Research Article

A Modified Halpern-Type Iterative Method of a System of Equilibrium Problems and a Fixed Point for a Totally Quasi-$\phi$-Asymptotically Nonexpansive Mapping in a Banach Space

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The purpose of this paper is to introduce the modified Halpern-type iterative method by the generalized $f$-projection operator for finding a common solution of fixed-point problem of a totally quasi-$\phi$-asymptotically nonexpansive mapping and a system of equilibrium problems in a uniform smooth and strictly convex Banach space with the Kadec-Klee property. Consequently, we prove the strong convergence for a common solution of above two sets. Our result presented in this paper generalize and improve the result of Chang et al., (2012), and some others.

1. Introduction

In 1953, Mann [1] introduced the following iteration process which is now known as Mann’s iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad (1.1)$$

where $T$ is nonexpansive, the initial guess element $x_1 \in C$ is arbitrary, and $\{\alpha_n\}$ is a sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. In an
infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence (see [2, 3]).

Later, in 1967, Halpern [4] considered the following algorithm:

\[ x_1 \in C, \quad x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.2} \]

where \( T \) is nonexpansive. He proved the strong convergence theorem of \( \{x_n\} \) to a fixed point of \( T \) under some control condition \( \{\alpha_n\} \). Many authors improved and studied the result of Halpern [4] such as Qin et al. [5], Wang et al. [6], and reference therein.

In 2008-2009, Takahashi and Zembayashi [7, 8] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of the Banach spaces.

On the other hand, Li et al. [9] introduced the following hybrid iterative scheme for approximation fixed points of relatively nonexpansive mapping using the generalized \( f \)-projection operator in a uniformly smooth real Banach space which is also uniformly convex. They obtained strong convergence theorem for finding an element in the fixed point set of \( T \).


Very recently, Chang et al. [13] extended the results of Qin et al. [5] and Wang et al. [6] to consider a modification to the Halpern-type iteration algorithm for a total quasi-\( \phi \)-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of Banach spaces.

The purpose of this paper is to be motivated and inspired by the works mentioned above, we introduce a modified Halpern-type iterative method by using the new hybrid projection algorithm of the generalized \( f \)-projection operator for solving the common solution of fixed point for totally quasi-\( \phi \)-asymptotically nonexpansive mappings and the system of equilibrium problems in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. The results presented in this paper improve and extend the corresponding ones announced by many others.

2. Preliminaries and Definitions

Let \( E \) be a real Banach space with dual \( E^* \), and let \( C \) be a nonempty closed and convex subset of \( E \). Let \( \{\theta_i\}_{i \in \Gamma} : C \times C \to \mathbb{R} \) be a bifunction, where \( \Gamma \) is an arbitrary index set. The system of equilibrium problems is to find \( x \in C \) such that

\[ \theta_i(x, y) \geq 0, \quad i \in \Gamma, \forall y \in C. \tag{2.1} \]
If \( \Gamma \) is a singleton, then problem (2.1) reduces to the *equilibrium problem*, which is to find \( x \in C \) such that

\[
\theta(x, y) \geq 0, \quad \forall y \in C.
\]  

(2.2)

A mapping \( T \) from \( C \) into itself is said to be *nonexpansive* if

\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.
\]  

(2.3)

\( T \) is said to be *asymptotically nonexpansive* if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \) as \( n \to \infty \) such that

\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C.
\]  

(2.4)

\( T \) is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences \( \nu_n, \mu_n \) with \( \nu_n \to 0, \mu_n \to 0 \) as \( n \to \infty \) and a strictly increasing continuous function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \varphi(0) = 0 \) such that

\[
\|T^n x - T^n y\| \leq \|x - y\| + \nu_n \varphi(\|x - y\|) + \mu_n, \quad \forall x, y \in C, \quad \forall n \geq 1.
\]  

(2.5)

A point \( x \in C \) is a *fixed point* of \( T \) provided \( Tx = x \). Denote by \( F(T) \) the fixed point set of \( T \); that is, \( F(T) = \{x \in C : Tx = x\} \). A point \( p \) in \( C \) is called an *asymptotic fixed point* of \( T \) if \( C \) contains a sequence \( \{x_n\} \) which converges weakly to \( p \) such that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). The asymptotic fixed point set of \( T \) is denoted by \( \hat{F}(T) \).

The *normalized duality mapping* \( J : E \to 2^{E^*} \) is defined by \( J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\} \). If \( E \) is a Hilbert space, then \( J = I \), where \( I \) is the identity mapping. Consider the functional defined by

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,
\]  

(2.6)

where \( J \) is the normalized duality mapping and \( \langle \cdot, \cdot \rangle \) denote the duality pairing of \( E \) and \( E^* \). If \( E \) is a Hilbert space, then \( \phi(y, x) = \|y - x\|^2 \). It is obvious from the definition of \( \phi \) that

\[
(\|y\|^2 - \|x\|^2) \leq \phi(y, x) \leq (\|y\|^2 + \|x\|^2), \quad \forall x, y \in E.
\]  

(2.7)

A mapping \( T \) from \( C \) into itself is said to be *\( \phi \)-nonexpansive* \([14, 15]\) if

\[
\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.
\]  

(2.8)

\( T \) is said to be *quasi-\( \phi \)-nonexpansive* \([14, 15]\) if \( F(T) \neq \emptyset \) and

\[
\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T).
\]  

(2.9)
$T$ is said to be \textit{asymptotically $\phi$-nonexpansive} [15] if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq k_n \phi(x, y), \quad \forall x, y \in C. \quad (2.10)$$

$T$ is said to be \textit{quasi-$\phi$-asymptotically nonexpansive} [15] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall x \in C, \ p \in F(T), \ \forall n \geq 1. \quad (2.11)$$

$T$ is said to be \textit{totally quasi-$\phi$-asymptotically nonexpansive}, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\nu_n, \mu_n$ with $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, \ \forall x \in C, \ p \in F(T). \quad (2.12)$$

A mapping $T$ from $C$ into itself is said to be \textit{closed} if for any sequence $\{x_n\} \subset C$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} T x_n = y_0$, then $T x_0 = y_0$.

Alber [16] introduced the \textit{generalized projection} $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution of the minimization problem:

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x). \quad (2.13)$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping $f$ (see, e.g., [16–20]). If $E$ is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C$ becomes the metric projection $P_C : H \to C$. If $C$ is a nonempty, closed, and convex subset of a Hilbert space $H$, then $P_C$ is nonexpansive. This fact actually characterizes Hilbert spaces, and consequently, it is not available in more general Banach spaces. Later, Wu and Huang [21] introduced a new generalized $f$-projection operator in the Banach space. They extended the definition of the generalized projection operators and proved some properties of the generalized $f$-projection operator. Next, we recall the concept of the generalized $f$-projection operator. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined by

$$G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi \rangle + \|\varpi\|^2 + 2\rho f(y), \quad (2.14)$$

where $y \in C, \varpi \in E^*$, $\rho$ is positive number, and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From the definition of $G$, Wu and Huang [21] proved the following properties:

(1) $G(y, \varpi)$ is convex and continuous with respect to $\varpi$ when $y$ is fixed;

(2) $G(y, \varpi)$ is convex and lower semicontinuous with respect to $y$ when $\varpi$ is fixed.
Definition 2.1. Let \( E \) be a real Banach space with its dual \( E^* \). Let \( C \) be a nonempty, closed, and convex subset of \( E \). We say that \( \pi^F_C : E^* \to 2^C \) is a generalized \( f \)-projection operator if

\[
\pi^F_C(x) = \left\{ u \in C : G(u, x) = \inf_{y \in C} G(y, x), \quad \forall x \in E^* \right\}.
\] (2.15)

A Banach space \( E \) with norm \( \| \cdot \| \) is called strictly convex if \( \| (x+y)/2 \| < 1 \) for all \( x, y \in E \) with \( \| x \| = \| y \| = 1 \) and \( x \neq y \). Let \( U = \{ x \in E : \| x \| = 1 \} \) be the unit sphere of \( E \). A Banach space \( E \) is called smooth if the limit \( \lim_{t \to 0} ((\| x+ty \| - \| x \|)/t) \) exists for each \( x, y \in U \). It is also called uniformly smooth if the limit exists uniformly for all \( x, y \in U \). The modulus of smoothness of \( E \) is the function \( \rho_E : [0, \infty) \to [0, \infty) \) defined by \( \rho_E(t) = \sup \{ (\| x+y \| + \| x-y \|)/2 - 1 : \| x \| = 1, \| y \| \leq t \} \). The modulus of convexity of \( E \) (see [22]) is the function \( \delta_E : [0, 2] \to [0, 1] \) defined by \( \delta_E(\varepsilon) = \inf \{ 1 - \| (x+y)/2 \| : x, y \in E, \| x \| = \| y \| = 1, \| x-y \| \geq \varepsilon \} \). In this paper we denote the strong convergence and weak convergence of a sequence \( \{ x_n \} \) by \( x_n \to x \) and \( x_n \rightharpoonup x \), respectively.

Remark 2.2. The basic properties of \( E, E^*, J, \) and \( J^{-1} \) (see [18]) are as follows.

(i) If \( E \) is an arbitrary Banach space, then \( J \) is monotone and bounded.

(ii) If \( E \) is a strictly convex, then \( J \) is strictly monotone.

(iii) If \( E \) is a smooth, then \( J \) is single valued and semicontinuous.

(iv) If \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \).

(v) If \( E \) is reflexive smooth and strictly convex, then the normalized duality mapping \( J \) is single valued, one-to-one, and onto.

(vi) If \( E \) is a reflexive strictly convex and smooth Banach space and \( J \) is the duality mapping from \( E \) into \( E^* \), then \( J^{-1} \) is also single valued, bijective, and is also the duality mapping from \( E^* \) into \( E \), and thus \( JJ^{-1} = I_E \) and \( J^{-1}J = I_{E^*} \).

(vii) If \( E \) is uniformly smooth, then \( E \) is smooth and reflexive.

(viii) \( E \) is uniformly smooth if and only if \( E^* \) is uniformly convex.

(ix) If \( E \) is a reflexive and strictly convex Banach space, then \( J^{-1} \) is norm-weak*-continuous.

Remark 2.3. If \( E \) is a reflexive, strictly convex, and smooth Banach space, then \( \phi(x, y) = 0 \), if and only if \( x = y \). It is sufficient to show that if \( \phi(x, y) = 0 \) then \( x = y \). From (2.6), we have \( \| x \| = \| y \| \). This implies that \( \langle x, Jy \rangle = \| x \|^2 = \| Jy \|^2 \). From the definition of \( J \), one has \( Jx = Jy \). Therefore, we have \( x = y \) (see [18, 20, 23] for more details).

Recall that a Banach space \( E \) has the Kadec-Klee property [18, 20, 24], if for any sequence \( \{ x_n \} \subset E \) and \( x \in E \) with \( x_n \rightharpoonup x \) and \( \| x_n \| \to \| x \| \), then \( \| x_n - x \| \to 0 \) as \( n \to \infty \). It is well known that if \( E \) is a uniformly convex Banach space, then \( E \) has the Kadec-Klee property.

We also need the following lemmas for the proof of our main results.

Lemma 2.4 (see Change et al. [25]). Let \( C \) be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space \( E \) with the Kadec-Klee property. Let \( T : C \to C \) be a closed
Lemma 2.5 (see Wu and Hung [21]). Let $E$ be a real reflexive Banach space with its dual $E^*$ and $C$ a nonempty, closed, and convex subset of $E$. The following statement hold:

1. $\pi_C^f(\varpi)$ is a nonempty, closed and convex subset of $C$ for all $\varpi \in E^*$;
2. if $E$ is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f(\varpi)$ if and only if
\[
(x - y, \varpi - Jx) + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;
\]
3. if $E$ is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then $\pi_C^f(\varpi)$ is single-valued mapping.

Lemma 2.6 (see Fan et al. [26]). Let $E$ be a real reflexive Banach space with its dual $E^*$ and $C$ be a nonempty, closed and convex subset of $E$. If $E$ is strictly convex, then $\pi_C^f(\varpi)$ is single valued.

Recall that $J$ is single-valued mapping when $E$ is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution in (2.14) gives
\[
G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + \rho f(y).
\]

Now we consider the second generalized $f$ projection operator in Banach space (see [9]).

Definition 2.7. Let $E$ be a real smooth Banach space, and let $C$ be a nonempty, closed, and convex subset of $E$. We say that $\Pi_C^f : E \to 2^C$ is generalized $f$-projection operator if
\[
\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \quad \forall x \in E \right\}.
\]

Lemma 2.8 (see Deimling [27]). Let $E$ be a Banach space, and let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that
\[
f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E.
\]

Lemma 2.9 (see Li et al. [9]). Let $E$ be a reflexive smooth Banach space, and let $C$ be a nonempty, closed, and convex subset of $E$. The following statements hold:

1. $\Pi_C^f x$ is nonempty, closed and convex subset of $C$ for all $x \in E$;
2. for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if
\[
\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C;
\]
3. if $E$ is strictly convex, then $\Pi_C^f$ is single-valued mapping.
Lemma 2.10 (see Li et al. [9]). Let $E$ be a real reflexive smooth Banach space, let $C$ be a nonempty, closed, and convex subset of $E$, $x \in E$, and let $\tilde{x} \in \Pi^E_Cx$. Then

$$\phi(y, \tilde{x}) + G(\tilde{x}, Jx) \leq G(y, Jx), \quad \forall y \in C.$$  \hspace{1cm} (2.21)

Remark 2.11. Let $E$ be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$, then Lemma 2.10 reduces to the property of the generalized projection operator considered by Alber [16].

If $f(y) \geq 0$ for all $y \in C$ and $f(0) = 0$, then the definition of totally quasi-$\phi$-asymptotically nonexpansive $T$ is equivalent to if $F(T) \neq \emptyset$, and there exist nonnegative real sequences $\nu_n$, $\mu_n$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$G(p, T^n x) \leq G(p, x) + \nu_n \zeta G(p, x) + \mu_n, \quad \forall n \geq 1, \ \forall x \in C, \ p \in F(T).$$  \hspace{1cm} (2.22)

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that $\theta$ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) $\theta$ is monotone; that is, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$  \hspace{1cm} (2.23)

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

For example, let $A$ be a continuous and monotone operator of $C$ into $E^*$ and define

$$\theta(x, y) = \langle Ax, y - x \rangle, \quad \forall x, y \in C.$$  \hspace{1cm} (2.24)

Then, $\theta$ satisfies (A1)–(A4). The following result is in Blum and Oettli [28].

Lemma 2.12 (see Blum and Oettli [28]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $\theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$\theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$  \hspace{1cm} (2.25)
Lemma 2.13 (see Takahashi and Zembayashi [8]). Let \( C \) be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space \( E \), and let \( \theta \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfying conditions (A1)–(A4). For all \( r > 0 \) and \( x \in E \), define a mapping \( T^\theta_r : E \to C \) as follows:

\[
T^\theta_r x = \left\{ z \in C : \theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}.
\]  

Then the following hold:

1. \( T^\theta_r \) is single-valued;
2. \( T^\theta_r \) is a firmly nonexpansive-type mapping [29]; that is, for all \( x, y \in E \),

\[
\langle T^\theta_r x - T^\theta_r y, JT^\theta_r x - JT^\theta_r y \rangle \leq \langle T^\theta_r x - T^\theta_r y, Jx - Jy \rangle;
\]

3. \( F(T^\theta_r) = EP(\theta) \);
4. \( EP(\theta) \) is closed and convex.

Lemma 2.14 (see Takahashi and Zembayashi [8]). Let \( C \) be a closed convex subset of a smooth, strictly convex, and reflexive Banach space \( E \), let \( \theta \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfying (A1)–(A4), and let \( r > 0 \). Then, for \( x \in E \) and \( q \in F(T^\theta_r) \),

\[
\phi(q, T^\theta_r x) + \phi(T^\theta_r x, x) \leq \phi(q, x).
\]

3. Main Result

Theorem 3.1. Let \( C \) be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space \( E \) with the Kadec-Klee property. For each \( j = 1, 2, \ldots, m \), let \( \theta_j \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) which satisfies conditions (A1)–(A4). Let \( S : C \to C \) be a closed totally quasi-\( \phi \)-asymptotically nonexpansive mappings with nonnegative real sequences \( \nu_n, \mu_n \) with \( \nu_n \to 0, \mu_n \to 0 \) as \( n \to \infty \), and a strictly increasing continuous function \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \psi(0) = 0 \). Let \( f : E \to \mathbb{R} \) be a convex and lower semicontinuous function with \( C \subset \text{int} (D(f)) \) such that \( f(x) \geq 0 \) for all \( x \in C \) and \( f(0) = 0 \). Assume that \( \forall := F(S) \cap \bigcap_{j=1}^{m} EP(\theta_j) \neq \emptyset \). For an initial point \( x_1 \in E \) and \( C_1 = C \), one define the sequence \( \{x_n\} \) by

\[
u_n = \mathcal{T}^{\theta_{m-1}}_{\mathcal{T}^{\theta_{m-2}}_{\mathcal{T}^{\theta_{m-3}}_{\mathcal{T}^{\theta_{m-4}}_{\ldots \mathcal{T}^{\theta_1}}}}},
\]

\[
z_n = f^{-1}(\alpha_n Jx_1 + (1 - \alpha_n) J\mathcal{S}^n u_n),
\]

\[
C_{n+1} = \{ v \in C_n : G(v, Jz_n) \leq G(v, Ju_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \xi_n \},
\]

\[
x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad n \in \mathbb{N},
\]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\), \( \xi_n = \nu_n \sup_{q \in \forall} \psi(G(q, x_n)) + \mu_n \) and \( \{\tau_{j,n}\} \subset \{d, \infty\} \) for some \( d > 0 \). If \( \lim_{n \to \infty} \alpha_n = 0 \), then \( \{x_n\} \) converges strongly to \( \Pi_{E}^f x_0 \).
First, we show that \( C_n \) is closed and convex for all \( n \in \mathbb{N} \).

Clearly \( C_1 = C \) is closed and convex. Suppose that \( C_n \) is closed and convex for all \( n \in \mathbb{N} \). For any \( v \in C_n \), we know that \( G(v, Jz_n) \leq G(v, Jx_n) + \zeta_n \) is equivalent to

\[
2(v, Jx_n - Jz_n) \leq \|x_n\|^2 - \|z_n\|^2 + \zeta_n. \tag{3.2}
\]

This shows that \( q \in C_{n+1} \) which implies that \( F \subset C_{n+1} \), and hence, \( F \subset C_n \) for all \( n \in \mathbb{N} \) and the sequence \{\( x_n \)\} is well defined. From \( x_n = \Pi_{C_n}^f x_1 \), we see that

\[
\langle x_n - q, Jx_1 - Jx_n \rangle + \rho f(q) - \rho f(x_n) \geq 0, \quad \forall q \in C_n. \tag{3.5}
\]

Since \( F \subset C_n \) for each \( n \in \mathbb{N} \), we arrive at

\[
\langle x_n - q, Jx_1 - Jx_n \rangle + \rho f(q) - \rho f(x_n) \geq 0, \quad \forall q \in F. \tag{3.6}
\]

Hence, the sequence \{\( x_n \)\} is well defined.
Step 3. We will show that $x_n \to p \in \mathcal{F} := F(S) \cap \bigcap_{j=1}^n \text{EP}(\theta_j)$.

Let $f : E \to \mathbb{R}$ be convex and lower semicontinuous function, follows from Lemma 2.8, there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that
\[
f(y) \geq \langle y, x^* \rangle + \alpha, \quad \forall y \in E.
\]
(3.7)

Since $x_n \in C_n \subset E$, it follows that
\[
G(x_n, Jx_1) = \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\alpha f(x_n)
\]
\[
\geq \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\alpha (x_n, x^*) + 2\alpha
\]
\[
= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^* \rangle + \|x_1\|^2 + 2\alpha
\]
\[
\geq \|x_n\|^2 - 2\|x_n\| \|Jx_1 - \rho x^*\| + \|x_1\|^2 + 2\alpha
\]
\[
= (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\alpha.
\]
(3.8)

For $q \in \mathcal{F}$ and $x_n = \Pi_{C_n}^{f} x_1$, we have
\[
G(q, Jx_1) \geq G(x_n, Jx_1) \geq (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\alpha.
\]
(3.9)

This shows that $\{x_n\}$ is bounded and so is $\{G(x_n, Jx_1)\}$. From the fact that $x_{n+1} = \Pi_{C_{n+1}}^{f} x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}^{f} x_1$, it follows from Lemma 2.10 that
\[
0 \leq (\|x_{n+1}\| - \|x_n\|)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_1) - G(x_n, Jx_1).
\]
(3.10)

That is, $\{G(x_n, Jx_1)\}$ is nondecreasing. Hence, we obtain that $\lim_{n \to \infty} G(x_n, Jx_1)$ exists. Taking $n \to \infty$, we obtain
\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0.
\]
(3.11)

Since $E$ is reflexive, $\{x_n\}$ is bounded, and $C_n$ is closed and convex for all $n \in \mathbb{N}$. Without loss of generality, we can assume that $x_n \to p \in C_n$. From the fact that $x_n = \Pi_{C_n}^{f} x_1$, we get that
\[
G(x_n, Jx_1) \leq G(p, Jx_1), \quad \forall n \in \mathbb{N}.
\]
(3.12)

Since $f$ is convex and lower semicontinuous, we have
\[
\liminf_{n \to \infty} G(x_n, Jx_1) = \liminf_{n \to \infty} \left\{ \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\alpha f(x_n) \right\}
\]
\[
\geq \|p\|^2 - 2\langle p, Jx_1 \rangle + \|x_1\|^2 + 2\alpha f(p)
\]
\[
= G(x_n, Jx_1).
\]
(3.13)
By (3.12) and (3.13), we get

$$G(p, Jx_1) \leq \liminf_{n \to \infty} G(x_n, Jx_1) \leq \limsup_{n \to \infty} G(x_n, Jx_1) \leq G(p, Jx_1).$$

That is, \(\lim_{n \to \infty} G(x_n, Jx_1) = G(p, Jx_1)\); this implies that \(\|x_n\| \to \|p\|\); by virtue of the Kadec-Klee property of \(E\), we obtain that

$$\lim_{n \to \infty} x_n = p.$$

We also have

$$\lim_{n \to \infty} x_{n+1} = p.$$

From (3.15), we get that

$$\lim_{n \to \infty} \zeta_n = \lim_{n \to \infty} \left( \nu_n \sup_{q \in F} \psi \left( G(q, x_n) + \mu_n \right) \right) = 0.$$

(a) We show that \(p \in \cap_{j=1}^m \text{EP}(\theta_j)\).

Since \(x_{n+1} = \Pi_{C_{n+1}}^F x_1 \in C_{n+1} \subset C_n\) and the definition of \(C_{n+1}\), we have

$$G(x_{n+1}, Jx_n) \leq \alpha_n G(x_{n+1}, Jx_1) + (1 - \alpha_n) G(x_{n+1}, Jx_n) + \zeta_n$$

is equivalent to

$$\phi(x_{n+1}, x_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \zeta_n.$$

From (3.11), (3.15), and (3.17), it follows that

$$\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0.$$

From (2.7), we have

$$\left( \|x_{n+1}\| - \|u_n\| \right)^2 \to 0.$$

Since \(\|x_{n+1}\| \to \|p\|\), we have

$$\|u_n\| \to \|p\| \quad \text{as} \quad n \to \infty.$$

It follow that

$$\|Ju_n\| \to \|Jp\| \quad \text{as} \quad n \to \infty.$$
That is, \( \{ \| J u_n \| \} \) is bounded in \( E^* \) and \( E^* \) is reflexive; we assume that \( J u_n \rightharpoonup u^* \in E^* \). In view of \( J(E) = E^* \), there exists \( u \in E \) such that \( Ju = u^* \). It follows that

\[
\phi(x_{n+1}, u_n) = \| x_{n+1} \|^2 - 2 \langle x_{n+1}, J y_n \rangle + \| u_n \|^2 \\
= \| x_{n+1} \|^2 - 2 \langle x_{n+1}, J u_n \rangle + \| J u_n \|^2.
\] (3.24)

Taking \( \lim \inf_{n \to \infty} \) on both sides of the equality above and \( \| \cdot \| \) is the weak lower semicontinuous, it yields that

\[
0 \geq \| p \|^2 - 2 \langle p, u^* \rangle + \| u^* \|^2 \\
= \| p \|^2 - 2 \langle p, Ju \rangle + \| Ju \|^2 \\
= \| p \|^2 - 2 \langle p, Ju \rangle + \| u \|^2 \\
= \phi(p, u).
\] (3.25)

That is, \( p = u \), which implies that \( u^* = Jp \). It follows that \( J u_n \rightharpoonup Jp \in E^* \). From (3.23) and the Kadec-Klee property of \( E^* \) we have \( J u_n \to Jp \) as \( n \to \infty \). Note that \( J^{-1} : E^* \to E \) is norm-weak *-continuous; that is, \( u_n \rightharpoonup p \). From (3.22) and the Kadec-Klee property of \( E \), we have

\[
\lim_{n \to \infty} u_n = p.
\] (3.26)

For \( q \in F \subset C_n \), by nonexpansiveness, we observe that

\[
\phi(q, u_n) = \phi(q, K_n^m x_n) \\
\leq \phi(q, K_n^{m-1} x_n) \\
\leq \phi(q, K_n^{m-2} x_n) \\
\vdots \\
\leq \phi(q, K_n^j x_n).
\] (3.27)

By Lemma 2.14, we have for \( j = 1, 2, 3, \ldots, m \)

\[
\phi(K_n^j x_n, x_n) \leq \phi(q, x_n) - \phi(q, K_n^j x_n) \leq \phi(q, x_n) - \phi(q, u_n).
\] (3.28)

Since \( x_n, u_n \to p \) as \( n \to \infty \), we get \( \phi(K_n^j x_n, x_n) \to 0 \) as \( n \to \infty \), for \( j = 1, 2, 3, \ldots, m \). From (2.7), it follow that

\[
\left( \| K_n^j x_n \| - \| x_n \| \right)^2 \to 0.
\] (3.29)
Since \( \|x_n\| \to \|p\| \), we also have

\[
\left\| K_n^j x_n \right\| \to \|p\| \quad \text{as} \quad n \to \infty. \tag{3.30}
\]

Since \( \{ K_n^j x_n \} \) is bounded and \( E \) is reflexive, without loss of generality we assume that \( K_n^j y_n \rightharpoonup h \). We know that \( C_n \) is closed and convex for each \( n \geq 1 \) it is obvious that \( h \in C_n \). Again since

\[
\phi(K_n^j x_n, x_n) = \left\| K_n^j x_n \right\|^2 - 2\left\langle K_n^j x_n, Jx_n \right\rangle + \|x_n\|^2, \tag{3.31}
\]

taking \( \liminf_{n \to \infty} \) on the both sides of equality above, we have

\[
0 \geq \|h\|^2 - 2\langle h, Jp \rangle + \|p\|^2 = \phi(h, p). \tag{3.32}
\]

That is, \( h = p \), for all \( j = 1, 2, 3, \ldots, m \); it follow that

\[
K_n^j x_n \rightharpoonup p; \tag{3.33}
\]

from (3.30), (3.33), and the Kadec-Klee property, it follows that

\[
\lim_{n \to \infty} K_n^j x_n = p, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.34}
\]

By using triangle inequality, we have

\[
\left\| x_n - K_n^j x_n \right\| \leq \left\| x_n - p \right\| + \left\| p - K_n^j u_n \right\|. \tag{3.35}
\]

Since \( x_n, K_n^j x_n \to p \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \left\| x_n - K_n^j x_n \right\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.36}
\]

Again by using triangle inequality, we have

\[
\left\| K_n^j x_n - K_n^{j-1} x_n \right\| \leq \left\| K_n^j x_n - x_n \right\| + \left\| x_n - K_n^{j-1} x_n \right\|. \tag{3.37}
\]

From (3.36), we also have

\[
\lim_{n \to \infty} \left\| K_n^j x_n - K_n^{j-1} x_n \right\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.38}
\]

Since \( J \) is uniformly norm-to-norm continuous, we obtain

\[
\lim_{n \to \infty} \left\| J K_n^j x_n - J K_n^{j-1} x_n \right\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.39}
\]
From $r_{j,n} > 0$, we have $\|J_{n}^{j}x_{n} - J_{n}^{k-1}x_{n}\|/r_{j,n} \to 0$ as $n \to \infty$ for all $j = 1, 2, 3, \ldots, m$, and

$$\theta_j(K_{n}^{j}y_{n}, y) + \frac{1}{r_{j,n}}(y - K_{n}^{j}x_{n}, J_{n}^{j}y_{n} - J_{n}^{k-1}x_{n}) \geq 0, \forall y \in C. \quad (3.40)$$

By (A2), that

$$\left\|y - K_{n}^{j}y_{n}\right\| \geq \frac{1}{r_{j,n}}(y - K_{n}^{j}x_{n}, J_{n}^{j}y_{n} - J_{n}^{k-1}x_{n}) \geq -\theta_j(K_{n}^{j}x_{n}, y) \geq \theta_j(y, K_{n}^{j}x_{n}), \forall y \in C,$$

and $K_{n}^{j}x_{n} \to p$ as $n \to \infty$, we get $\theta_j(y, p) \leq 0$, for all $y \in C$. For $0 < t < 1$, define $y_{t} = ty + (1 - t)p$, then $y_{t} \in C$ which imply that $\theta_j(y_{t}, p) \leq 0$. From (A1), we obtain that

$$0 = \theta_j(y_{t}, y_{t}) \leq t\theta_j(y_{t}, y) + (1 - t)\theta_j(y_{t}, p) \leq t\theta_j(y_{t}, y). \quad (3.42)$$

We have that $\theta_j(y_{t}, y) \geq 0$. From (A3), we have $\theta_j(p, y) \geq 0$, for all $y \in C$ and $j = 1, 2, 3, \ldots, m$. That is, $p \in EP(\theta_j)$, for all $j = 1, 2, 3, \ldots, m$. This imply that $p \in \cap_{j=1}^{m} EP(\theta_j)$.

(b) We show that $p \in F(S)$.

Since $x_{n+1} = \Pi_{C_{m+1}}^{j=1} x_{1} \in C_{n+1} \subset C_{n}$ and the definition of $C_{n+1}$, we have

$$G(x_{n+1}, Jz_{n}) \leq \alpha_{n}G(x_{n+1}, Jx_{1}) + (1 - \alpha_{n})G(x_{n+1}, Jx_{n}) + \xi_{n} \quad (3.43)$$

is equivalent to

$$\phi(x_{n+1}, z_{n}) \leq \alpha_{n}\phi(x_{n+1}, x_{1}) + (1 - \alpha_{n})\phi(x_{n+1}, x_{n}) + \xi_{n}. \quad (3.44)$$

Following (3.11), (3.15), and (3.17), we get that

$$\lim_{n \to \infty} \phi(x_{n+1}, z_{n}) = 0. \quad (3.45)$$

From (2.7), we also have

$$\|z_{n}\| \to \|p\| \quad \text{as } n \to \infty. \quad (3.46)$$

It follows that

$$\|Jz_{n}\| \to \|Jp\| \quad \text{as } n \to \infty. \quad (3.47)$$
This implies that \( \{\|Jz_n\|\} \) is bounded in \( E^* \). Since \( E \) is reflexive and \( E^* \) is also reflexive, we can assume that \( Jz_n \rightharpoonup z^* \in E^* \). In view of the reflexive of \( E \), we see that \( J(E) = E^* \). There exists \( z \in E \) such that \( Jz = z^* \). It follows that

\[
\phi(x_{n+1}, z_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n \rangle + \|z_n\|^2
\]

\[
= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n \rangle + \|Jz_n\|^2.
\]

Taking \( \liminf_{n \to \infty} \) on both sides of the equality above and in view of the weak lower semicontinuity of norm \( \| \cdot \| \), it yields that

\[
0 \geq \|p\|^2 - 2\langle p, z^* \rangle + \|z^*\|^2
\]

\[
= \|p\|^2 - 2\langle p, Jz \rangle + \|Jz\|^2
\]

\[
= \|p\|^2 - 2\langle p, Jz \rangle + \|z\|^2
\]

\[
= \phi(p, z);
\]

That is \( p = z \), which implies that \( z^* = Jp \). It follows that \( Jz_n \rightharpoonup Jp \in E^* \). From (3.47) and the Kadec-Klee property of \( E^* \) we have \( Jz_n \rightharpoonup Jp \) as \( n \to \infty \). Since \( J^{-1} : E^* \to E \) is norm-weak *-continuous, \( z_n \rightharpoonup p \) as \( n \to \infty \). From (3.46) and the Kadec-Klee property of \( E \), we have

\[
\lim_{n \to \infty} Jz_n = Jp.
\]

Since \( \{x_n\} \) is bounded, then a mapping \( S \) is also bounded. From the condition \( \lim_{n \to \infty} \alpha_n = 0 \), we have that

\[
\|Jz_n - JS^n u_n\| = \lim_{n \to \infty} \alpha_n \|Jx_1 - JS^n u_n\| = 0.
\]

From (3.47), we get

\[
\|JS^n u_n\| \to \|Jp\| \quad \text{as} \quad n \to \infty.
\]

Since \( J^{-1} : E^* \to E \) is norm-weak *-continuous,

\[
S^n u_n \rightharpoonup p \quad \text{as} \quad n \to \infty.
\]

On the other hand, we observe that

\[
\|S^n u_n\| - \|p\| = \|JS^n u_n\| - \|Jp\| \leq \|J(S^n u_n) - Jp\|.
\]

In view of (3.52), we obtain \( \|S^n u_n\| \to \|p\| \). Since \( E \) has the Kadec-Klee property, we get

\[
S^n u_n \rightharpoonup p \quad \text{for each} \quad n \in \mathbb{N}.
\]
From $S^n u_n \to p$, we get $S^{n+1} u_n \to p$; that is, $SS^n u_n \to p$. In view of closeness of $S$, we have $Sp = p$. This implies that $p \in F(S)$. From (a) and (b), it follows that $p \in \cap_{j=1}^m \text{EP}(\theta_j) \cap F(S)$.

Step 4. We will show that $p = \Pi^f_q x_1$.

Since $\mathcal{F}$ is closed and convex set from Lemma 2.9, we have $\Pi^f_q x_1$ which is single valued, denoted by $v$. By definition $x_n = \Pi^f_{C_n} x_1$ and $v \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_1) \leq G(v, Jx_1), \quad \forall n \geq 1. \quad (3.56)$$

By the definition of $G$ and $f$, we know that, for each given $x$, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to $\xi$. So

$$G(p, Jx_1) \leq \liminf_{n \to \infty} G(x_n, Jx_1) \leq \limsup_{n \to \infty} G(x_n, Jx_1) \leq G(v, Jx_1). \quad (3.57)$$

From the definition of $\Pi^f_q x_1$ and since $p \in \mathcal{F}$, we conclude that $v = p = \Pi^f_q x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed.

Setting $v_n \equiv 0$ and $\mu_n \equiv 0$ in Theorem 3.1, then we have the following corollary.

**Corollary 3.2.** Let $C$ be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space $E$ with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let $\theta_j$ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4). Let $S : C \to C$ be a closed and quasi-$\phi$-asymptotically nonexpansive mappings, and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \text{int} (D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (\cap_{j=1}^m \text{EP}(\theta_j)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, we define the sequence $\{x_n\}$ by

$$u_n = T_{r_{\alpha_n}}^0 T_{r_{\alpha_{n-1}}}^0 \cdots T_{r_{\alpha_1}}^0 x_n,$$

$$z_n = J^{-1}(\alpha_n f x_1 + (1 - \alpha_n) f S^n u_n),$$

$$C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, Jx_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \xi_n \},$$

$$x_{n+1} = \Pi^f_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \quad (3.58)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\xi_n = v_n \sup_{q \in \mathcal{F}} \varphi(G(q, x_n)) + \mu_n$, and $\{r_{\alpha_n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi^f_q x_1$.

Let $E$ be a real Banach space, and let $C$ be a nonempty closed convex subset of $E$. Given a mapping $A : C \to E^*$, let $\theta(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then $x^* \in \text{EP}(\theta)$ if and only if $\langle Ax^*, y - x^* \rangle \geq 0$ for all $y \in C$; that is, $x^*$ is a solution of the classical variational inequality problem. The set of this solution is denoted by $VI(A, C)$. For each $r > 0$ and $x \in E$, we define
the mapping $T^d_x$ by
\[
T^d_x = \left\{ z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.
\] (3.59)

Hence, we obtain the following corollary.

**Corollary 3.3.** Let $C$ be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space $E$ with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let $\{A_j\}$ be a continuous monotone mapping of $C$ into $E'$. Let $S : C \to C$ be a closed totally quasi-$\phi$-asymptotically nonexpansive mappings with nonnegative real sequences $\nu_n$, $\mu_n$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \text{int}(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (r_j^{m} VI(A_j, C)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, one defines the sequence $\{x_n\}$ by
\[
\begin{align*}
\nu_n &= T^{\nu_{n-1}}_{r_{m-1}} T^{\nu_{m-2}}_{r_{m-2}} \cdots T^{\nu_1}_{r_1} x_n, \\
\mu_n &= J^{-1}(\alpha_n Jx + (1 - \alpha_n) J\nu_n), \\
C_{n+1} &= \{v \in C_n : G(v, Jz_n) \leq G(v, J\nu_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, J\nu_n) + \zeta_n\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{align*}
\] (3.60)

where $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \varphi(G(q, x_n)) + \mu_n$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{r_{j,n}\} \subset [\delta, \infty)$ for some $\delta > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

If $f(x) = 0$ for all $x \in E$, we have $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C x = \Pi_C x$. From Theorem 3.1, we obtain the following corollary.

**Corollary 3.4.** Let $C$ be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space $E$ with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let $\theta_j$ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). Let $S : C \to C$ be a closed totally quasi-$\phi$-asymptotically nonexpansive mappings with nonnegative real sequences $\nu_n$, $\mu_n$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (r_j^{m} \text{EP}(\theta_j)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, we define the sequence $\{x_n\}$ by
\[
\begin{align*}
\nu_n &= T^{\nu_{n-1}}_{r_{m-1}} T^{\nu_{m-2}}_{r_{m-2}} \cdots T^{\nu_1}_{r_1} x_n, \\
\mu_n &= J^{-1}(\alpha_n Jx + (1 - \alpha_n) J\nu_n), \\
C_{n+1} &= \{v \in C_n : G(v, Jz_n) \leq G(v, J\nu_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, J\nu_n) + \zeta_n\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{align*}
\] (3.61)

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \varphi(G(q, x_n)) + \mu_n$, and $\{r_{j,n}\} \subset [\delta, \infty)$ for some $\delta > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$. 


Remark 3.5. Our main result extends and improves the result of Chang et al. [13] in the following sense.

(i) From the algorithm we used new method replace by the generalized $f$-projection method which is more general than generalized projection.

(ii) For the problem, we extend the result to a common problem of fixed point problems and equilibrium problems.

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