Research Article

Jacobi Elliptic Solutions for Nonlinear Differential Difference Equations in Mathematical Physics

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We put a direct new method to construct the rational Jacobi elliptic solutions for nonlinear differential difference equations which may be called the rational Jacobi elliptic functions method. We use the rational Jacobi elliptic function method to construct many new exact solutions for some nonlinear differential difference equations via the lattice equation and the discrete nonlinear Schrödinger equation with a saturable nonlinearity. The proposed method is more effective and powerful to obtain the exact solutions for nonlinear differential difference equations.

1. Introduction

It is well known that the investigation of differential difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, and many others, has played an important role in the study of modern physics. Unlike difference equations which are fully discretized, DDEs are semidiscretized with some (or all) of their special variables discretized, while time is usually kept continuous. DDEs also play an important role in numerical simulations of nonlinear partial differential equations (NLPDEs), queuing problems, and discretization in solid state and quantum physics.

Since the work of Fermi et al. in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs.
Zhang et al. [18] and Aslan [19] used the (G'/G)-expansion method in some physically important nonlinear DDEs. Xu and Li [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently, S. Zhang and H.-Q. Zhang [20] and Gepreel [21] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of the integral discrete nonlinear Schrödinger equation. The main objective of this paper is to put a direct new method to construct the rational Jacobi elliptic solutions for nonlinear DDEs. We use this method to calculate the exact wave solutions for some nonlinear DDEs. For a given nonlinear DDEs

\[ \Delta \left( u_{n+p_1}(x), \ldots, u_{n+p_k}(x), u'_{n+p_1}(x), \ldots, u'_{n+p_k}(x), \ldots, u^{(r)}_{n+p_1}(x), \ldots, u^{(r)}_{n+p_k}(x), \ldots, v_{n+p_1}(x), \ldots, v'_{n+p_1}(x), \ldots, v'_{n+p_k}(x), \ldots, v^{(r)}_{n+p_1}(x), \ldots, v^{(r)}_{n+p_k}(x), \ldots \right) = 0, \]  

(2.1)

where \( \Delta = (\Delta_1, \ldots, \Delta_g), x = (x_1, x_2, \ldots, x_m), n = (n_1, \ldots, n_Q), \) and \( g, m, Q, p_1, \ldots, p_k \) are integers, \( u^{(r)}_i, v^{(r)}_i \) denotes the set of all \( r \)th order derivatives of \( u, v \) with respect to \( x \).

The main steps of the algorithm for the rational Jacobi elliptic functions method to solve nonlinear DDEs are outlined as follows.

**Step 1.** We seek the traveling wave solutions of the following form:

\[ u_n(x) = U(\xi_n), \quad v_n(x) = V(\xi_n), \ldots, \]  

(2.2)

where

\[ \xi_n = \sum_{i=1}^Q d_i n_i + \sum_{j=1}^m c_j x_j + \xi_0, \]  

(2.3)

d_i (i = 1, \ldots, Q), c_j (j = 1, \ldots, m), and the phase \( \xi_0 \) are constants to be determined later. The transformations in (2.2) are reduced (2.1) to the following ordinary differential difference equations

\[ \Omega \left( U(\xi_{n+p_1}), \ldots, U(\xi_{n+p_k}), U'(\xi_{n+p_1}), \ldots, U'(\xi_{n+p_k}), \ldots, U^{(r)}(\xi_{n+p_1}), \ldots, U^{(r)}(\xi_{n+p_k}), \ldots, V(\xi_{n+p_1}), \ldots, V(\xi_{n+p_k}), V'(\xi_{n+p_1}), \ldots, V'(\xi_{n+p_k}), \ldots, V^{(r)}(\xi_{n+p_1}), \ldots, V^{(r)}(\xi_{n+p_k}), \ldots \right) = 0, \]  

(2.4)

where \( \Omega = (\Omega_1, \ldots, \Omega_g) \). The transformations in (2.2) help in the calculation of the iteration relations between \( u_n(x), u_{n-1}(x), \) and \( u_{n+1}(x) \). For example, Langmuir chains equation
\[ \frac{du_n(t)}{dt} = u_n(t)(u_{n+1}(t) - u_{n-1}(t)) \] under the wave transformation \( u_n(t) = U(\xi_n), \xi_n = dn + ct + \xi_0 \) takes the form \( cU'(\xi_n) = U(\xi_n)(U(\xi_n + d) - U(\xi_n - d)) \).

**Step 2.** We suppose the rational series expansion solutions of (2.4) in the following form:

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{F'(\xi_n)}{F(\xi_n)} \right)^i, \quad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{F'(\xi_n)}{F(\xi_n)} \right)^i, \tag{2.5}
\]

where \( \alpha_i \) (\( i = 0, 1, \ldots, K \)), and \( \beta_i \) (\( i = 0, 1, \ldots, L \)) are constants to be determined later, and \( F(\xi_n) \) satisfies a discrete Jacobi elliptic differential equation

\[
F'^2(\xi_n) = e_0 + e_1 F^2(\xi_n) + e_2 F^4(\xi_n), \tag{2.6}
\]

where \( e_0, e_1, \) and \( e_2 \) are arbitrary constants.

**Step 3.** Since the general solution of the proposed (2.6) is difficult to obtain and so the iteration relations corresponding to the general exact solutions. So that we discuss the solutions of the proposed discrete Jacobi elliptic differential equation (2.6) at some special cases to \( e_0, e_1 \) and \( e_2 \) to cover all the Jacobi elliptic functions as follows:

**Type 1.** if \( e_0 = 1, e_1 = -(1 + m^2), e_2 = m^2 \). In this case (2.6) has the solution \( F(\xi_n) = sn(\xi_n, m) \), where \( sn(\xi_n, m) \) is the Jacobi elliptic sine function, and \( m \) is the modulus.

The Jacobi elliptic functions satisfy the following properties:

\[
\begin{align*}
[sn(\xi_n, m)]' &= cn(\xi_n, m)dn(\xi_n, m), & [cn(\xi_n, m)]' &= -sn(\xi_n, m)dn(\xi_n, m), \\
[dn(\xi_n, m)]' &= -m^2 sn(\xi_n, m)cn(\xi_n, m), & [cs(\xi_n, m)]' &= -ns(\xi_n, m)ds(\xi_n, m), \\
[sd(\xi_n, m)]' &= -nd(\xi_n, m)ds(\xi_n, m), & [dc(\xi_n, m)]' &= (1 - m^2) nc(\xi_n, m) sc(\xi_n, m),
\end{align*}
\]

where \( cn(\xi_n, m) \), and \( dn(\xi_n, m) \) are the Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind. The other Jacobi elliptic functions can be generated by \( sn(\xi_n, m) \), \( cn(\xi_n, m) \), and \( dn(\xi_n, m) \) as follows:

\[
\begin{align*}
cd(\xi_n, m) &= \frac{cn(\xi_n, m)}{dn(\xi_n, m)}, & dc(\xi_n, m) &= \frac{dn(\xi_n, m)}{cn(\xi_n, m)}, & nc(\xi_n, m) &= \frac{1}{cn(\xi_n, m)}, & nd(\xi_n, m) &= \frac{1}{dn(\xi_n, m)}, \\
cs(\xi_n, m) &= \frac{cn(\xi_n, m)}{sn(\xi_n, m)}, & cs(\xi_n, m) &= \frac{sn(\xi_n, m)}{cn(\xi_n, m)}, & sc(\xi_n, m) &= \frac{1}{sn(\xi_n, m)}, & sd(\xi_n, m) &= \frac{1}{dn(\xi_n, m)}.
\end{align*}
\tag{2.8}
\]

\[
\begin{align*}
\text{sn}(\xi_1 \pm \xi_2, m) &= \frac{sn(\xi_1, m)cn(\xi_2, m)dn(\xi_2, m) \pm sn(\xi_2, m)cn(\xi_1, m)dn(\xi_1, m)}{1 - m^2 sn^2(\xi_1, m) sn^2(\xi_2, m)}, \\
\text{cn}(\xi_1 \pm \xi_2, m) &= \frac{cn(\xi_1, m)cn(\xi_2, m) \mp sn(\xi_1, m)sn(\xi_2, m)dn(\xi_1, m)dn(\xi_2, m)}{1 - m^2 sn^2(\xi_1, m) sn^2(\xi_2, m)}, \\
\text{dn}(\xi_1 \pm \xi_2, m) &= \frac{dn(\xi_1, m)dn(\xi_2, m) \mp sn(\xi_1, m)sn(\xi_2, m)cn(\xi_1, m)cn(\xi_2, m)}{1 - m^2 sn^2(\xi_1, m) sn^2(\xi_2, m)}.
\tag{2.9}
\end{align*}
\]
In this case from using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

\[ U(\xi_n) = \sum_{i=0}^{K} a_i \left( \frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i, \]

\[ V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i, \ldots \] (2.10)

Further by using the properties of Jacobi elliptic functions, the iterative relations can be written in the following form:

\[ U(\xi_{n+1}) = \sum_{i=0}^{K} a_i \left( \frac{F'(\xi_{n+1})}{F(\xi_{n+1})} \right)^i, \]

\[ V(\xi_{n+1}) = \sum_{i=0}^{L} \beta_i \left( \frac{F'(\xi_{n+1})}{F(\xi_{n+1})} \right)^i, \ldots, \] (2.11)

where

\[ \frac{F'(\xi_{n+1})}{F(\xi_{n+1})} = \frac{1}{M_1} \left\{ \pm cn(d, m)cn(\xi_n, m)dn(\xi_n, m)dn(d, m) \pm m^2 sn(d, m)sn(\xi_n, m) \right. \]

\[ + \left. 2m^2 sn(d, m)sn^3(\xi_n, m) \mp 2m^2 sn^3(d, m)sn(\xi_n, m) \pm m^2 sn^3(d, m)sn^3(\xi_n, m) \right. \]

\[ + \left. sn(d, m)sn(\xi_n, m) \pm m^2 sn^3(d, m)sn^3(\xi_n, m) \right. \]

\[ + \left. m^2 sn^2(d, m)sn^2(\xi_n, m)dn(\xi_n, m)dn(d, m)cn(d, m)cn(\xi_n, m) \right\}, \] (2.12)

\[ M_1 = -cn(\phi, m)dn(\phi, m)sn(\xi_n, m) \mp sn(\phi, m)dn(\xi_n, m)cn(\xi_n, m) + m^2 sn^3(\xi_n, m) \]

\[ \times sn^2(\phi, m)cn(\phi, m)dn(\phi, m) \pm m^2 sn^2(\xi_n, m)sn^3(\phi, m)cn(\xi_n, m)dn(\xi_n, m), \] (2.13)

\[ d = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sk}d_k, \text{ } p_{sj} \text{ is the } j \text{th component of shift vector } p_s. \]

Type 2. if \( e_0 = 1 - m^2, \ e_1 = 2m^2 - 1, \ e_2 = -m^2 \). In this case, (2.6) has the solution \( F(\xi_n) = cn(\xi_n, m) \). From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

\[ U(\xi_n) = \sum_{i=0}^{K} a_i \left( -\frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i, \]

\[ V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( -\frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i, \ldots. \] (2.14)
Type 3. If \( e_0 = m^2 - 1 \), \( e_1 = 2 - m^2 \), \( e_2 = -1 \). In this case, (2.6) has the solution \( F(\xi_n) = dn(\xi_n, m) \). From using the properties of Jacobi elliptic functions the series expansion solutions (2.5) take the following form

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{-m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{-m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i, \ldots.
\]  

(2.15)

Type 4. If \( e_0 = 1 - m^2 \), \( e_1 = 2 - m^2 \), \( e_2 = 1 \). In this case, (2.6) has the solution \( F(\xi_n) = cs(\xi_n, m) \), then the series expansion solutions (2.5) take the following form

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{-ns(\xi_n, m) ds(\xi_n, m)}{cs(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{-ns(\xi_n, m) ds(\xi_n, m)}{cs(\xi_n, m)} \right)^i, \ldots.
\]  

(2.16)

Equation (2.16) can be written in the following form:

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{-dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{-dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \ldots.
\]  

(2.17)

Type 5. If \( e_0 = 1 \), \( e_1 = 2m^2 - 1 \), and \( e_2 = m^2(m^2 - 1) \). In this case, (2.6) has the solution \( F(\xi_n) = sd(\xi_n, m) \), then the series expansion solutions (2.5) take the following form

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{nd(\xi_n, m) cd(\xi_n, m)}{sd(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{nd(\xi_n, m) cd(\xi_n, m)}{sd(\xi_n, m)} \right)^i, \ldots.
\]  

(2.18)

Equation (2.18) can be written in the following form:

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \ldots.
\]  

(2.19)

Type 6. If \( e_0 = m^2 \), \( e_1 = -(m^2 + 1) \), and \( e_2 = 1 \). In this case, (2.6) has the solution \( F(\xi_n) = dc(\xi_n, m) \), then the series expansion solutions (2.5) take the following form

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{(1 - m^2) nc(\xi_n, m) sc(\xi_n, m)}{dc(\xi_n, m)} \right)^i, \\
V(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{(1 - m^2) nc(\xi_n, m) sc(\xi_n, m)}{dc(\xi_n, m)} \right)^i, \ldots.
\]  

(2.20)
Equation (2.20) can be written in the following form:

\[
U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left( \frac{(1 - m^2) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \quad V_n(\xi_n) = \sum_{i=0}^{L} \beta_i \left( \frac{(1 - m^2) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \ldots
\]

(2.21)

From the properties of the Jacobi elliptic functions, we can deduce the iterative relation to the above kind of solutions from Types 2–6 as we show in Type 1.

Equations (2.10)–(2.21) lead to getting all formulas of solutions from Types 1–6 as different. Consequently, we will discuss all solutions from Types 1–6.

Step 4. Determine the degree \(K, L, \ldots\) of (2.5) by balancing the nonlinear term(s) and the highest-order derivatives of \(U(\xi_n), V(\xi_n), \ldots\) in (2.4). It should be noted that the leading terms \(U(\xi_{n+p}), V(\xi_{n+p}), \ldots, p \neq 0\) will not affect the balance because we are interested in balancing the terms of \(F(\xi_n) / F(\xi_m)\).

Step 5. Substituting \(U(\xi_n), V(\xi_n), \ldots\) in each type form 1–6 and the given values of \(K, L, \ldots\) into (2.4). Cleaning the denominator and collecting all terms with the same degree of \(sn(\xi_n, m), dn(\xi_n, m), cn(\xi_n, m)\) together, the left hand side of (2.4) is converted into a polynomial in \(sn(\xi_n, m), dn(\xi_n, m), \) and \(cn(\xi_n, m)\). Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for \(\alpha_i, \beta_i, \delta_i, \) and \(\epsilon_i\).

Step 6. Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica. We end up with explicit expressions for \(\alpha_i, \beta_i, \delta_i, \) and \(\epsilon_i\).

Step 7. Substituting \(\alpha_i, \beta_i, \delta_i, \) and \(\epsilon_i\) into \(U(\xi_n), V(\xi_n), \ldots\) in the corresponding type from 1–6, we can finally obtain the exact solutions for (2.1).

3. Applications

In this section, we apply the proposed rational Jacobi elliptic functions method to construct the traveling wave solutions for some nonlinear DDEs via the lattice equation and the discrete nonlinear Schrödinger equation with a saturable nonlinearity which are very important in the mathematical physics and have been paid attention to by many researchers.

3.1. Example 1. The Lattice Equation

In this section, we study the lattice equation which takes the following form [22–25]

\[
\frac{du_n(t)}{dt} = \left( a + \beta u_n + \gamma u_n^2 \right) (u_{n-1} - u_{n+1}),
\]

(3.1)
where $\alpha, \beta$, and $\gamma$ are nonzero constants. The equation contains hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation, and Langmuir chain equation:

(i) (1+1) dimensional Hybrid lattice equation [25]:

$$\frac{du_n(t)}{dt} = \left(1 + \beta u_n + \gamma u_n^2\right)\left(u_{n-1} - u_{n+1}\right); \quad (3.2)$$

(ii) mKdV lattice equation [25]:

$$\frac{du_n(t)}{dt} = \left(\alpha - u_n^2\right)\left(u_{n-1} - u_{n+1}\right); \quad (3.3)$$

(iii) modified Volterra equation [24]:

$$\frac{du_n(t)}{dt} = u_n^2\left(u_{n-1} - u_{n+1}\right); \quad (3.4)$$

(iv) Langmuir chain equation [25]:

$$\frac{du_n(t)}{dt} = u_n\left(u_{n+1} - u_{n-1}\right). \quad (3.5)$$

According to the above steps, to seek traveling wave solutions of (3.1), we construct the transformation

$$u_n(t) = U(\xi_n), \quad \xi_n = dn + c_1 t + \xi_0, \quad (3.6)$$

where $d$, $c_1$, and $\xi_0$ are constants. The transformation in (3.6) permits us to convert (3.1) into the following form:

$$c_1 U'(\xi_n) = \left(\alpha + \beta U(\xi_n) + \gamma U^2(\xi_n)\right)\left(U(\xi_n - d) - U(\xi_n + d)\right), \quad (3.7)$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest-order derivative and the nonlinear term in (3.7), we get $K = 1$. Thus, the solution of (3.7) has the following form:

$$U(\xi_n) = \alpha_1 \left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \alpha_0, \quad (3.8)$$

where $\alpha_0$ and $\alpha_1$ are constants to be determined later, and $F(\xi_n)$ satisfies a discrete Jacobi elliptic ordinary differential (2.6). When, we discuss the solutions of the Jacobi elliptic differential difference (2.6), we get the following types.
Type 1. If \( e_0 = 1, e_1 = -(1 + m^2), \) and \( e_2 = m^2 \). In this case, the series expansion solution of (3.7) has the form:

\[
U(\xi_n) = a_0 + \frac{a_1 cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)}.
\] (3.9)

With help of Maple, we substitute (3.9) and (2.12) into (3.7), cleaning the denominator and collecting all terms with the same degree of \( sn(\xi_n, m), dn(\xi_n, m), \) and \( cn(\xi_n, m) \) together, the left hand side of (3.7) is converted into polynomial in \( sn(\xi_n, m), dn(\xi_n, m), \) and \( cn(\xi_n, m) \). Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for \( a_0, \alpha_1, d, \) and \( c_1 \). Solving the set of algebraic equations by using Maple or Mathematica, we have

\[
a_0 = -\frac{\beta}{2\gamma}, \quad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m)dn(d, m)}}{2\gamma cn(d, m)dn(d, m)}, \quad c_1 = -\frac{(4\alpha \gamma - \beta^2)sn(d, m)}{2\gamma cn(d, m)dn(d, m)}.
\] (3.10)

From (3.9) and (3.10), the solution of (3.7) takes the following form:

\[
U(\xi_n) = \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m)cn(\xi_n, m)dn(\xi_n, m)}}{2\gamma cn(d, m)dn(d, m)sn(\xi_n, m)} - \frac{\beta}{2\gamma},
\] (3.11)

where \( \xi_n = dn - ((4\alpha \gamma - \beta^2)sn(d, m) / [2\gamma cn(d, m)dn(d, m)])t + \xi_0 \).

Type 2. If \( e_0 = 1 - m^2, e_1 = 2m^2 - 1, \) and \( e_2 = -m^2 \). In this case, the series expansion solution of (3.7) has the form:

\[
U(\xi_n) = a_0 - \alpha_1 \frac{sn(\xi_n)dn(\xi_n)}{cn(\xi_n, m)}.
\] (3.12)

With the help of Maple, we substitute (3.12) into (3.7), cleaning the denominator and collecting all terms with the same degree of \( sn(\xi_n, m), dn(\xi_n, m), \) and \( cn(\xi_n, m) \) together, the left hand side of (3.7) is converted into polynomial in \( sn(\xi_n, m), dn(\xi_n, m), \) and \( cn(\xi_n, m) \). Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for \( a_0, \alpha_1, d, \) and \( c_1 \). Solving the set of algebraic equations by using Maple or Mathematica, we get

\[
a_0 = -\frac{\beta}{2\gamma}, \quad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m)dn(d, m)}}{2\gamma cn(d, m)}, \quad c_1 = -\frac{(4\alpha \gamma - \beta^2)dn(d, m)sn(d, m)}{2\gamma cn(d, m)}.
\] (3.13)

In this case the solution of (3.7) takes the following form:

\[
U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m)dn(d, m)sn(\xi_n, m)dn(\xi_n, m)}}{2\gamma cn(d, m)cn(\xi_n, m)},
\] (3.14)

where \( \xi_n = dn - ((4\alpha \gamma - \beta^2)dn(d, m)sn(d, m) / [2\gamma cn(d, m)])t + \xi_0 \).
Type 3. if \( e_0 = m^2 - 1, \ e_1 = 2 - m^2, \text{ and } e_2 = -1 \). In this case, the series expansion solution of (3.7) has the form:

\[
U(\xi_n) = a_0 - \frac{m^2 \alpha_1 sn(\xi_n) cn(\xi_n)}{dn(\xi_n)}. 
\]

(3.15)

Consequently, by using Maple or Mathematica, we obtain the following results:

\[
\begin{align*}
\alpha_0 &= -\frac{\beta}{2\gamma}, \\
\alpha_1 &= \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m) cn(d, m)}}{2\gamma dn(d, m)}, \\
c_1 &= -\frac{(4\alpha \gamma - \beta^2) cn(d, m) sn(d, m)}{2\gamma dn(d, m)}.
\end{align*}
\]

(3.16)

In this case, the solution takes the following form:

\[
U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha \gamma m^2 sn(d, m) cn(d, m) sn(\xi_n, m) cn(\xi_n, m)}}{2\gamma dn(d, m) dn(\xi_n, m)},
\]

(3.17)

where \( \xi_n = dn - ((4\alpha \gamma - \beta^2) cn(d, m) sn(d, m) / [2\gamma dn(d, m)]) t + \xi_0 \).

Type 4. if \( e_0 = 1 - m^2, \ e_1 = 2 - m^2, \text{ and } e_2 = 1 \). In this case, the series expansion solution of (3.7) has the form:

\[
U_n(\xi_n) = a_0 - \frac{\alpha_1 ns(\xi_n) ds(\xi_n)}{cs(\xi_n)}. 
\]

(3.18)

Consequently, using the Maple or Mathematica we get the following results:

\[
\begin{align*}
\alpha_0 &= -\frac{\beta}{2\gamma}, \\
\alpha_1 &= \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m) cn(d, m)}}{2\gamma dn(d, m)}, \\
c_1 &= -\frac{(4\alpha \gamma - \beta^2) cn(d, m) sn(d, m)}{2\gamma dn(d, m)}.
\end{align*}
\]

(3.19)

In this case, the solution of (3.7) takes the following form:

\[
U_n(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha \gamma sn(d, m) cn(d, m) ns(\xi_n, m) ds(\xi_n, m)}}{2\gamma dn(d, m) cs(\xi_n, m)},
\]

(3.20)

where \( \xi_n = dn - ((4\alpha \gamma - \beta^2) cn(d, m) sn(d, m) / [2\gamma dn(d, m)]) t + \xi_0 \).

Type 5. if \( e_0 = 1, \ e_1 = 2m^2 - 1, \text{ and } e_2 = m^2(m^2 - 1) \). In this case, the series expansion solution of (3.7) has the form:

\[
U(\xi_n) = a_0 + \frac{\alpha_1 nd(\xi_n) cd(\xi_n)}{sd(\xi_n)}. 
\]

(3.21)
Consequently, by using Maple or Mathematica, we get the following results:

\[
\alpha_0 = -\frac{\beta}{2\gamma}, \quad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma} \text{sn}(d,m) \text{dn}(d,m)}{2\gamma \text{cn}(d,m)}, \quad c_1 = \frac{(\beta^2 - 4\alpha \gamma) \text{sn}(d,m) \text{dn}(d,m)}{2\gamma \text{cn}(d,m)}.
\]

(3.22)

In this case, the solution takes of (3.7) the following form:

\[
U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2 - 4\alpha \gamma} \text{sn}(d,m) \text{dn}(d,m) \text{cd}(\xi_n, m) \text{sd}(\xi_n, m)}{2\gamma \text{cn}(d,m) \text{sn}(d,m)},
\]

(3.23)

where \(\xi_n = dn + ((\beta^2 - 4\alpha \gamma) \text{sn}(d,m) \text{dn}(d,m) / [2\gamma \text{cn}(d,m)]) t + \xi_0\).

Type 6. if \(e_0 = m^2\), \(e_1 = -(m^2 + 1)\), and \(e_2 = 1\). In this case, the series expansion solution of (3.7) has the form:

\[
U(\xi_n) = \alpha_0 + \frac{(1-m^2) \alpha_1 \text{n}\text{c}(\xi_n) \text{s}\text{c}(\xi_n)}{\text{d}\text{c}(\xi_n)}.
\]

(3.24)

Consequently, by using Maple or Mathematica, we get the following results:

\[
\alpha_0 = -\frac{\beta}{2\gamma}, \quad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma} \text{sn}(d,m)}{2\gamma \text{cn}(d,m) \text{dn}(d,m)}, \quad c_1 = \frac{(\beta^2 - 4\alpha \gamma) \text{sn}(d,m)}{2\gamma \text{cn}(d,m) \text{dn}(d,m)}.
\]

(3.25)

In this case, the solution of (3.7) takes the following form:

\[
U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2 - 4\alpha \gamma} (1-m^2) \text{sn}(d,m) \text{nc}(\xi_n, m) \text{sc}(\xi_n, m)}{2\gamma \text{dn}(d,m) \text{cn}(d,m) \text{dc}(\xi_n, m)},
\]

(3.26)

where \(\xi_n = dn + ((\beta^2 - 4\alpha \gamma) \text{sn}(d,m) / [2\gamma \text{cn}(d,m) \text{dn}(d,m)]) t + \xi_0\).

### 3.2. Example 2. The Discrete Nonlinear Schrodinger Equation

The discrete nonlinear Schrodinger equation (DNSE) is one of the most fundamental nonlinear lattice models [8]. It arises in nonlinear optics as a model of infinite wave guide arrays [26] and has been recently implemented to describe Bose-Einstein condensates in optical lattices. The class of DNSE model with saturable nonlinearity is also of particular interest in their own right, due to a feature first unveiled in [27]. In this section, we study the DNSE with a saturable nonlinearity [28, 29] having the form

\[
\frac{\partial \psi_n}{\partial t} + (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \frac{v|\psi_n|^2}{1 + \mu|\psi_n|^2} \psi_n = 0,
\]

(3.27)
which describes optical pulse propagations in various doped fibers, \(\psi_n\) is a complex valued wave function at sites \(n\) while \(v\) and \(\mu\). We make the transformation

\[
\psi_n = \phi(\xi_n)e^{-i(\alpha n + \beta)}, \quad \xi_n = \alpha n + \beta,
\]

where \(\sigma, \rho, \alpha,\) and \(\beta\) are arbitrary real constants. The transformation (3.28) permits us converting (3.27) into the following nonlinear difference equation

\[
(\sigma - 2)\phi(\xi_n) + \phi(\xi_{n+1}) + \phi(\xi_{n-1}) + \frac{v\phi^3(\xi_n)}{1 + \mu\phi^2(\xi_n)} = 0.
\]

We assume that (3.29) has a solution of the form:

\[
\phi(\xi_n) = U(\xi_n) = \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \alpha_0,
\]

where \(\alpha_1,\) and \(\alpha_0\) are constants to be determined later and \(F(\xi_n)\) satisfying a discrete Jacobi elliptic differential equation (2.6). When, we discuss the solutions of (2.6), we have the following types.

Type 1. If \(e_0 = 1,\) \(e_1 = -(1 + m^2),\) and \(e_2 = m^2.\) In this case, the series expansion solution of (3.29) has the form:

\[
U(\xi_n) = \alpha_1\frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} + \alpha_0.
\]

With the help of Maple, we substitute (3.31) and (2.12) into (3.29), cleaning the denominator and collecting all terms with the same order of \(cn(\xi_n, m), dn(\xi_n, m),\) and \(sn(\xi_n, m)\) together, the left hand side of (3.29) is converted into polynomial in \(cn(\xi_n, m), dn(\xi_n, m),\) and \(sn(\xi_n, m).\) Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for \(\alpha_0,\) \(\alpha_1,\) \(\sigma,\) \(\rho,\) \(\alpha,\) and \(\beta.\) Solving the set of algebraic equations by using Maple or Mathematica, we obtain

\[
\begin{align*}
\alpha_0 &= 0, \\
\alpha_1 &= \frac{sn(a, m)}{\sqrt{-\mu}cn(a, m)dn(a, m)}, \\
\nu &= -2\mu(m^2sn^4(a, m) - 1), \\
\sigma &= \frac{-2sn^2(a, m)}{cn^2(a, m)dn^2(a, m)}[m^2cn^2(a, m) + dn^2(a, m)], \\
\mu &< 0.
\end{align*}
\]

In this case, the solution of (3.27) takes the following form:

\[
\psi_n = \frac{sn(a, m)cn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}cn(a, m)dn(a, m)sn(\xi_n, m)} \exp\left\{-i\left[-2\mu sn^2(a, m) \left[ m^2cn^2(a, m) + dn^2(a, m) \right] \right] + \beta \right\},
\]

where \(\xi_n = \alpha n + \beta.\)
Type 2. If \( e_0 = 1 - m^2, e_1 = 2m^2 - 1, \) and \( e_2 = -m^2. \) In this case the solution of (3.29) has the form:

\[
U(\xi_n) = a_0 - a_1 \frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)}.
\]  (3.34)

Consequently, by using Maple or Mathematica, we get the following results:

\[
a_0 = 0, \quad a_1 = \frac{sn(a, m)dn(a, m)}{\sqrt{-\mu}cn(a, m)}, \quad \nu = \frac{2\mu(m^2sn^4(a, m) - 2m^2sn^2(a, m) + 1)}{cn^2(a, m)},
\]

\[
\sigma = -\frac{2sn^2(a, m)[m^2sn^2(a, m) + 1 - 2m^2]}{cn^2(a, m)}, \quad \mu < 0.
\]  (3.35)

In this case, the solution takes the following form:

\[
\psi_n = -\frac{sn(a, m)dn(a, m)sn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}cn(a, m)cn(\xi_n, m)}
\times \exp \left\{-i \left[ \frac{-2sn^2(a, m)[m^2sn^2(a, m) + 1 - 2m^2]}{cn^2(a, m)} + \rho \right] \right\}.
\]  (3.36)

Type 3. If \( e_0 = m^2 - 1, e_1 = 2 - m^2, \) and \( e_2 = -1. \) In this case, the series expansion solution of (3.29) has the form:

\[
U(\xi_n) = a_0 - \frac{m^2a_1sn(\xi_n)cn(\xi_n)}{dn(\xi_n)}.
\]  (3.37)

Consequently, by using Maple or Mathematica, we get the following results:

\[
a_0 = 0, \quad a_1 = \frac{sn(a, m)cn(a, m)}{\sqrt{-\mu} \, dn(a, m)}, \quad \nu = \frac{2\mu(m^2sn^4(a, m) - 2sn^2(a, m) + 1)}{dn^2(a, m)},
\]

\[
\sigma = -\frac{2sn^2(a, m)[m^2sn^2(a, m) - 2 + m^2]}{dn^2(a, m)}, \quad \mu < 0.
\]  (3.38)

In this case, the solution takes the following form:

\[
\psi_n = -\frac{m^2sn(a, m)cn(a, m)sn(\xi_n, m)cn(\xi_n, m)}{\sqrt{-\mu}dn(a, m)dn(\xi_n, m)}
\times \exp \left\{-i \left[ \frac{-2sn^2(a, m)[m^2sn^2(a, m) - 2 + m^2]}{dn^2(a, m)} + \rho \right] \right\}.
\]  (3.39)
Type 4. if $e_0 = 1 - m^2$, $e_1 = 2 - m^2$, and $e_2 = 1$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = a_0 - \frac{a_1 ns(\xi_n) ds(\xi_n)}{cs(\xi_n)}. \quad (3.40)$$

After some calculation, the solution of (3.27) takes the following form:

$$\psi_n = -\frac{sn(a, m)c(n(a, m)ns(\xi_n, m)ds(\xi_n, m))}{\sqrt{-\mu}dn(a, m)cn(\xi_n, m)} \times \text{Exp} \left\{ -i \left[ -2t sn^2(a, m) \frac{m^2sn^2(a, m) - 2 + m^2}{dn^2(a, m)} + \rho \right] \right\}, \quad (3.41)$$

where $\nu = 2\mu(m^2 sn^4(a, m) - 2sn^2(a, m) + 1)/dn^2(a, m)$.

Type 5. if $e_0 = 1$, $e_1 = 2m^2 - 1$, and $e_2 = m^2(m^2 - 1)$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = a_0 + \frac{a_1 nd(\xi_n) cd(\xi_n)}{sd(\xi_n)}. \quad (3.42)$$

After some calculation, the solution of (3.27) takes the following form:

$$\psi_n = -\frac{sn(a, m)dn(a, m)cd(\xi_n, m)nd(\xi_n, m))}{\sqrt{-\mu}cn(a, m)sd(\xi_n, m)} \times \text{Exp} \left\{ -i \left[ -2tsn^2(a, m) \frac{m^2sn^2(a, m) + 1 - 2m^2}{cn^2(a, m)} + \rho \right] \right\}, \quad (3.43)$$

where $\nu = 2\mu(m^2 sn^4(a, m) - 2sn^2(a, m) + 1)/cn^2(a, m)$.

Type 6. if $e_0 = m^2$, $e_1 = -(m^2 + 1)$, and $e_2 = 1$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = a_0 + \frac{(1 - m^2)a_1 nc(\xi_n) sc(\xi_n)}{dc(\xi_n)}. \quad (3.44)$$

After some calculation, the solution of (3.27) takes the following form:

$$\psi_n = \frac{(1 - m^2)sn(a, m)nc(\xi_n, m)sc(\xi_n, m))}{\sqrt{-\mu}cn(a, m)dn(a, m)dc(\xi_n, m)} \times \text{Exp} \left\{ -i \left[ -2tsn^2(a, m) \frac{m^2cn^2(a, m) + dn^2(a, m)}{cn^2(a, m)dn^2(a, m)} + \rho \right] \right\}, \quad (3.45)$$

where $\nu = -2\mu(m^2 sn^4(a, m) - 1)/[cn^2(a, m)dn^2(a, m)]$. 
Remark 3.1. (1) The formulas of the exact solutions from Types 1–6 are different, and consequently, we must discuss the exact solutions in all types from 1–6.

(2) The values of $\alpha_i$, $\beta_i$, $d_i$, and $c_i$ in Examples 1 and 2 have a unique determination in all types of this method.

4. Conclusion

In this paper, we put a direct method to calculate the rational Jacobi elliptic solutions for the nonlinear difference differential equations via the lattice equation and the discrete nonlinear Schrödinger equation with a saturable nonlinearity. As a result, many new and more rational Jacobi elliptic solutions are obtained, from which hyperbolic function solutions and trigonometric function solutions are derived when the modulus $m \to 1$ and $m \to 0$.

References


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