) and (2.22), we know that the following inequality holds for \( t > 0 \):

\[ \|x(t, \varphi_2) - x(t, \varphi_1)\| \leq M \|\varphi_2 - \varphi_1\| e^{-\lambda t}, \tag{2.24} \]

in which \( M = M_0/m_0 \).
We can always choose a positive integer $N$ such that $e^{-\lambda_{0}NT}M \leq 1/2$ and define a Poincaré mapping $P : C \rightarrow C$ by $P(\xi) = x_{T}(\xi)$; we have

$$\left\|P^{N}\varphi_{2} - P^{N}\varphi_{1}\right\| \leq \frac{1}{2}\left\|\varphi_{2} - \varphi_{1}\right\|,$$

(2.25)

which implies that $P^{N}$ is a contraction mapping. Similar to [10], using contraction mapping principle, we know that system (2.1) has a $T$-periodic solution which is globally exponentially stable. This completes the proof.

**Remark 2.6.** The result above also holds for (2.1) without impulses, and the existence and globally exponential stability of the periodic solution for (2.1) have nothing to do with amplification functions and inputs of the neuron. The results in [5] have more restrictions than Lemma 2.5 in this paper because conditions for the ones in [5] are relevant to amplification functions.

### 3. Periodic Solutions of CGBAMNNs with Distributed Delays and Impulses

Consider the following periodic CGBAMNNs model with distributed delays:

$$\dot{x}_{i}(t) = -a_{i}(x_{i}(t)) \left[ b_{i}(t,x_{i}(t)) - \sum_{j=1}^{m} p_{ij}(t)f_{j}(\rho_{j}y_{j}(t)) \right. \left. - \sum_{j=1}^{m} u_{ij}(t) \int_{0}^{+\infty} k_{ij}(s)f_{j}(\rho_{j}y_{j}(t-s))ds - I_{i}(t) \right], \quad t > 0, \ t \neq t_{k},$$

$$\Delta x_{i}(t_{k}) = -\gamma_{ik}x_{i}(t_{k}), \quad t = t_{k}, \ k \in Z^{+},$$

$$\dot{y}_{j}(t) = -c_{j}(y_{j}(t)) \left[ d_{j}(t,y_{j}(t)) - \sum_{i=1}^{n} q_{ji}(t)g_{i}(\tilde{\rho}_{i}x_{i}(t)) \right. \left. - \sum_{i=1}^{n} v_{ji}(t) \int_{0}^{+\infty} \tilde{k}_{ji}(s)g_{i}(\tilde{\rho}_{i}x_{i}(t-s))ds - J_{j}(t) \right], \quad t > 0, \ t \neq t_{k},$$

$$\Delta y_{j}(t_{k}) = -\delta_{jk}y_{j}(t_{k}), \quad t = t_{k}, \ k \in Z^{+}$$

for $1 \leq i \leq n$, $1 \leq j \leq m$, and $Z^{+} = \{1,2,\ldots\}$; $x_{i}(t)$ and $y_{j}(t)$ denote the state variable of the $i$th neuron from the neural field $F_{X}$ and the $j$th neuron from the neural field $F_{Y}$ at time $t$; $f_{j}(\cdot)$ and $g_{i}(\cdot)$ denote the signal functions of the $j$th neuron from the neural field $F_{Y}$ and the $i$th neuron from the neural field $F_{X}$ at time $t$; respectively, $I_{i}$ and $J_{j}$ denote inputs of the $i$th neuron from the neural field $F_{X}$ and the $j$th neuron from the neural field $F_{Y}$ at time $t$; respectively, $a_{i}(\cdot)$ and $c_{j}(\cdot)$ represent amplification functions; $b_{i}(t,\cdot)$ and $d_{j}(t,\cdot)$ are appropriately behaved functions; $p_{ij}(t), q_{ji}(t), u_{ij}(t),$ and $v_{ji}(t)$ are the connection weights; $\rho_{j}, \tilde{\rho}_{i}$ are positive constants, which correspond to the neuronal gains associated with the neuronal activations; $k_{ij}$ and $\tilde{k}_{ji}$ correspond to the delay kernel functions; $u_{ij}(t), v_{ji}(t),$ $p_{ij}(t), q_{ji}(t),$ $I_{i}(t),$ and $J_{j}(t)$ are all continuously periodic functions on $[0, +\infty)$ with common period $T > 0.$
\[ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-), \quad \Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-); \quad t_k \text{ is called impulsive moment and satisfies} \quad 0 < t_1 < t_2 < \cdots, \lim_{k \to +\infty} t_k = +\infty; \quad x_i(t^-), y_j(t^-) \text{ and } x_i(t^+), y_j(t^+) \text{ denote the left-hand and right-hand limits at } t_k; \text{ respectively, we always assume } x_i(t_k^-) = x_i(t_k), y_j(t_k^-) = y_j(t_k), \quad x_i(t_k^+) = x_i(t_k), \text{ and } y_j(t_k^+) = y_j(t_k), \quad k \in \mathbb{Z}^+. \]

For system (3.1), we assume the following.

(H8) Amplification functions \(a_i(\cdot)\) and \(c_i(\cdot)\) are continuous and there exist constants \(\underline{a}_i, \overline{a}_i\) and \(\underline{c}_i, \overline{c}_i\) such that \(0 < \underline{a}_i \leq a_i(x(t)) \leq \overline{a}_i, \quad 0 < \underline{c}_i \leq c_i(y(t)) \leq \overline{c}_i, \quad 1 \leq i \leq n, 1 \leq j \leq m.\)

(H9) \(b_i(t, u), d_i(t, u)\) are \(T\)-periodic about the first argument, there exist continuous, \(T\)-periodic functions \(\alpha_i(t)\) and \(\beta_j(t)\) such that
\[
\frac{b_i(t, x) - b_i(t, y)}{x - y} \geq \alpha_i(t) > 0, \quad \frac{d_i(t, x) - d_i(t, y)}{x - y} \geq \beta_j(t) > 0
\]
for all \(x \neq y, 1 \leq i \leq n, 1 \leq j \leq m.\)

(H10) For activation functions \(f_j(\cdot)\) and \(g_i(\cdot)\), there exist constant \(L_j\) and \(\tilde{L}_i\) such that
\[
L_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|, \quad \tilde{L}_i = \sup_{x \neq y} \left| \frac{g_i(x) - g_i(y)}{x - y} \right|, \quad \forall x \neq y \in \mathbb{R}, 1 \leq i \leq n, 1 \leq j \leq m.
\]

(H11) The kernel functions \(k_{ij}(s)\) and \(\tilde{k}_{ij}(s)\) are nonnegative continuous functions on \([0, +\infty)\) and satisfy
\[
\int_0^{+\infty} se^{\lambda s} k_{ij}(s) ds < +\infty, \quad \int_0^{+\infty} se^{\lambda s} \tilde{k}_{ij}(s) ds < +\infty, \quad \lambda \in [0, r_{ij}], \quad \lambda \in [0, \tilde{r}_{ij}]; \text{ respectively, } 0 < r_{ij} < +\infty, \quad 0 < \tilde{r}_{ij} < +\infty, \quad K_{ij}(0) = 1, \quad \tilde{K}_{ij}(0) = 1, \quad \lim_{\lambda \to r_{ij}} K_{ij}(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to \tilde{r}_{ij}} \tilde{K}_{ij}(\lambda) = +\infty.
\]

(H12) There exists positive integer \(k_0\) such that \(t_{k+k_0} = t_k + T\) and \(y_{i(k+k_0)} = y_{ik}, \quad \delta_j(k+k_0) = \delta_{jk}\) hold.

We assume that system (3.1) has the following initial conditions:
\[
x_i(s) = \varphi_i(s), \quad y_j(s) = \phi_j(s), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m, \quad -\infty \leq s \leq 0,
\]
where \(\varphi = (\varphi, \phi) \in \text{PC}((-\infty, 0], \mathbb{R}^{n+m}), \varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s)), \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_m(s)).\)

Let \(Z(t, \varphi) = (x(t, \varphi), y(t, \varphi))\) denote any solution of the system (3.1) with initial value \(\varphi = (\varphi, \phi) \in \text{PC}, \quad x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \ldots, x_{n+\xi}(t, \varphi)), \quad y(t, \varphi) = (y_1(t, \varphi), y_2(t, \varphi), \ldots, y_m(t, \varphi)).\)
Theorem 3.1. Under assumptions \((H_6)-(H_{12})\), there exists a \(T\)-periodic solution which is asymptotically stable, if the following conditions hold.

\((H_{13})\) The following \(\mathcal{M}\) is a nonsingular \(M\)-matrix, and

\[
\mathcal{M} = \begin{pmatrix} A & -C \\ \bar{C} & \bar{A} \end{pmatrix},
\]

in which

\[
A = \text{diag}(a_1, a_2, \ldots, a_n), \quad \bar{A} = \text{diag}(\beta_1, \beta_2, \ldots, \beta_m),
\]

\[
C = (c_{ij})_{m \times n}, \quad \bar{C} = (\bar{c}_{ij})_{n \times m}, \quad \bar{e}_{ij} = \left(\bar{q}_{ji} + \bar{v}_{ji}\right)\bar{\rho}_iL_i, \quad e_{ij} = \left(\bar{v}_{ij} + \bar{u}_{ij}\right)\rho_jL_j.
\]

\((H_{14})\) \(a_i((1 - \gamma_i)s) \geq |1 - \gamma_i|a_i(s), c_j((1 - \delta_jk)s) \geq |1 - \delta_jk|c_j(s), \forall s \in \mathbb{R}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m.\)

Proof. Let

\[
\begin{align*}
x_{n+j}(t) &= y_j(t), \quad a_{n+j}(t, x_{n+j}(t)) = c_j(t, y_j(t)), \quad b_{n+j}(t, x_{n+j}(t)) = d_j(t, y_j(t)), \\
p_{n+j}(t) &= q_{ji}(t), \quad p_{i,n+j}(t) = p_{i,j}(t), \quad u_{n+j}(t) = v_{ji}(t), \quad u_{i,n+j}(t) = u_{i,j}(t), \\
S_i(x_i(t)) &= g_i(x_i(t)), \quad S_{n+j}(x_{n+j}(t)) = f_j(x_j(t)), \quad q_{n+j}(s) = q_j(s), \\
I_{n+j} = J_j(t), \quad k_{n+j}(s) = k_{ji}(s), \quad k_j(s) = k_{i,n+j}(s), \\
a_{n+j}(t) = \beta_j(t), \quad \bar{L}_{n+j} = L_j, \quad \bar{\rho}_{n+j} = \rho_j.
\end{align*}
\]

It follows that system (3.1) can be rewritten as

\[
\begin{align*}
\dot{x}_i(t) &= -a_i(x_i(t)) \left[ b_j(t, x_i(t)) - \sum_{j=1}^{m} p_{i,n+j}(t) S_{n+j}(\bar{\rho}_{n+j} x_{n+j}(t)) \right] \\
&\quad - \sum_{j=1}^{m} u_{i,n+j}(t) \int_{0}^{+\infty} k_{i,n+j}(s) S_{n+j}(\bar{\rho}_{n+j} x_{n+j}(t - s)) ds - I_i(t), \quad t \neq t_k, \\
\Delta x_i(t_k) &= -\gamma_{ik} x_i(t_k), \quad t = t_k, \ k \in \mathbb{Z}^+, \\
\dot{x}_{n+j}(t) &= -a_{n+j}(x_{n+j}(t)) \left[ b_{n+j}(t, x_{n+j}(t)) - \sum_{i=1}^{n} p_{n+j,i}(t) S_i(\bar{\rho}_i x_i(t)) \right] \\
&\quad - \sum_{i=1}^{n} u_{n+j,i}(t) \int_{0}^{+\infty} k_{n+j,i}(s) S_i(\bar{\rho}_i x_i(t - s)) ds - I_{n+j}(t), \quad t \neq t_k, \\
\Delta x_{n+j}(t_k) &= -\gamma_{n+j,k} x_{n+j}(t_k), \quad t = t_k, \ k \in \mathbb{Z}^+,
\end{align*}
\]

for \(1 \leq i \leq n, \ 1 \leq j \leq m.\)
Initial conditions are given by
\[ x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0], \ i = 1, 2, \ldots, (n + m). \] (3.10)

Thus system (3.9) is a special case of system (2.1) in mathematical form, under conditions (H_8)–(H_{14}), we obtain from Lemma 2.5 that system (3.9) has a T-periodic solution which is globally exponentially stable if \( a_i((1 - \gamma_k)s) \geq |1 - \gamma_k|a_i(s) \) and the following matrix \( \mathcal{M}' \) is a \( M \)-matrix, and
\[ \mathcal{M}' = A' - C', \] (3.11)

where
\[
A' = \text{diag}(a_1, a_2, \ldots, a_{n+m}),
\]
\[
D' = \begin{pmatrix}
0 & \cdots & 0 & w'_{1,n+1} & \cdots & w'_{1,n+m} \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & w'_{n,n+1} & \cdots & w'_{n,n+m} \\
w'_{n+1,1} & \cdots & w'_{n+1,n} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
w'_{n+m,1} & \cdots & w'_{n+m,n} & 0 & \cdots & 0
\end{pmatrix}
\] (3.12)
in which \( w'_{ij} = (\bar{p}_{ij} + \bar{u}_{ij})\tilde{p}_{ij}\).

Then, we know from (3.8) and (3.11) that Theorem 3.1 holds.

If \( a_i(x_i(t)) = c_j(y_j(t)) = 1, \ b_i(t, x_i(t)) = b_i(t)x_i(t) \) and \( d_j(t, y_j(t)) = d_j(t)y_j(t) \), where \( b_i(t) \) and \( d_j(t) \) are positive continuous T-periodic functions for \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \). System (3.1) reduces to the following Hopfield-type BAM neural networks model:
\[
\begin{align*}
\dot{x}_i(t) &= -b_i(t)x_i(t) + \sum_{j=1}^{m} p_{ij}(t)f_j(\rho_jy_j(t)) \\
&\quad + \sum_{j=1}^{m} u_{ij}(t) \int_{0}^{\infty} k_{ij}(s)f_j(\rho_jy_j(t-s))ds + I_i(t), \quad t > 0, \ t \neq t_k, \\
\Delta x_i(t_k) &= -\gamma_{ik}x_i(t_k), \quad t = t_k, \ k \in \mathbb{Z}^+, \\
\dot{y}_j(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} q_{ji}(t)g_i(\tilde{p}_ix_i(t)) \\
&\quad + \sum_{i=1}^{n} v_{ji}(t) \int_{0}^{\infty} \tilde{k}_{ji}(s)g_i(\tilde{p}_ix_i(t-s))ds + f_j(t), \quad t > 0, \ t \neq t_k, \\
\Delta y_j(t_k) &= -\delta_{jk}y_j(t_k), \quad t = t_k, \ k \in \mathbb{Z}^+.
\end{align*}
\] (3.13)

**Corollary 3.2.** Under assumptions (H_8)–(H_{12}), there exists a T-periodic solution which is globally asymptotically stable, if the following conditions hold.
(\textbf{H}_{13}) The following $\mathcal{M}$ is a nonsingular $M$-matrix, and

$$
\mathcal{M} = \begin{pmatrix} A & -C \\ -\bar{C} & \bar{A} \end{pmatrix},
$$

in which

$$
A = \text{diag}(b_1, b_2, \ldots, b_n), \quad \bar{A} = \text{diag}(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_m), \\
C = (e_{ij})_{m \times n'}, \quad \bar{C} = (\bar{e}_{ij})_{n \times m'}, \quad e_{ij} = (\bar{p}_{ij} + \bar{u}_{ij})\rho_j L_{ji}, \quad \bar{e}_{ij} = (\bar{q}_{ji} + \bar{v}_{ji})\bar{\rho}_i \bar{L}_{ij}.
$$

(\textbf{H}_{14}) $0 \leq \gamma_{ik} \leq 2, \ 0 \leq \delta_{jk} \leq 2$ for $i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m, \ k \in \mathbb{Z}^+.$

\textbf{Proof.} As $b_i(t, x_i(t)) = b_i(t)x_i(t)$ and $d_j(t, y_j(t)) = d_j(t)y_j(t),$ we obtain $a_i(t) = b_i(t)$ and $\beta_j(t) = d_j(t)$ in (\textbf{H}_2), (\textbf{H}_{13}) implies (\textbf{H}_{13}) holds. Since $a_i(x_i(t)) = c_i(y_i(t)) \equiv 1,$ then condition (\textbf{H}_{14}) reduces to (\textbf{H}_{14}). Corollary 3.2 Holds from Theorem 3.1.

Remark 3.3. The conditions for the existence and globally exponential stability of the periodic solution of (3.1) without impulses have nothing to do with inputs of the neuron and amplification functions. The results in [13, 14] have more restrictions than Theorem 3.1 in this paper because conditions for the ones in [13, 14] are relevant to amplification functions and inputs of neurons our results should be better. In addition, Corollary 3.2 is similar to Theorem 2.1 in [15]; our results generalize the results in [15].

Remark 3.4. In view of proof of Theorem 3.1, since CGBAMNNs model is a special case of CGNNs model in form as BAM neural networks model is a special case of Hopfield neural networks model, many results of CGBAMNNs can be directly obtained from the ones of CGNNs, needing no repetitive discussions. Since system (3.1) reduces to autonomous system, Theorem 3.1 still holds, which means that system (3.1) has an equilibrium which is globally asymptotically stable; we know that many results in [18] can be directly obtained from the results in [19].

\section*{4. Two Simple Examples}

\textbf{Example 4.1.} Consider the following CGNNs model with distributed delays:

$$
\begin{align*}
\dot{x}_1(t) &= -2 \left[ x_1(t) - 0.2 \tanh(x_1(t)) - \sin t \int_0^{+\infty} e^{-s} \tanh(x_2(t - s))ds \right], \\
\dot{x}_2(t) &= -2(2 + \cos(x_2(t))) \left[ x_2(t) - 0.3 \int_0^{+\infty} e^{-s} \tanh(x_1(t - s))ds - 5 \right].
\end{align*}
$$

(4.1)

Obviously, system (4.1) satisfies (\textbf{H}_1)--(\textbf{H}_5).
Figure 1: Time response of state variables $x_1$, $x_2$ and phase plot in space $(t, x_1, x_2)$ for system (4.1).

Note that

\[ M = \begin{pmatrix} 1 & -1.2 \\ -0.3 & 1 \end{pmatrix}, \] (4.2)

it is a nonsingular $M$-matrix and system (4.1) also satisfies condition ($H_6$). According to Lemma 2.5, system (4.1) has a $2\pi$-periodic solution which is globally exponentially stable. Figure 1 shows the dynamic behaviors of system (4.1) with initial condition $x_0$. However, it is easy to check that system (4.1) does not satisfy Theorem 4.3 or 4.4 in [5], so theorems in [5] cannot be used to ascertain the existence and stability of periodic solutions of system (4.1).

Example 4.2. Consider the following CGBAMNNs model with distributed delays and impulses:

\[
\begin{align*}
\dot{x}_1(t) &= -(2 + \sin(x_1(t))) \left[2x_1(t) - \sin t \int_0^{+\infty} e^{-s} |y_1(t-s)| ds - 1\right], & t > 0, t \neq t_k, \\
\Delta x_1(t_k) &= -\gamma_1 k x_i(t_k), & t = t_k, k \in \mathbb{Z}^+, \\
\dot{y}_1(t) &= -(3 + \cos(y_1(t))) \left[(3 + \cos t)y_1(t) - \sin t \int_0^{+\infty} e^{-s}|x_1(t-s)| ds - 1\right], & t > 0, t \neq t_k, \\
\Delta y_1(t_k) &= -\delta_1 k y_1(t_k), & t = t_k, k \in \mathbb{Z}^+, 
\end{align*}
\] (4.3)

where $t_k = \pi k, k \in \mathbb{Z}^+$.

Obviously, system (4.3) satisfies ($H_8$)–($H_{12}$).
Case 1. \( \gamma_{1k} = 0, \delta_{1k} = 0. \) Note that

\[
\mathcal{M} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.4}
\]

it is a nonsingular \( M \)-matrix and system (4.3) also satisfies condition (H13). According to Theorem 3.1, system (4.3) without impulses has a \( 2\pi \)-periodic solution which is globally exponentially stable. Figure 2 shows the dynamic behaviors of system (4.3) with initial condition \((0,1,0).\)

However, it is easy to check that system (4.3) without impulses does not satisfy Theorem 1 in [13] and theorems in [14]; so theorems in [13, 14] cannot be used to ascertain the existence and stability of periodic solutions of system (4.3).

Case 2. \( \gamma_{1k} = 0.7, \delta_{1k} = (1 - 0.5 \sin(t_{k} + 1)). \) Note that \( a_{1}(s) = 2 + \sin s, c_{1}(s) = 3 + \cos s, \) and \( a_{1} / a_{1} = 0.5 > |1 - \gamma_{1k}| = 0.3 \) and \( |1 - \delta_{1k}| < 0.5 < c_{1} / c_{1} = 2 / 3, \) which means condition (H14) also holds for system (4.3). Hence, system (4.3) with impulses still has that there exists a \( 2\pi \)-periodic solution which is globally asymptotically stable. Figure 3 shows the dynamic behaviors of system (4.3) with initial condition \((0.1,0.2).\)

This example illustrates the feasibility and effectiveness of the main results obtained in this paper, and it also shows that the conditions for the existence and globally exponential stability of the periodic solutions of CGBAMNNs without impulses have nothing to do with inputs of the neurons and amplification functions. If impulsive perturbations exist, the periodic solutions still exist and they are globally exponentially stable when we give some restrictions on impulsive perturbations.
5. Conclusions

A class of CGBAMNNs with distributed delays and impulses are investigated by using suitable Lyapunov functional, the properties of $M$-matrix, and some suitable mathematical transformation in this paper. Sufficient conditions to guarantee the uniqueness and global exponential stability of the periodic solutions of such networks are established without assuming the boundedness of the activation functions. Lemma 2.5 improves the results in [5], and Theorem 3.1 improves the results in [13, 14] and generalize the results in [15]. In addition, we point that CGBAMNNs model is a special case of CGNNs model; many results of CGBAMNNs can be directly obtained from the ones of CGNNs, needing no repetitive discussions. Our results are new, and two examples have been provided to demonstrate the effectiveness of our results.

Appendix

The source program (MATLAB 7.0) of Figure 1 is given as follows [14].

```matlab
clear
T=70;
N=7000;
h=T/N;
m=40/h;
for i=1:m
U(:,i)=[0.1; 0.2];
end
```
The source program (MATLAB 7.0) of Figures 2 and 3 is given as follows [14].

clear
T=70;
N=7000;
h=T/N;
m=40/h;
for i=1:m
U(:,i)=[0.1;0.2];
end
for i=(m+1):(N+m)
r(i)=i*h-40;
x(i)=r(i);
I=2+
J=2+cos(U(2,i-1));
A=[-1,0;0,-J];
B=[0,sin(x(i))*I;0.3*J,0];
U(:,i)=h*A*[(U(1,i-1)-0.2*tanh(U(1,i-1)))+U(:,i-1)];
P(:,1)=[0;0];
for k=1:m
P(:,1)=P(:,1)+h*exp(-(40-(k-1)*h))*(tanh(U(1,i-m+k-1)))+(tanh(U(2,i-m+k-1))));
end
U(:,i)=U(:,i)+B*h*[(P(1,1));(P(2,1))]+h*[0;5*J];
end
y=U(1,:);
z=U(2,:);
hold on
plot(r,y,'-')
hold on
plot(r,z)
hold on
plot3(r,y,z)
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References


