Research Article

Implicit and Explicit Iterations with Meir-Keeler-Type Contraction for a Finite Family of Nonexpansive Semigroups in Banach Spaces

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We introduce an implicit and explicit iterative schemes for a finite family of nonexpansive semigroups with the Meir-Keeler-type contraction in a Banach space. Then we prove the strong convergence for the implicit and explicit iterative schemes. Our results extend and improve some recent ones in literatures.

1. Introduction

Let $C$ be a nonempty subset of a Banach space $E$ and $T : C \to C$ be a mapping. We call $T$ nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. The set of all fixed points of $T$ is denoted by $\text{Fix}(T)$, that is, $\text{Fix}(T) = \{x \in C : x = Tx\}$.

One parameter family $\mathcal{T} = \{T(t) : t \geq 0\}$ is said to a semigroup of nonexpansive mappings or nonexpansive semigroup on $C$ if the following conditions are satisfied:

1. $T(0)x = x$ for all $x \in C$;
2. $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
3. for each $t \geq 0$, $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$;
4. for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of all nonnegative reals, into $C$ is continuous.
We denote by $\text{Fix}(\mathcal{T})$ the set of all common fixed points of semigroup $\mathcal{T}$, that is, $\text{Fix}(\mathcal{T}) = \{ x \in C : T(t)x = x, \ 0 \leq t < \infty \}$ and $\mathbb{N}$ by the set of natural numbers.

Now, we recall some recent work on nonexpansive semigroup in literatures. In [1], Shioji and Takahashi introduced the following implicit iteration for a nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \ \forall n \in \mathbb{N}, \quad (1.1)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$. Under the certain conditions on $\{\alpha_n\}$ and $\{t_n\}$, they proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly to an element in $\text{Fix}(\mathcal{T})$.

In [2], Suzuki introduced the following implicit iteration for a nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \ \forall n \in \mathbb{N}, \quad (1.2)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$. Under the conditions that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \alpha_n/T_n = 0$, he proved that $\{x_n\}$ defined by (1.2) converges strongly to an element of $\text{Fix}(\mathcal{T})$. Later on, Xu [3] extended the iteration (1.2) to a uniformly convex Banach space that admits a weakly sequentially continuous duality mapping. Song and Xu [4] also extended the iteration (1.2) to a reflexive and strictly convex Banach space.

In 2007, Chen and He [5] studied the following implicit and explicit viscosity approximation processes for a nonexpansive semigroup in a reflexive Banach space admitting a weakly sequentially continuous duality mapping:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad y_{n+1} = \beta_n f(y_n) + (1 - \beta_n)T(t_n)y_n, \ \forall n \in \mathbb{N}, \quad (1.3)$$

where $f$ is a contraction, $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$. They proved the strong convergence for the above iterations under some certain conditions on the control sequences.

Recently, Chen et al. [6] introduced the following implicit and explicit iterations for nonexpansive semigroups in a reflexive Banach space admitting a weakly sequentially continuous duality mapping:

$$y_n = \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \quad x_n = \beta_n f(x_n) + (1 - \beta_n)y_n, \ \forall n \in \mathbb{N}, \quad (1.4)$$

$$y_n = \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \quad x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \ \forall n \in \mathbb{N}, \quad (1.5)$$

where $f$ is a contraction, $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$. They proved that $\{x_n\}$ defined by (1.4) and (1.5) converges strongly to an element $q$ of $\text{Fix}(\mathcal{T})$, which is the unique solution of the following variation inequality problem:

$$\langle (f - I), j(x - q) \rangle \leq 0, \ \forall x \in \text{Fix}(\mathcal{T}). \quad (1.6)$$
For more convergence theorems on implicit and explicit iterations for nonexpansive semigroups, refer to [7–13].

In this paper, we introduce an implicit and explicit iterative process by a generalized contraction for a finite family of nonexpansive semigroups in a Banach space. Then we prove the strong convergence for the iterations and our results extend the corresponding ones of Suzuki [2], Xu [3], Chen and He [5], and Chen et al. [6].

2. Preliminaries

Let $E$ be a Banach space and $E^*$ the duality space of $E$. We denote the normalized mapping from $E$ to $2^{E^*}$ by $J$ defined by

$$J(x) = \{ j \in E^* : \langle x, j(x) \rangle = \| x \|^2 = \| j \| \}, \quad \forall x \in E, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. For any $x, y \in E$ with $j(x) \in J(x)$ and $j(x + y) \in J(x + y)$, it is well known that the following inequality holds:

$$\| x \|^2 + 2 \langle y, j(x) \rangle \leq \| x + y \|^2 \leq \| x \|^2 + 2 \langle y, j(x + y) \rangle. \quad (2.2)$$

The dual mapping $J$ is called weakly sequentially continuous if $J$ is single valued, and \{ $x_n$ $\rightarrow$ $x$ in $E$, where $\rightarrow$ denotes the weak convergence, then $J(x_n)$ weakly star converges to $J(x)$ [14–16]. A Banach space $E$ is called to satisfy Opial’s condition [17] if for any sequence \{ $x_n$ $\}$ in $E$, $x_n \rightharpoonup x$,

$$\limsup_{n \to \infty} \| x_n - x \| < \limsup_{n \to \infty} \| x_n - y \|, \quad \forall y \in E \text{ with } x \neq y. \quad (2.3)$$

It is known that if $E$ admits a weakly sequentially continuous duality mapping $J$, then $E$ is smooth and satisfies Opial’s condition [14].

A function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be an $L$-function if $\varphi(0) = 0$, $\varphi(t) > 0$ for any $t > 0$, and for every $t > 0$ and $s > 0$, there exists $u > s$ such that $\varphi(t) \leq s$, for all $t \in [s, u]$. This implies that $\varphi(t) < t$ for all $t > 0$.

Let $f : C \rightarrow C$ be a mapping. $f$ is said to be a $(\varphi, L)$-contraction if there exists a $L$-function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\| f(x) - f(y) \| < \varphi(\| x - y \|)$ for all $x, y \in C$ with $x \neq y$. Obviously, if $\varphi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$, then $f$ is a contraction. $f$ is called a Meir-Keeler-type mapping if for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in C$, if $\epsilon < \| x - y \| < \epsilon + \delta$, then $\| f(x) - f(y) \| < \epsilon$.

In this paper, we always assume that $\varphi(t)$ is continuous, strictly increasing and $\lim_{t \to \infty} \varphi(t) = \infty$, where $\eta(t) = t - \varphi(t)$, is strictly increasing and onto.

The following lemmas will be used in next section.

**Lemma 2.1** (see [18]). Let $(X, d)$ be a metric space and $f : X \rightarrow X$ be a mapping. The following assertions are equivalent:

1. $f$ is a Meir-Keeler-type mapping,
2. there exists an $L$-function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f$ is a $(\varphi, L)$-contraction.
Lemma 2.2 (see [19]). Let $E$ be a Banach space and $C$ be a convex subset of $E$. Let $T : C \to C$ be a nonexpansive mapping and $f$ be a $(\varphi, L)$-contraction. Then the following assertions hold:

(i) $T \circ f$ is a $(\varphi, L)$-contraction on $C$ and has a unique fixed point in $C$;

(ii) for each $\alpha \in (0, 1)$, the mapping $x \mapsto \alpha f(x) + (1 - \alpha)Tx$ is of Meir-Keeler-type and it has a unique fixed point in $C$.

Lemma 2.3 (see [20]). Let $E$ be a Banach space and $C$ be a convex subset of $E$. Let $f : C \to C$ be a Meir-Keeler-type contraction. Then for each $\epsilon > 0$ there exists $r \in (0, 1)$ such that, for each $x, y \in C$ with $\|x - y\| \geq \epsilon$, $\|f(x) - f(y)\| \leq r\|x - y\|$.

Lemma 2.4 (see [21]). Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let $T_m : C \to C$ be a nonexpansive mapping for each $1 \leq m \leq r$, where $r$ is some integer. Suppose that $\cap_{m=1}^{r} \text{Fix}(T_m)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{r} \lambda_n = 1$. Then the mapping $S : C \to C$ defined by

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x, \quad \forall x \in C,$$  \hspace{1cm} (2.4)

is well defined, nonexpansive and $\text{Fix}(S) = \cap_{m=1}^{r} \text{Fix}(T_m)$ holds.

Lemma 2.5 (see [22]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \in \mathbb{N},$$  \hspace{1cm} (2.5)

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\lim_{n \to \infty} \gamma_n = 0$;

(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(iii) $\lim \sup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

3. Main Results

In this section, by a generalized contraction mapping we mean a Meir-Keeler-type mapping or $(\varphi, L)$- contraction. In the rest of the paper we suppose that $\varphi$ from the definition of the $(\varphi, L)$-contraction is continuous, strictly increasing and $\eta(t)$ is strictly increasing and onto, where $\eta(t) = t - \varphi(t)$, for all $t \in \mathbb{R}^+$. As a consequence, we have the $\eta(t)$ is a bijection on $\mathbb{R}^+$.

Theorem 3.1. Let $C$ be a nonempty closed convex subset of a reflexive Banach space $E$ which admits a weakly sequentially continuous duality mapping $J$ from $E$ into $E^*$. For every $i = 1, \ldots, N (N \geq 1)$, let $\mathcal{T}_i = \{T_i(t) : t \geq 0\}$ be a semigroup of nonexpansive mappings on $C$ such that $\mathcal{F} = \cap_{i=1}^{N} \text{Fix}(\mathcal{T}_i) \neq \emptyset$ and $f : C \to C$ be a generalized contraction on $C$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1)$ and $\{\gamma_n\} \subset (0, \infty)$ be
the sequences satisfying $\lim_{n \to \infty} t_n = \lim_{n \to \infty} (\alpha_n / t_n) = 0$ and $\lim sup_{n \to \infty} \beta_n < 1$. Let $\{x_n\}$ be a sequence generated by

$$x_n = \alpha_n f(x_n) + \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} y_{in},$$

(3.1)

$$y_{in} = \beta_n x_n + (1 - \beta_n) T_i(t_n) x_n, \quad i = 1, \ldots, N.\,$$

Then $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution to the following variational inequality:

$$\langle (f - I)x^*, j(x - x^*) \rangle \leq 0, \quad \forall x \in \mathcal{F}. \quad (3.2)$$

Proof. First, we show that the sequence $\{x_n\}$ generated by (3.1) is well defined. For every $n \in \mathbb{N}$ and $i = 1, \ldots, N$, let $U_{in} = \beta_n I + (1 - \beta_n) T_i(t_n)$ and define $W_n : C \to C$ by

$$W_n x = \alpha_n f(x) + (1 - \alpha_n) G_n x, \quad \forall x \in C, \quad (3.3)$$

where $G_n x = (1/N) \sum_{i=1}^{N} U_{in} x$. Since $U_{in}$ is nonexpansive, $G_n$ is nonexpansive. By Lemma 2.2 we see that $W_n$ is a Meir-Keeler-type contraction for each $n \in \mathbb{N}$. Hence, each $W_n$ has a unique fixed point, denoted as $x_n$, which uniquely solves the fixed point equation (3.3). Hence $\{x_n\}$ generated by (3.1) is well defined.

Now we prove that $\{x_n\}$ generated by (3.1) is bounded. For any $p \in \mathcal{F}$, we have

$$\|y_{in} - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T_i(t_n) x_n - p\| \leq \|x_n - p\|. \quad (3.4)$$

Using (3.4), we get

$$\|x_n - p\|^2 = \left\langle \alpha_n f(x_n) + \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} y_{in} - p, j(x_n - p) \right\rangle$$

$$= \alpha_n \langle f(x_n) - f(p), j(x_n - p) \rangle + \alpha_n \langle f(p) - p, j(x_n - p) \rangle$$

$$+ \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \langle y_{in} - p, j(x_n - p) \rangle$$

$$\leq \alpha_n \psi(\|x_n - p\|) \|x_n - p\| + \alpha_n \|f(p) - p\| \|x_n - p\|$$

$$+ \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \|y_{in} - p\| \|x_n - p\| \quad (3.5)$$

$$= \alpha_n \psi(\|x_n - p\|) \|x_n - p\| + \alpha_n \|f(p) - p\| \|x_n - p\|$$

$$+ (1 - \alpha_n) \|x_n - p\|^2$$
and hence

\[ \|x_n - p\| \leq \eta(\|x_n - p\|) + \|f(p) - p\|, \quad (3.6) \]

which implies that

\[ \eta(\|x_n - p\|) = \|x_n - p\| - \eta(\|x_n - p\|) \leq \|f(p) - p\|. \quad (3.7) \]

Hence

\[ \|x_n - p\| \leq \eta^{-1}(\|f(p) - p\|). \quad (3.8) \]

This shows that \( \{x_n\} \) is bounded, and so are \( \{T_i(t_n) x_n\}, \{f(x_n)\} \) and \( \{y_{im}\} \).

Since \( E \) is reflexivity and \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_j}\} \subset \{x_n\} \) such that \( x_{n_j} \to x^* \) for some \( x^* \in C \) as \( j \to \infty \). Now we prove that \( x^* \in \mathcal{F} \). For any fixed \( t > 0 \), we have

\[
\sum_{i=1}^{N} \left\| x_{n_j} - T_i(t)x^* \right\| \leq \sum_{i=1}^{N} \left[ \sum_{k=0}^{[t/t_{n_j}]-1} \left| T_i \left( \frac{t}{t_{n_j}} \right) x_{n_j} - T_i \left( \frac{t}{t_{n_j}}k\right) x_{n_j} \right| \right]
\leq \sum_{i=1}^{N} \left\| T_i \left( \frac{t}{t_{n_j}} \right) x_{n_j} - x_{n_j} \right\| + \left\| x_{n_j} - x^* \right\| + \left\| T_i \left( t - \frac{t}{t_{n_j}} \right) x_{n_j} - x_{n_j} \right\|
\leq \sum_{i=1}^{N} \left\| T_i \left( \frac{t}{t_{n_j}} \right) x_{n_j} - x_{n_j} \right\| + \left\| x_{n_j} - x^* \right\| + \max \left\{ \|T_i(s)x^* - x^*\| : 0 \leq s \leq t_{n_j} \right\}
\leq \frac{N \alpha_{n_j} \left( t/t_{n_j} \right)}{(1 - \alpha_{n_j}) (1 - \beta_{n_j})} \left\| x_{n_j} - f \left( x_{n_j} \right) \right\| + N \left\| x_{n_j} - x^* \right\|
+ \sum_{i=1}^{N} \max \left\{ \|T_i(s)x^* - x^*\| : 0 \leq s \leq t_{n_j} \right\}
\leq \frac{N t}{(1 - \alpha_{n_j}) (1 - \beta_{n_j}) t_{n_j}} \alpha_{n_j} \left\| x_{n_j} - f \left( x_{n_j} \right) \right\| + N \left\| x_{n_j} - x^* \right\|
+ \sum_{i=1}^{N} \max \left\{ \|T_i(s)x^* - x^*\| : 0 \leq s \leq t_{n_j} \right\}. \quad (3.9) \]
By hypothesis on \( \{t_n\}, \{\alpha_n\}, \{\beta_n\} \), we have
\[
\lim_{j \to \infty} \frac{N t}{\left(1 - \alpha_n\right)\left(1 - \beta_n\right)} t_{n_j} = 0. \tag{3.10}
\]

Further, from (3.9) we get
\[
\limsup_{j \to \infty} \sum_{i=1}^{N} \left\| x_{n_j} - T_i(t)x^* \right\| \leq \limsup_{j \to \infty} \left\| x_{n_j} - x^* \right\|. \tag{3.11}
\]

Since \( E \) admits a weakly sequentially duality mapping, we see that \( E \) satisfies Opial’s condition. Thus if \( x^* \notin \mathcal{F} \), we have
\[
\limsup_{j \to \infty} N \left\| x_{n_j} - x^* \right\| < \limsup_{j \to \infty} \sum_{i=1}^{N} \left\| x_{n_j} - T_ix^* \right\|. \tag{3.12}
\]

This contradicts (3.11). So \( x^* \in \mathcal{F} \).

In (3.5), replacing \( p \) with \( x^* \) and \( n \) with \( n_j \), we see that
\[
\left\| x_{n_j} - x^* \right\|^2 = \alpha_{n_j} \left< f\left(x_{n_j}\right) - f(x^*), j\left(x_{n_j} - x^*\right) \right> + \alpha_{n_j} \left< f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right>
+ \frac{1 - \alpha_{n_j}}{N} \sum_{i=1}^{N} \left< y_{in_j} - x^*, j\left(x_{n_j} - x^*\right) \right>
\leq \alpha_{n_j} \psi\left(\left\| x_{n_j} - x^* \right\|\right) \left\| x_{n_j} - x^* \right\| + \alpha_{n_j} \left< f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right>
+ \frac{1 - \alpha_{n_j}}{N} \sum_{i=1}^{N} \left\| y_{in_j} - x^* \right\| \left\| x_{n_j} - x^* \right\|
\leq \alpha_{n_j} \psi\left(\left\| x_{n_j} - x^* \right\|\right) \left\| x_{n_j} - x^* \right\| + \alpha_{n_j} \left< f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right>
+ \left(1 - \alpha_{n_j}\right) \left\| x_n - p \right\|^2, \tag{3.13}
\]

which implies that
\[
\left\| x_{n_j} - x^* \right\| \left( \psi\left(\left\| x_{n_j} - x^* \right\|\right) - \left\| x_{n_j} - x^* \right\| \right) \leq \left< f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right>. \tag{3.14}
\]

Now we prove that \( \{x_n\} \) is relatively sequentially compact. Since \( j \) is weakly sequentially continuous, we have
\[
\lim_{j \to \infty} \left\| x_{n_j} - x^* \right\| \left( \psi\left(\left\| x_{n_j} - x^* \right\|\right) - \left\| x_{n_j} - x^* \right\| \right) \leq 0, \tag{3.15}
\]
which implies that
\[
\lim_{j \to \infty} \|x_{n_j} - x^*\| = 0,
\]
or
\[
\lim_{j \to \infty} \left( \psi \left( \|x_{n_j} - x^*\| \right) - \|x_{n_j} - x^*\| \right) = 0.
\]  \hspace{1cm} (3.16)

If \( \lim_{j \to \infty} \|x_{n_j} - x^*\| = 0 \), then \( \{x_n\} \) is relatively sequentially compact. If \( \lim_{j \to \infty} \psi(\|x_{n_j} - x^*\|) - \|x_{n_j} - x^*\| = 0 \), we have \( \lim_{j \to \infty} \|x_{n_j} - x^*\| = \lim_{j \to \infty} \psi(\|x_{n_j} - x^*\|) \). Since \( \psi \) is continuous, \( \lim_{j \to \infty} \|x_{n_j} - x^*\| = \psi(\|x_{n_j} - x^*\|) \). By the definition of \( \psi \), we conclude that \( \lim_{j \to \infty} \|x_{n_j} - x^*\| = 0 \), which implies that \( \{x_n\} \) is relatively sequentially compact.

Next, we prove that \( x^* \) is the solution to (3.2). Indeed, for any \( x \in \mathcal{F} \), we have
\[
\|x_n - x\|^2 = \langle \alpha_n(f(x_n) - x_n + x_n - x), j(x_n - x) \rangle + \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \langle y_{in} - x, j(x_n - x) \rangle
\]
\[
= \alpha_n \langle f(x_n) - x_n, j(x_n - x) \rangle + \alpha_n \langle x_n - x, j(x_n - x) \rangle
\]
\[
+ \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \left\{ \beta_n \langle x_n - x, j(x_n - x) \rangle + (1 - \beta_n) \langle T_i(t_n)x_n - x, j(x_n - x^*) \rangle \right\}
\]
\[
\leq \alpha_n \langle f(x_n) - x_n, j(x_n - x) \rangle + \alpha_n \|x_n - x\|^2
\]
\[
+ \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \left[ \beta_n \|x_n - x\|^2 + (1 - \beta_n) \|T_i(t_n)x_n - x\| \|x_n - x\| \right]
\]
\[
\leq \alpha_n \langle f(x_n) - x_n, j(x_n - x) \rangle + \alpha_n \|x_n - x\|^2
\]
\[
+ \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} \left[ \beta_n \|x_n - x\|^2 + (1 - \beta_n) \|x_n - x\| \right]
\]
\[
= \alpha_n \langle f(x_n) - x_n, j(x_n - x) \rangle + \|x_n - x\|^2.
\]  \hspace{1cm} (3.17)

Therefore,
\[
\langle f(x_n) - x_n, j(x - x_n) \rangle \leq 0.
\]  \hspace{1cm} (3.18)

Since \( x_{n_j} \to x^* \) and \( j \) is weakly sequentially continuous, we have
\[
\langle f(x^*) - x^*, j(x - x^*) \rangle = \lim_{j \to \infty} \langle f(x_{n_j}) - x_{n_j}, j(x - x_{n_j}) \rangle \leq 0.
\]  \hspace{1cm} (3.19)

This shows that \( x^* \) is the solution of the variational inequality (3.2).

Finally, we prove that \( x^* \) is the unique solution of the variational inequality (3.2). Assume that \( \tilde{x} \in \mathcal{F} \) with \( \tilde{x} \neq x^* \) is another solution of (3.2). Then there exists \( \epsilon > 0 \) such that \( \|\tilde{x} - x^*\| > \epsilon \). By Lemma 2.3 there exists \( r \in (0, 1) \) such that \( \|f(\tilde{x}) - f(x^*)\| \leq r\|\tilde{x} - x^*\| \). Since both \( \tilde{x} \) and \( x^* \) are the solution of (3.2), we have
\[
\langle f(x^*) - x^*, j(\tilde{x} - x^*) \rangle \leq 0,
\]
\[
\langle f(\tilde{x}) - \tilde{x}, j(x^* - \tilde{x}) \rangle \leq 0.
\]  \hspace{1cm} (3.20)
Adding the above inequalities, we get
\[0 < (1 - r)e^2 < (1 - r)\|\tilde{x} - x^*\|^2 \leq \langle (I - f)x^* - (I - f)\tilde{x}, j(x^* - \tilde{x})\rangle \leq 0,\] (3.21)
which is a contradiction. Therefore, we must have \(\tilde{x} = x^*\), which implies that \(x^*\) is the unique solution of (3.2).

In a similar way it can be shown that each cluster point of sequence \(\{x_n\}\) is equal to \(x^*\). Therefore, the entire sequence \(\{x_n\}\) converges strongly to \(x^*\). This completes the proof. \(\square\)

If letting \(\beta_n = 0\) for all \(n \in \mathbb{N}\) in Theorem 3.1, then we get the following.

**Corollary 3.2.** Let \(C\) be a nonempty closed convex subset of a reflexive Banach space \(E\) which admits a weakly sequentially continuous duality mapping \(J\) from \(E\) into \(E^*\). For every \(i = 1, \ldots, N\) \((N \geq 1)\), let \(T_i = \{T_i(t) : t \geq 0\}\) be a semigroup of nonexpansive mappings on \(C\) such that \(\mathcal{F} = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset\) and \(f : C \rightarrow C\) be a generalized contraction on \(C\). Let \(\{\alpha_n\} \subset [0, 1)\) and \(\{t_n\} \subset (0, \infty)\) be sequences satisfying \(\lim_{n \to \infty} t_n = \lim_{n \to \infty} (\alpha_n/t_n) = 0\). Let \(\{x_n\}\) be a sequence generated by
\[x_n = \alpha_n f(x_n) + \frac{1 - \alpha_n}{N} \sum_{i=1}^{N} T_i(t_n)x_n.\] (3.22)
Then \(\{x_n\}\) converges strongly to a point \(x^* \in \mathcal{F}\), which is the unique solution to the following variational inequality:
\[\langle (f - I)x^*, j(x - x^*) \rangle \leq 0, \quad \forall x \in \mathcal{F}.\] (3.23)

**Theorem 3.3.** Let \(C\) be a nonempty closed convex subset of a reflexive and strictly convex Banach space \(E\) which admits a weakly sequentially continuous duality mapping \(J\) from \(E\) into \(E^*\). For every \(i = 1, \ldots, N\) \((N \geq 1)\), let \(T_i = \{T_i(t) : t \geq 0\}\) be a semigroup of nonexpansive mappings on \(C\) such that \(\mathcal{F} = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset\) and \(f : C \rightarrow C\) be a generalized contraction on \(C\). Let \(\{\alpha_n\}, \{\beta_n\} \subset [0, 1)\) and \(\{t_n\} \subset (0, \infty)\) be the sequences satisfying \(\lim_{n \to \infty} t_n = \lim_{n \to \infty} (\beta_n/t_n) = 0\). Let \(\{x_n\}\) be a sequence generated
\[y_{in} = \alpha_n x_n + (1 - \alpha_n)T_i(t_n)x_n, \quad i = 1, \ldots, N,\]
\[x_{n+1} = \beta_n f(x_n) + \frac{1 - \beta_n}{N} \sum_{i=1}^{N} y_{in}, \quad \forall n \in \mathbb{N}.\] (3.24)
Then \(\{x_n\}\) converges strongly to a point \(x^* \in \mathcal{F}\), which is the unique solution of variational inequality (3.2).

**Proof.** Let \(p \in \mathcal{F}\) and \(M = \max \{\|x_1 - p\|, \eta^{-1}(\|f(p) - p\|)\}\). Now we show by induction that
\[\|x_n - p\| \leq M, \quad \forall n \in \mathbb{N}.\] (3.25)
It is obvious that (3.25) holds for \( n = 1 \). Suppose that (3.25) holds for some \( n = k \), where \( k > 1 \). Observe that

\[
\|y_{ik} - p\| = \|\alpha_k (x_k - p) + (1 - \alpha_k) (T_i(t_k)x_k - p)\| \\
\leq \alpha_k \|x_k - p\| + (1 - \alpha_k) \|T_i(t_k)x_k - p\| \leq \|x_k - p\|. \tag{3.26}
\]

Now, by using (3.24) and (3.26), we have

\[
\|x_{k+1} - p\| = \left\| \beta_k (f(x_k) - p) + \frac{1 - \beta_k}{N} \sum_{i=1}^{N} (y_{ik} - p) \right\| \\
\leq \beta_k \|f(x_k) - f(p)\| + \beta_k \|f(p) - p\| + \frac{1 - \beta_k}{N} \sum_{i=1}^{N} \|y_{ik} - p\| \\
\leq \beta_k \psi(\|x_k - p\|) + \beta_k \|f(p) - p\| + (1 - \beta_k) \|x_k - p\| \\
\leq \beta_k \psi(M) + \beta_k \eta(M) + (1 - \beta_k) M \\
\leq \beta_k \psi(M) + \beta_k (M - \phi(M)) + (1 - \beta_k) M = M. \tag{3.27}
\]

By induction we conclude that (3.25) holds for all \( n \in \mathbb{N} \). Therefore, \( \{x_n\} \) is bounded and so are \( \{f(x_n)\}, \{y_{in}\}, \{T_i(t_n)x_n\} \).

For each \( i = 1, \ldots, N \) and \( n \in \mathbb{N} \), define the mapping \( U(t_n) = (1/N) \sum_{i=1}^{N} S_i(t_n) \), where \( S_i(t_n) = \alpha_i I + (1 - \alpha_i) T_i(t_n) \). Then we rewrite the sequence (3.24) to

\[
x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) U(t_n)x_n. \tag{3.28}
\]

Obviously, each \( U(t_n) \) is nonexpansive. Since \( \{x_n\} \) is bounded and \( E \) is reflexive, we may assume that some subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) converges weakly to \( p \). Next we show that \( p \in \mathcal{F} \).

Put \( x_j = x_{n_j}, \beta_j = \beta_{n_j}, \) and \( t_j = t_{n_j} \) for each \( j \in \mathbb{N} \). Fix \( t > 0 \). By (3.28) we have

\[
\|x_j - U(t)p\| = \sum_{k=0}^{\lfloor t/t_j \rfloor - 1} \|U((k+1)t_j)x_j - U(kt_j)x_j\| \\
+ \|U\left(\frac{t}{t_j}t_j\right)x_j - U\left(\frac{t}{t_j}t_j\right)p\| + \|U\left(\frac{t}{t_j}t_j\right)p - U(t)p\|
\]
\[ \begin{aligned} &\leq \left[ \frac{t}{t_j} \right] \| U(t_j)x_j - x_{j+1} \| + \| x_{j+1} - p \| + \| U \left( t - \left[ \frac{t}{t_j} \right] t_j \right) p - p \| \\ &= \left[ \frac{t}{t_j} \right] \beta_j \| U(t_j)x_j - f(x_j) \| + \| x_{j+1} - p \| + \| U \left( t - \left[ \frac{t}{t_j} \right] t_j \right) p - p \| \\ &\leq \frac{t \beta_j}{t_j} \| U(t_j)x_j - f(x_j) \| + \| x_{j+1} - p \| + \max \{ \| U(s)p - p \| : 0 \leq s \leq t_j \}. \end{aligned} \] (3.29)

So, for all \( j \in \mathbb{N} \), we have

\[ \limsup_{j \to \infty} \| x_j - U(t)p \| \leq \limsup_{j \to \infty} \| x_{j+1} - p \| = \limsup_{j \to \infty} \| x_j - p \|. \] (3.30)

Since \( E \) has a weakly sequentially continuous duality mapping satisfying Opals’ condition, this implies \( p = U(t)p \). By Lemma 2.4, we have \( \text{Fix}(U(t)) = \cap_{j=1}^{N} \text{Fix}(T_j(t)) \) for each \( t > 0 \). Therefore, \( p \in \mathcal{F} \). In view of the variational inequality (3.2) and the assumption that duality mapping \( J \) is weakly sequentially continuous, we conclude that

\[ \limsup_{n \to \infty} \langle (f - I)q, j(x_{n+1} - q) \rangle = \lim_{j \to \infty} \langle (f - I)q, j(x_{n+1} - q) \rangle = \langle (I - f)q, j(p - q) \rangle \leq 0. \] (3.31)

Finally, we prove that \( x_n \to q \) as \( n \to \infty \). Suppose that \( \| x_n - q \| \to 0 \). Then there exists \( \varepsilon > 0 \) and subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) such that \( \| x_{n_j} - q \| \geq \varepsilon \) for all \( j \in \mathbb{N} \). Put \( x_j = x_{n_j}, \beta_j = \beta_{n_j}, \) and \( t_j = t_{n_j} \). By Lemma 2.3 one has \( \| f(x_j) - f(q) \| \leq r \| x_j - q \| \) for all \( j \in \mathbb{N} \). Now, from (2.2) and (3.28) we have

\[ \| x_{j+1} - q \|^2 = \| (1 - \beta_n)(U(t_j)x_j - q) + \beta_n(f(x_j) - q) \|^2 \]
\[ \leq (1 - \beta_j)^2 \| U(t_j)x_j - q \|^2 + 2\beta_j \langle f(x_j) - q, j(x_{j+1} - q) \rangle \]
\[ \leq (1 - \beta_j)^2 \| x_j - q \|^2 + 2\beta_n \langle f(x_j) - f(q), j(x_{j+1} - q) \rangle + 2\beta_j \langle f(q) - q, j(x_{j+1} - q) \rangle \]
\[ \leq (1 - \beta_j)^2 \| x_j - q \|^2 + 2\beta_j r \| x_j - q \| \| x_{j+1} - q \| + 2\beta_n \langle f(q) - q, j(x_{j+1} - q) \rangle \]
\[ \leq (1 - \beta_j)^2 \| x_j - q \|^2 + \beta_j r \left( \| x_j - q \|^2 + \| x_{j+1} - q \|^2 \right) + 2\beta_j \langle f(q) - q, j(x_{j+1} - q) \rangle \]
\[ = \left( (1 - \beta_j)^2 + \beta_j r \right) \| x_j - q \|^2 + \beta_j r \| x_{j+1} - q \|^2 + 2\beta_j \langle f(q) - q, j(x_{j+1} - q) \rangle. \] (3.32)
It follows that
\[
\|x_{j+1}\| \leq \frac{1 - (2 - r)\beta_j + \beta_j^2}{1 - \beta_j r} \|x_j - q\|^2 + \frac{2\beta_j}{1 - \beta_j r} \langle f(q) - q, j(x_j + q) \rangle
\leq \frac{1 - \beta_j r - 2(1 - r)\beta_j}{1 - \beta_j r} \|x_j - q\|^2 + \frac{2\beta_j}{1 - \beta_j r} \langle f(q) - q, j(x_j + q) \rangle + \beta_j M
\]
\[
= \left(1 - \frac{2(1 - r)\beta_j}{1 - \beta_j r}\right) \|x_j - q\|^2 + \frac{2\beta_j}{1 - \beta_j r} \langle f(q) - q, j(x_j + q) \rangle + \beta_j M
\]
\[
\leq (1 - 2(1 - r)\beta_j) \|x_j - q\|^2 + \beta_j \left(\frac{2}{1 - r} \langle f(q) - q, j(x_j + q) \rangle + \beta_j M\right),
\] (3.33)

where \(M\) is a constant.

Let \(\gamma_j = 2(1 - r)\beta_j\) and \(\delta_j = \beta_j((2/(1 - r))(f(q) - q, j(x_j + q)) + \beta_j M)\). It follows from (3.33) that
\[
\|x_{j+1} - q\| \leq (1 - \gamma_j) \|x_j - q\| + \delta_j.
\] (3.34)

It is easy to see that \(\gamma_j \to 0, \sum_{j=1}^{\infty} \gamma_j = \infty\) and (noting (3.28))
\[
\limsup_{j \to \infty} \frac{\delta_j}{\gamma_j} = \limsup_{j \to \infty} \frac{1}{(1 - r)^2} \langle f(q) - q, j(x_j + q) \rangle + \frac{M}{2(1 - r)} \beta_j,
\]
\[
\limsup_{n \to \infty} \frac{1}{(1 - r)^2} \langle f(q) - q, j(x_j + q) \rangle \leq 0.
\] (3.35)

Using Lemma 2.5, we conclude that \(\|x_j - q\| \to 0\) as \(j \to \infty\). It is a contradiction. Therefore, \(x_n \to q\) as \(n \to \infty\). This completes the proof. \(\square\)

If letting \(\alpha_n = 0\) for all \(n \in \mathbb{N}\) in Theorem 3.3, then we get the following.

**Corollary 3.4.** Let \(C\) be a nonempty closed convex subset of a reflexive and strictly convex Banach space \(E\) which admits a weakly sequentially continuous duality mapping \(J\) from \(E\) into \(E^*\). For every \(i = 1, \ldots, N(N \geq 1)\), let \(\mathcal{T}_i = \{T_i(t) : t \geq 0\}\) be a semigroup of nonexpansive mappings on \(C\) such that \(\mathcal{F} = \cap_{i=1}^{N} \text{Fix} (\mathcal{T}_i) \neq \emptyset\) and \(f : C \to C\) be a generalized contraction on \(C\). Let \(\{\beta_n\} \subset (0, 1)\) and \(\{t_n\} \subset (0, \infty)\) be sequences satisfying \(\lim_{n \to \infty} t_n = \lim_{n \to \infty} (\beta_n / t_n) = 0\). Let \(\{x_n\}\) be a sequence generated
\[
x_{n+1} = \beta_n f(x_n) + \frac{1 - \beta_n}{N} \sum_{i=1}^{N} T_i(t_n)x_n, \quad \forall n \in \mathbb{N}.
\] (3.36)

Then \(\{x_n\}\) converges strongly to a point \(x^* \in \mathcal{F}\), which is the unique solution of variational inequality (3.2).
Remark 3.5. Theorem 3.1 and Corollary 3.2 extend the corresponding ones of Suzuki [2], Xu [3], and Chen and He [5] from one nonexpansive semigroup to a finite family of nonexpansive semigroups. But Theorem 3.3 and Corollary 3.4 are not the extension of Theorem 3.2 of Chen and He [5] since Banach space in Theorem 3.3 and Corollary 3.4 is required to be strictly convex. But if letting \( N = 1 \) in Theorem 3.3 and Corollary 3.4, we can remove the restriction on strict convexity and hence they extend Theorem 3.2 of Chen and He [5] from a contraction to a generalized contraction.

Remark 3.6. Our Theorem 3.1 extends and improves Theorems 3.2 and 4.2 of Song and Xu [4] from a nonexpansive semigroup to a finite family of nonexpansive semigroups and a contraction to a generalized contraction. Our conditions on the control sequences are different with ones of Song and Xu [4].

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References


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