Research Article

A Note on Approximating Curve with 1-Norm Regularization Method for the Split Feasibility Problem

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Inspired by the very recent results of Wang and Xu (2010), we study properties of the approximating curve with 1-norm regularization method for the split feasibility problem (SFP). The concept of the minimum-norm solution set of SFP in the sense of 1-norm is proposed, and the relationship between the approximating curve and the minimum-norm solution set is obtained.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. The problem under consideration in this paper is formulated as finding a point $x$ satisfying the property:

$$x \in C, \quad Ax \in Q,$$  \hspace{2cm} (1.1)

where $A : H_1 \to H_2$ is a bounded linear operator. Problem (1.1), referred to by Censor and Elfving [1] as the split feasibility problem (SFP), attracts many authors’ attention due to its application in signal processing [1]. Various algorithms have been invented to solve it (see [2–13] and references therein).

Using the idea of Tikhonov’s regularization, Wang and Xu [14] studied the properties of the approximating curve for the SFP. They gave the concept of the minimum-norm solution of the SFP (1.1) and proved that the approximating curve converges strongly...
to the minimum-norm solution of the SFP (1.1). Together with some properties of this approximating curve, they introduced a modification of Byrne’s CQ algorithm [2] so that strong convergence is guaranteed and its limit is the minimum-norm solution of SFP (1.1).

In the practical application, \( H_1 \) and \( H_2 \) are often \( \mathbb{R}^N \) and \( \mathbb{R}^M \), respectively. Moreover, scientists and engineers are more willing to use 1-norm regularization method in the calculation process (see, e.g., [15–18]). Inspired by the above results of Wang and Xu [14], we study properties of the approximating curve with 1-norm regularization method. We also define the concept of the minimum-norm solution set of SFP (1.1) in the sense of 1-norm. The relationship between the approximating curve and the minimum-norm solution set is obtained.

2. Preliminaries

Let \( X \) be a normed linear space with norm \( \| \cdot \| \), and let \( X^* \) be the dual space of \( X \). We use the notation \( \langle x, f \rangle \) to denote the value of \( f \in X^* \) at \( x \in X \). In particular, if \( X \) is a Hilbert space, we will denote it by \( H \), and \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) are the inner product and its induced norm, respectively.

We recall some definitions and facts that are needed in our study.

Let \( P_C \) denote the projection from \( H \) onto a nonempty closed convex subset \( C \) of \( H \); that is,

\[
P_Cx = \arg \min_{y \in C} \| x - y \|, \quad x \in H.
\] (2.1)

It is well known that \( P_Cx \) is characterized by the inequality

\[
\langle x - P_Cx, y - P_Cx \rangle \leq 0, \quad \forall y \in C.
\] (2.2)

Definition 2.1. Let \( \varphi : X \to \mathbb{R} \cup \{ +\infty \} \) be a convex functional, \( x_0 \in \text{dom}(\varphi) = \{ x \in X : \varphi(x) < +\infty \} \). Set

\[
\partial \varphi(x_0) = \{ \xi \in X^* : \varphi(x) \geq \varphi(x_0) + \langle x - x_0, \xi \rangle, \forall x \in X \}.
\] (2.3)

If \( \partial \varphi(x_0) \neq \emptyset \), \( \varphi \) is said to be subdifferentiable at \( x_0 \) and \( \partial \varphi(x_0) \) is called the subdifferential of \( \varphi \) at \( x_0 \). For any \( \xi \in \partial \varphi(x_0) \), we say \( \xi \) is a subgradient of \( \varphi \) at \( x_0 \).

Lemma 2.2. There holds the following property:

\[
\partial(\| x \|) = \begin{cases} 
\{ x^* \in X^* : \| x^* \| = 1, \langle x, x^* \rangle = \| x \| \}, & x \neq 0, \\
\{ x^* \in X^* : \| x^* \| \leq 1 \}, & x = 0,
\end{cases}
\] (2.4)

where \( \partial(\| x \|) \) denotes the subdifferential of the functional \( \| x \| \) at \( x \in X \).

Proof. The process of the proof will be divided into two parts.
Case 1. In the case of \( x = 0 \), for any \( x^* \in X^* \) such that \( \|x^*\| \leq 1 \) and any \( y \in X \), there holds the inequality

\[
\|y\| \geq \langle y, x^* \rangle = \|x\| + \langle y - x, x^* \rangle,
\]

so we have \( x^* \in \partial(\|x\|) \), and thus,

\[
\{x^* \in X^* : \|x^*\| \leq 1 \} \subset \partial(\|x\|).
\]

Conversely, for any \( x^* \in \partial(\|x\|) \), we have from the definition of subdifferential that

\[
\|y\| \geq \|x\| + \langle y - x, x^* \rangle = \langle y, x^* \rangle, \quad \forall y \in X,
\]

\[
\|y\| = \|y - x\| \geq \langle -y, x^* \rangle = -\langle y, x^* \rangle.
\]

Consequently,

\[
|\langle y, x^* \rangle| \leq \|y\|, \quad \forall y \in X,
\]

and this implies that \( \|x^*\| \leq 1 \). Thus, we have verified that

\[
\partial(\|x\|) \subset \{x^* \in X^* : \|x^*\| \leq 1 \}.
\]

Combining (2.6) and (2.9), we immediately obtain

\[
\partial(\|x\|) = \{x^* \in X^* : \|x^*\| \leq 1 \}.
\]

Case 2. If \( x \neq 0 \), for any \( x^* \in \{x^* \in X^* : \|x^*\| = 1, \langle x, x^* \rangle = \|x\| \} \), we obviously have

\[
\langle y - x, x^* \rangle = \langle y, x^* \rangle - \|x\| \leq \|y\| - \|x\|, \quad \forall y \in X,
\]

which means that \( x^* \in \partial(\|x\|) \), and thus,

\[
\{x^* \in X^* : \|x^*\| = 1, \langle x, x^* \rangle = \|x\| \} \subset \partial(\|x\|).
\]

Conversely, if \( x^* \in \partial(\|x\|) \), we have

\[
\langle -x, x^* \rangle \leq 0 - \|x\| = -\|x\|, \quad \langle x, x^* \rangle \leq 2\|x\| - \|x\| = \|x\|;
\]

hence,

\[
\langle x, x^* \rangle = \|x\|.
\]
On the other hand, using (2.14), we get
\[ \|y\| \geq \|x\| + \langle y - x, x^* \rangle = \|x\| + \langle y, x^* \rangle - \langle x, x^* \rangle = \langle y, x^* \rangle, \quad \forall y \in X, \tag{2.15} \]
and consequently,
\[
\|y\| = \| - y \| \geq \|x\| + \langle -y - x, x^* \rangle \\
= \|x\| - \langle y, x^* \rangle - \langle x, x^* \rangle \\
= - \langle y, x^* \rangle; \tag{2.16}
\]
that is,
\[ -\|y\| \leq \langle y, x^* \rangle. \tag{2.17} \]
Equation (2.17) together with (2.15) implies that
\[ |\langle y, x^* \rangle| \leq \|y\|, \quad \forall y \in X; \tag{2.18} \]
hence, \(\|x^*\| \leq 1\). Note that (2.14) implies that \(\|x^*\| \geq \langle x, x^* \rangle / \|x\| = 1\); we assert that
\[ \|x^*\| = 1. \tag{2.19} \]
Thus we have from (2.14) and (2.19) that
\[ \{x^* \in X^*: \|x^*\| = 1, \langle x, x^* \rangle = \|x\| \} \supset \partial(\|x\|). \tag{2.20} \]
The proof is finished by combining (2.12) and (2.20).

\(\|\cdot\|_\infty\) and \(\|\cdot\|_1\) will stand for \(\infty\)-norm and 1-norm of any Euclidean space; respectively, that is, for any \(x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l\), we have
\[ \|x\|_\infty = \max_{1 \leq j \leq l} |x_j|, \quad \|x\|_1 = \sum_{j=1}^l |x_j|. \tag{2.21} \]

**Corollary 2.3.** In \(l\)-dimensional Euclidean space \(\mathbb{R}^l\), there holds the following result:
\[
\partial(\|x\|_1) = \begin{cases} 
\{\xi \in \mathbb{R}^l : \|\xi\|_\infty = 1, \langle x, \xi \rangle = \|x\|_1\}, & x \neq 0, \\
\{\xi \in \mathbb{R}^l : \|\xi\|_\infty \leq 1\}, & x = 0,
\end{cases} \tag{2.22}
\]
Let $H$ be a Hilbert space and $f : H \to \mathbb{R}$ a functional. Recall that

(i) $f$ is convex if $f((1 - \lambda)x + \lambda y) \leq \lambda f(x) + (1 - \lambda)f(y)$, for all $0 < \lambda < 1$, for all $x, y \in H$;

(ii) $f$ is strictly convex if $f((1 - \lambda)x + \lambda y) < \lambda f(x) + (1 - \lambda)f(y)$, for all $0 < \lambda < 1$, for all $x, y \in H$ with $x \neq y$;

(iii) $f$ is coercive if $f(x) \to \infty$ whenever $\|x\| \to \infty$. See [19] for more details about convex functions.

The following lemma gives the optimality condition for the minimizer of a convex functional over a closed convex subset.

**Lemma 2.4** (see [20]). Let $H$ be a Hilbert space and $C$ a nonempty closed convex subset of $H$. Let $f : H \to \mathbb{R}$ be a convex and subdifferentiable functional. Then $x \in C$ is a solution of the problem

$$\min_{x \in C} f(x)$$

if and only if there exists some $\xi \in \partial f(x)$ satisfying the following optimality condition:

$$\langle \xi, v - x \rangle \geq 0, \quad \forall v \in C.$$  \hspace{1cm} (2.24)

**3. Main Results**

It is well known that SFP (1.1) is equivalent to the minimization problem

$$\min_{x \in C} \| (I - P_\mathcal{Q}) A x \|^2.$$ \hspace{1cm} (3.1)

Using the idea of Tikhonov’s regularization method, Wang and Xu [14] studied the minimization problem in Hilbert spaces:

$$\min_{x \in C} \| (I - P_\mathcal{Q}) A x \|^2 + \alpha \|x\|^2,$$ \hspace{1cm} (3.2)

where $\alpha > 0$ is the regularization parameter.

In what follows, $H_1$ and $H_2$ in SFP (1.1) are restricted to $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, and $\| \cdot \|$ will stand for the usual 2-norm of any Euclidean space $\mathbb{R}^l$; that is, for any $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_l^2}. \hspace{1cm} (3.3)$$

Inspired by the above work of Wang and Xu, we study properties of the approximating curve with 1-norm regularization scheme for the SFP, that is, the following minimization problem:

$$\min_{x \in C} \frac{1}{2} \| (I - P_\mathcal{Q}) A x \|^2 + \alpha \|x\|_1,$$ \hspace{1cm} (3.4)
where $\alpha > 0$ is the regularization parameter. Let

$$f_\alpha(x) = \frac{1}{2} \| (I - P_Q) A x \|_2^2 + \alpha \| x \|_1.$$  \hfill (3.5)

It is easy to see that $f_\alpha$ is convex and coercive, so problem (3.4) has at least one solution. However, the solution of problem (3.4) may not be unique since $f_\alpha$ is not necessarily strictly convex. Denote by $S_\alpha$ the solution set of problem (3.4); thus we can assert that $S_\alpha$ is a nonempty closed convex set but may contain more than one element. The following simple example illustrates this fact.

**Example 3.1.** Let $C = \{(x, y) : x + y = 1\}$, $Q = \{(x, y) : x + y = 1/2\}$ and

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$ \hfill (3.6)

Then $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a bounded linear operator. Obviously, $S_\alpha = \{(x, y) : x + y = 1, x \geq 0, y \geq 0\}$ and it contains more than one element.

**Proposition 3.2.** For any $\alpha > 0$, $x_\alpha \in S_\alpha$ if and only if there exists some $\xi \in \partial \| x \|_1$ satisfying the following inequality:

$$\langle A^* (I - P_Q) A x_\alpha + \alpha \xi, v - x_\alpha \rangle \geq 0, \quad \forall v \in C.$$ \hfill (3.7)

**Proof.** Let

$$f(x) = \frac{1}{2} \| (I - P_Q) A x \|_2^2,$$ \hfill (3.8)

then

$$f_\alpha(x) = f(x) + \alpha \| x \|_1.$$ \hfill (3.9)

Since $f$ is convex and differentiable with gradient

$$\nabla f(x) = A^* (I - P_Q) A x,$$ \hfill (3.10)

$f_\alpha$ is convex, coercive, and subdifferentiable with the subdifferential

$$\partial f_\alpha(x) = \partial f(x) + \alpha \partial \{ \| x \|_1 \};$$ \hfill (3.11)

that is,

$$\partial f_\alpha(x) = A^* (I - P_Q) A x + \alpha \partial \{ \| x \|_1 \}.$$ \hfill (3.12)

By Corollary 2.3 and Lemma 2.4, the proof is finished. \qed
Theorem 3.3. Denote by \( x_\alpha \) an arbitrary element of \( S_\alpha \), then the following assertions hold:

(i) \( \|x_\alpha\|_1 \) is decreasing for \( \alpha \in (0, \infty) \);

(ii) \( \|(I - P_Q)A x_\alpha\| \) is increasing for \( \alpha \in (0, \infty) \).

Proof. Let \( \alpha > \beta > 0 \), for any \( x_\alpha \in S_\alpha \), \( x_\beta \in S_\beta \). We immediately obtain

\[
\frac{1}{2} \left\| (I - P_Q) A x_\alpha \right\|^2 + \alpha \|x_\alpha\|_1 \leq \frac{1}{2} \left\| (I - P_Q) A x_\beta \right\|^2 + \alpha \|x_\beta\|_1, \tag{3.13}
\]

\[
\frac{1}{2} \left\| (I - P_Q) A x_\beta \right\|^2 + \beta \|x_\beta\|_1 \leq \frac{1}{2} \left\| (I - P_Q) A x_\alpha \right\|^2 + \beta \|x_\alpha\|_1. \tag{3.14}
\]

Adding up (3.13) and (3.14) yields

\[
\alpha \|x_\alpha\|_1 + \beta \|x_\beta\|_1 \leq \alpha \|x_\beta\|_1 + \beta \|x_\alpha\|_1,
\]

which implies \( \|x_\alpha\|_1 \leq \|x_\beta\|_1 \). Hence (i) holds.

Using (3.14) again, we have

\[
\frac{1}{2} \left\| (I - P_Q) A x_\beta \right\|^2 \leq \frac{1}{2} \left\| (I - P_Q) A x_\alpha \right\|^2 + \beta \left( \|x_\alpha\|_1 - \|x_\beta\|_1 \right), \tag{3.16}
\]

which together with (i) implies

\[
\left\| (I - P_Q) A x_\beta \right\|^2 \leq \left\| (I - P_Q) A x_\alpha \right\|^2, \tag{3.17}
\]

and hence (ii) holds.

\[\square\]

Let \( \mathcal{F} = C \cap A^{-1}(Q) \), where \( A^{-1}(Q) = \{ x \in \mathbb{R}^N : A x \in Q \} \). In what follows, we assume that \( \mathcal{F} \neq \emptyset \); that is, the solution set of SFP (1.1) is nonempty. The fact that \( \mathcal{F} \) is nonempty closed convex set thus allows us to introduce the concept of minimum-norm solution of SFP (1.1) in the sense of norm \( \| \cdot \| \) (induced by the inner product).

Definition 3.4 (see [14]). An element \( x^\dagger \in \mathcal{F} \) is said to be the minimum-norm solution of SFP (1.1) in the sense of norm \( \| \cdot \| \) if \( \|x^\dagger\| = \inf_{x \in \mathcal{F}} \|x\| \). In other words, \( x^\dagger \) is the projection of the origin onto the solution set \( \mathcal{F} \) of SFP (1.1). Thus there exists only one minimum-norm solution of SFP (1.1) in the sense of norm \( \| \cdot \| \), which is always denoted by \( x^\dagger \).

We can also give the concept of minimum-norm solution of SFP (1.1) in other senses.

Definition 3.5. An element \( \bar{x} \in \mathcal{F} \) is said to be a minimum-norm solution of SFP (1.1) in the sense of 1-norm if \( \|\bar{x}\|_1 = \inf_{x \in \mathcal{F}} \|x\|_1 \). We use \( \mathcal{F}_1 \) to stand for all minimum-norm solutions of SFP (1.1) in the sense of 1-norm and \( \mathcal{F}_1 \) is called the minimum-norm solution set of SFP (1.1) in the sense of 1-norm.

Obviously, \( \mathcal{F}_1 \) is a closed convex subset of \( \mathcal{F} \). Moreover, it is easy to see that \( \mathcal{F}_1 \neq \emptyset \). Indeed, taking a sequence \( \{ x_n \} \subset \mathcal{F} \) such that \( \|x_n\|_1 \rightarrow \inf_{x \in \mathcal{F}} \|x\|_1 \) as \( n \rightarrow \infty \), then \( \{ x_n \} \)
is bounded. There exists a convergent subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \). Set \( \bar{x} = \lim_{k \to \infty} x_{n_k} \), then \( \bar{x} \in \mathcal{F} \) since \( \mathcal{F} \) is closed. On the other hand, using lower semicontinuity of the norm, we have

\[
\|\bar{x}\| \leq \lim_{k \to \infty} \|x_{n_k}\| = \inf_{x \in \mathcal{F}} \|x\|_1, \tag{3.18}
\]

and this implies that \( \bar{x} \in \mathcal{F}_1 \).

However, \( \mathcal{F}_1 \) may contain more than one elements, in general (see Example 3.1, \( \mathcal{F}_1 = \{(x, y) : x + y = 1, x, y \geq 0\} \)).

**Theorem 3.6.** Let \( \alpha > 0 \) and \( x_\alpha \in S_\alpha \). Then \( \omega(x_\alpha) \subset \mathcal{F}_1 \), where \( \omega(x_\alpha) = \{x : \exists x_{\alpha_k} \subset \{x_\alpha\}, x_{\alpha_k} \to x \ \text{weakly}\} \).

**Proof.** Taking \( \bar{x} \in \mathcal{F}_1 \) arbitrarily, for any \( \alpha \in (0, \infty) \), we always have

\[
\frac{1}{2} \|(I - PQ)Ax_\alpha\|^2 + \alpha \|x_\alpha\|_1 \leq \frac{1}{2} \|(I - PQ)A\bar{x}\|^2 + \alpha \|\bar{x}\|_1. \tag{3.19}
\]

Since \( \bar{x} \) is a solution of SFP (1.1), \( \|(I - PQ)A\bar{x}\| = 0 \). This implies that

\[
\frac{1}{2} \|(I - PQ)Ax_\alpha\|^2 + \alpha \|x_\alpha\|_1 \leq \alpha \|\bar{x}\|_1, \tag{3.20}
\]

then,

\[
\|x_\alpha\|_1 \leq \|\bar{x}\|_1; \tag{3.21}
\]

thus \( \{x_\alpha\} \) is bounded.

Take \( \omega \in \omega(x_\alpha) \) arbitrarily, then there exists a sequence \( \{\alpha_n\} \) such that \( \alpha_n \to 0 \) and \( x_{\alpha_n} \to \omega \) as \( n \to \infty \). Put \( x_{\alpha_n} = x_n \). By Proposition 3.2, we deduce that there exists some \( \xi_n \in \partial(\|x_n\|_1) \) such that

\[
\langle A^*(I - PQ)Ax_n + \alpha_n \xi_n, \bar{x} - x_n \rangle \geq 0. \tag{3.22}
\]

This implies that

\[
\langle (I - PQ)Ax_n, A(\bar{x} - x_n) \rangle \geq \alpha_n \langle \xi_n, x_n - \bar{x} \rangle. \tag{3.23}
\]

Since \( A\bar{x} \in Q \), the characterizing inequality (2.2) gives

\[
\langle (I - PQ)Ax_n, A\bar{x} - PQ(Ax_n) \rangle \leq 0, \tag{3.24}
\]

then,

\[
\|(I - PQ)Ax_n\|^2 \leq \langle (I - PQ)Ax_n, A(x_n - \bar{x}) \rangle. \tag{3.25}
\]
Combining (3.23) and (3.25), we have
\[
\| (I - P_Q) A x_n \|^2 \leq \alpha_n \langle \xi_n, \tilde{x} - x_n \rangle \\
\leq \alpha_n \| \xi_n \|_\infty \| \tilde{x} - x_n \|_1 \\
\leq 2 \alpha_n \| \tilde{x} \|_1.
\]

Consequently, we get
\[
\lim_{n \to \infty} \| (I - P_Q) A x_n \| = 0.
\]

Furthermore, noting the fact that \( x_n \to \omega \) and \( I - P_Q \) and \( A \) are all continuous operators, we have \( (I - P_Q) A \omega = 0 \); that is, \( A \omega \in Q \); thus, \( \omega \in \mathcal{F} \). Since \( \tilde{x} \) is a minimum-norm solution of SFP (1.1) in the sense of 1-norm, using (3.21) again, we get
\[
\| \omega \|_1 \leq \liminf_{n \to \infty} \| x_n \|_1 \leq \| \tilde{x} \|_1 = \min \{ \| x \|_1 : x \in \mathcal{F} \}.
\]

Thus we can assert that \( \omega \in \mathcal{F}_1 \) and this completes the proof. \( \square \)

**Corollary 3.7.** If \( \mathcal{F}_1 \) contains only one element \( \tilde{x} \), then \( x_\alpha \to \tilde{x}, (\alpha \to 0) \).

**Remark 3.8.** It is worth noting that the minimum-norm solution of SFP (1.1) in the sense of norm \( \| \cdot \| \) is very different from the minimum-norm solution of SFP (1.1) in the sense of 1-norm. In fact, \( x^\dagger \) may not belong to \( \mathcal{F}_1 \)! The following simple example shows this fact.

**Example 3.9.** Let \( C = \{ (x, y) : x + 2y \geq 2, x \geq 0, y \geq 0 \} \), \( Q = \{ (x, y) : x + y = 1, x \geq 0, y \geq 0 \} \), and
\[
A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.
\]

It is not hard to see that \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) is a bounded linear operator and \( A(x, y)^T = ((1/2)x, y)^T \), for all \((x, y) \in C\). Obviously, \( \mathcal{F} = \{ (x, y) : x + 2y = 2, x \geq 0, y \geq 0 \} \), \( x^\dagger = (2/5, 4/5) \), but \( \mathcal{F}_1 = \{ (0, 1) \} \). Hence, \( x^\dagger \notin \mathcal{F}_1 \).

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**References**

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