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Research Article

The Extended Hyperbolic Function Method for Generalized Forms of Nonlinear Heat Conduction and Huxley Equations

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The extended hyperbolic function method is used to derive abundant exact solutions for generalized forms of nonlinear heat conduction and Huxley equations. The extended hyperbolic function method provides abundant solutions in addition to the existing ones. Some previous results are supplemented and extended greatly.

1. Introduction

The quasi-linear diffusion equations with a nonlinear source arise in many scientific applications such as mathematical biology, diffusion process, plasma physics, combustion theory, neural physics, liquid crystals, chemical reactions, and mechanics of porous media. It is well known that wave phenomena of plasma media and fluid dynamics are modeled by kink-shaped tanh solution or by bell-shaped sech solutions.

The exact solution, if available, of nonlinear partial differential equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. It can also provide much physical information and more inside into the physical aspects of the nonlinear physical problem. During the past decades, much effort has been spent on the subject of obtaining the exact analytical solutions to the nonlinear evolution PDEs. Many powerful methods have been proposed such as inverse scattering transformation method [1], Bäcklund and Darboux transformation method [2, 3], Hirota bilinear method [4], Lie group reduction method [5], the tanh method [6], the tanh-coth method [7], the sine-cosine method [8, 9], homogeneous balance method [10–12], Jacobi elliptic function method [13, 14], extended tanh method [15, 16], F-expansion method and Exp-function method [17, 18], the first integral method and Riccati method [19, 20], as well as extended improved tanh-function method [21, 22]. With the development of symbolic computation, the tanh method, the Exp-function method, sine-Gordon equation expansion method, and all kinds of auxiliary equation methods attract more and more researchers. We present an effective extension to the projective Riccati equation method [19, 20] and extended improved tanh-function method [21, 22], namely, the extended hyperbolic function method in [23]. Our method can also be regarded as an extension of the recent works by Wazwaz [24–28].

The proposed method supply a unified formulation to construct abundant traveling wave solutions to nonlinear evolution partial differential equations of special physical significance. Furthermore, the presented method is readily computerized by using symbolic software Maple. Based on the extended hyperbolic function method and computer symbolic software, we develop a Maple software package "PDESolver."

The balancing parameter m plays an important role in the extended hyperbolic function method in that it should be a positive integer to derive a closed-form analytic solution. However, for noninteger values of m, we usually use a transformation formula to overcome this difficulty.

For illustration, we investigate generalized forms of the nonlinear heat conduction equation and Huxley equation expressed by

$$u_t - \alpha(u^n)_{xx} - u + u^n = 0, \tag{1.1}$$

$$u_t - \alpha u_{xx} - u(\beta - u^n)(u^n - 1) = 0, \tag{1.2}$$

respectively. Equation (1.1) is used to model flow of porous media. Equation (1.2) is used for nerve propagation in neuro-physics and wall propagation in liquid crystals. For $\alpha = 1,\ n = 1$, (1.2) becomes the FitzHugh-Nagumo equation. The FitzHugh-Nagumo equation described the dynamical behavior near the bifurcation point for the Rayleigh-Bénard convection of binary fluid mixtures [29]. Wazwaz studied (1.1) and (1.2) analytically by tanh method [26], the extended tanh method [27], the tanh-coth method [28], respectively. He obtained some exact traveling wave solutions for some $n > 1$. By combining a transformation with the extended hyperbolic function method, with the aid of the computer symbolic computational software package "PDESolver," we not only obtain all known exact solitary wave solutions, periodic wave solutions, and singular traveling wave solutions but also find abundant new exact solitary wave solutions, singular traveling wave solutions, and periodic traveling wave solutions of triangle function.

The paper is organized as follows: in Section 2, we briefly describe what is the extended hyperbolic function method and how to use it to derive the traveling solutions of nonlinear PDEs. In Section 3 and Section 4, we apply the extended hyperbolic function method to generalized forms of nonlinear heat conduction and Huxley equations and establish many rational form solitary wave, rational-form triangular periodic wave solutions. In the last section, we briefly make a summary to the results that we have obtained.

2. The Extended Hyperbolic Function Method

We now would like to outline the main steps of our method.

Consider the coupled Riccati equations:

$$f'(\xi) = -f(\xi)g(\xi), \qquad g'(\xi) = \varepsilon - r\varepsilon f(\xi) - g^2(\xi), \tag{2.1}$$

where $\varepsilon = \pm 1$ or $0, r$ is a constant. We can obtain the first integrals as follows:

$$g^2(\xi) = \varepsilon - 2r\varepsilon f(\xi) + Cf^2(\xi). \tag{2.2}$$

Step 1. For a given nonlinear PDE, say, in two variables:

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.3}$$

we seek for the following formal traveling wave solutions which are of important physical significance:

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t + \xi_0, \tag{2.4}$$

where k and ω are constants to be determined later and ξ_0 is an arbitrary constant.

Then, the nonlinear PDE (2.3) reduces to a nonlinear ODE:

$$Q(u, u', u'', \ldots) = 0, \tag{2.5}$$

where $'$ denotes $d/d\xi$.

Step 2. To seek for the exact solutions of system (2.5), we assume that the solution of the system (2.5) is of the following form.

(a) When $\varepsilon = \pm 1$ in (2.1), (2.2),

$$u(\xi) = \sum_{i=0}^{m} a_i f^i(\xi) + \sum_{j=1}^{m} b_j f^{j-1}(\xi) g(\xi), \tag{2.6}$$

where the coefficients a_i $(i = 0, 1, 2, \ldots, m)$ and b_j $(j = 1, 2, \ldots, m)$ are constants to be determined.

(b) When $\varepsilon = 0$ in (2.1),

$$u(\xi) = u(x,t) = \sum_{i=0}^{m} a_i (g(\xi))^i, \tag{2.7}$$

where $g'(\xi) = -g^2(\xi)$ and the coefficients a_i, $i = 0, 1, 2, \ldots, m$ are constants to be determined.

Substituting (2.6) (or (2.7)) into the simplified ODE (2.5) and making use of (2.1)-(2.2) (or $g'(\xi) = -g^2(\xi)$) repeatedly and eliminating any derivative of (f,g) and any power of g higher than one yield an equation in powers of f^i $(i = 0, 1, \ldots)$ and $f^i g$ $(j = 1, 2, \ldots)$.

Step 3. To determine the balance parameter m, we usually balance the linear terms of the highest-order derivative term in the resulting equation with the highest-order nonlinear terms. m is a positive integer, in most cases.

Step 4. With m determined, we collect all coefficients of powers f^i $(i = 0, 1, 2, \ldots)$ and $f^j g$ $(j = 1, 2, \ldots)$, (or the coefficients of the different powers g), in the resulting equation where these coefficients have to vanish. This will give a set of overdetermined algebraic equations with respect to the unknown variables k, ω, a_i $(i = 0, 1, 2, \ldots, m)$, b_j $(j = 1, 2, \ldots, m), r, a, b$. With the aid of Mathematica, we apply Wu-eliminating method [30] to solve the above overdetermined system of nonlinear algebraic equations, yielding the values of k, ω, a_i $(i = 0, 1, 2 \ldots, m)$, b_j $(j = 1, 2, \ldots, m), r, a, b$.

Step 5. We know that the coupled Riccati equations (2.1) admits the following general solutions.

(a) When $\varepsilon = 1$,

$$f(\xi) = \frac{1}{a\cosh\xi + b\sinh\xi + r}, \qquad g(\xi) = \frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}, \tag{2.8}$$

and then $g^2(\xi) = 1 - 2rf(\xi) + (b^2 - a^2 + r^2)f^2(\xi)$.

(b) When $\varepsilon = -1$,

$$f(\xi) = \frac{1}{a\cos\xi + b\sin\xi + r}, \qquad g(\xi) = \frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r}, \tag{2.9}$$

and then $g^2(\xi) = -1 + 2rf(\xi) + (b^2 + a^2 - r^2)f^2(\xi)$.

(c) When $\varepsilon = 0$,

$$f(\xi) = \pm\frac{1}{\sqrt{C}(\xi + C_1)}, \qquad g(\xi) = \frac{1}{\xi + C_1}, \tag{2.10}$$

where $C,\ C_1$ are two constant.

Having determined these parameters, and using (2.5) (or (2.6)), we obtain an analytic solution $u(x, t)$ in closed form.

If m is not an integer, then an appropriate transformation formula should be used to overcome this difficulty. This will be introduced in the forthcoming two sections.

3. Generalized Forms of the Nonlinear Heat Conduction Equation

In this section, we will use the extended hyperbolic function method to handle the generalized forms of the nonlinear heat conduction equation (1.1).

Using the wave variable $\xi = kx + \omega t + \xi_0$ carries (1.1) to

$$\omega u' - \alpha k^2 (u^n)'' - u + u^n = 0, \tag{3.1}$$

or, equivalently,

$$\omega u' - \alpha n(n-1)k^2 u^{n-2}(u')^2 - \alpha n k^2 u^{n-1}u'' - u + u^n = 0. \tag{3.2}$$

Balancing $u^{n-1}u''$ (or $u^{n-2}(u')^2$) with u' gives

$$(n-1)m + m + 2 = m + 1, \tag{3.3}$$

so that

$$m = -\frac{1}{n-1}. \tag{3.4}$$

To obtain a closed-form solution, m should be an integer. Therefore, we use the transformation

$$u(x,t) = v^{-1/(n-1)}(x,t), \tag{3.5}$$

and as a result (3.2) becomes

$$(1-n)\omega v^2 v' + k^2\alpha n(1-2n)(v')^2 + k^2 n(n-1)\alpha v v'' + (n-1)^2\left(v^2 - v^3\right) = 0. \tag{3.6}$$

Balancing vv'' with v^2v' gives

$$m + m + 2 = 2m + m + 1, \tag{3.7}$$

so that

$$m = 1, \tag{3.8}$$

Consequently, the extended hyperbolic function method allows us to set the following.

(1) In the case of $\varepsilon = \pm 1$,

$$v(\xi) = c + df(\xi) + eg(\xi). \tag{3.9}$$

(2) In the case of $\varepsilon = 0$,

$$v(\xi) = c + dg(\xi), \tag{3.10}$$

where $\xi = kx + \omega t + \xi_0$ and $c, d, e, k, \omega, \xi_0$ are constants to be determined.

Substituting (3.9) (or (3.10), resp.) into (3.6) and collecting the coefficients of f^i and $f^i g$ (or g^i, resp.) give the system of algebraic equations for k, ω, c, d, e. Solving the resulting system, we find the following nine sets of solutions.

(a) In the case of $\varepsilon = 1$, there are six sets of solutions:

(1)

$$k = \frac{n-1}{\sqrt{\alpha n}}, \quad r = r, \quad \omega = \frac{n-1}{n}, \quad c = \frac{1}{2}, \quad d = \frac{1}{2}, \quad e = \frac{1}{2}, \tag{3.11}$$

(2)

$$k = \frac{n-1}{\sqrt{\alpha n}}, \quad r = r, \quad \omega = \frac{n-1}{n}, \quad c = \frac{1}{2}, \quad d = \frac{-1}{2}, \quad e = \frac{1}{2}, \tag{3.12}$$

(3)

$$k = \frac{n-1}{\sqrt{\alpha n}}, \quad r = r, \quad \omega = \frac{-n+1}{n}, \quad c = \frac{1}{2}, \quad d = \frac{1}{2}, \quad e = \frac{-1}{2}, \tag{3.13}$$

(4)

$$k = \frac{n-1}{\sqrt{\alpha n}}, \quad r = r, \quad \omega = \frac{-n+1}{n}, \quad c = \frac{1}{2}, \quad d = \frac{-1}{2}, \quad e = \frac{-1}{2}, \tag{3.14}$$

(5)

$$k = \frac{n-1}{2\sqrt{\alpha n}}, \quad r = 0, \quad \omega = \frac{n-1}{2n}, \quad c = \frac{1}{2}, \quad d = 0, \quad e = \frac{1}{2}, \tag{3.15}$$

(6)

$$k = \frac{n-1}{2\sqrt{\alpha n}}, \quad r = 0, \quad \omega = -\frac{n-1}{2n}, \quad c = \frac{1}{2}, \quad d = 0, \quad e = \frac{-1}{2}. \tag{3.16}$$

(b) In the case of $\varepsilon = -1$, there are three sets of solutions:

(7)

$$k = \frac{\sqrt{-\alpha}(n-1)}{\alpha n}, \quad r = r, \quad \omega = \frac{(n-1)i}{n}, \quad c = \frac{1}{2}, \quad d = \frac{1}{2}i, \quad e = \frac{1}{2}i, \tag{3.17}$$

(8)

$$k = \frac{\sqrt{-\alpha}(n-1)}{\alpha n}, \quad r = r, \quad \omega = \frac{(n-1)i}{n}, \quad c = \frac{1}{2}, \quad d = \frac{-1}{2}i, \quad e = \frac{1}{2}i, \tag{3.18}$$

(9)

$$k=\frac{\sqrt{-\alpha}(n-1)}{2\alpha n},\qquad r=0,\qquad \omega=\frac{(n-1)}{2n}i,\qquad c=\frac{1}{2},\qquad d=0,\qquad e=\frac{1}{2}i. \tag{3.19}$$

(c) In the case of $\varepsilon=0$, there is no solution.

Recall that $u=v^{-1/(n-1)}$ and using (2.8), (3.9), (3.11)–(3.16), we obtain six sets of traveling wave solutions:

$$u_1(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)+r}{(a+b)(\cosh(\xi)+\sinh(\xi))+r+1}}, \tag{3.20}$$

where $\xi:=((n-1)/\sqrt{\alpha}n)x+((n-1)/n)t+\xi_0$;

$$u_2(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)+r}{(a+b)(\cosh(\xi)+\sinh(\xi))+r-1}}, \tag{3.21}$$

where $\xi:=((n-1)/\sqrt{\alpha}n)x+((n-1)/n)t+\xi_0$;

$$u_3(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)+r}{(a-b)(\cosh(\xi)-\sinh(\xi))+r+1}}, \tag{3.22}$$

where $\xi:=((n-1)/\sqrt{\alpha}n)x+((-n+1)/n)t+\xi_0$;

$$u_4(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)+r}{(a-b)(\cosh(\xi)-\sinh(\xi))+r-1}}, \tag{3.23}$$

where $\xi:=((n-1)/\sqrt{\alpha}n)x+((-n+1)/n)t+\xi_0$;

$$u_5(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)}{(a+b)(\cosh(\xi)+\sinh(\xi))}}, \tag{3.24}$$

where $\xi:=((n-1)/\sqrt{\alpha}n)x+((n-1)/2n)t+\xi_0$;

$$u_6(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+b\sinh(\xi)}{(a-b)(\cosh(\xi)-\sinh(\xi))}}, \tag{3.25}$$

where $\xi:=((n-1)/2\sqrt{\alpha}n)x-((n-1)/2n)t+\xi_0$.

Noting that $u=v^{-1/(n-1)}$ and using (2.9), (3.9), (3.17)–(3.19), we find three sets of complex solutions:

$$u_7(x,t)=\sqrt[n-1]{2\frac{a\cosh(\xi)+bi\sinh(\xi)+r}{(a+bi)(\cosh(\xi)+\sinh(\xi))+r+i}}, \tag{3.26}$$

where $\xi := (\sqrt{\alpha}(n-1)/\alpha n)x + ((n-1)/n)t + \xi_0$;

$$u_8(x,t) = \sqrt[n-1]{2\frac{a\cosh(\xi) + bi\sinh(\xi) + r}{(a+bi)(\cosh(\xi)+\sinh(\xi)) + r - i}}, \tag{3.27}$$

where $\xi := (\sqrt{\alpha}(n-1)/\alpha n)x + ((n-1)/n)t + \xi_0$;

$$u_9(x,t) = \sqrt[n-1]{2\frac{a\cosh(\xi) + bi\sinh(\xi)}{(a+bi)(\cosh(\xi)+\sinh(\xi))}}, \tag{3.28}$$

where $\xi := (\sqrt{\alpha}(n-1)/2\alpha n)x + ((n-1)/2n)t + \xi_0$.

Remark 3.1. Setting $a = 1,\ b = 0,\ \xi_0 = 0$ (or $a = 0,\ b = 1,\ \xi_0 = 0$, resp.) in solution (3.24), we obtain solutions (73), (or (74), resp.) of [26]. Setting $a = 1,\ b = 0, \xi_0 = 0$ (or $a = 0,\ b = 1,\ \xi_0 = 0$, resp.) in solution (3.25), we obtain solutions (71), (or (72), resp.) of [26]. Furthermore, in the case of $n = 2, \alpha = 1$, we obtain solutions (100) and (101) of [27]. In the case of $n = 3$, we obtain solutions (54)–(57) of [26] and solutions (111)-(112) of [27] again. So the known solutions of (1.1) obtained in previous works are some special cases of solutions (3.24), (3.25) presented in the paper. All other solutions obtained here are entirely new solutions first reported.

4. The Huxley Equation

In this section, we employ the extended hyperbolic function method to investigate the Huxley equation (1.2).

The Huxley equation (1.2) can be converted to

$$\omega u' - \alpha k^2 u'' - (\beta+1)u^{n+1} + u^{2n+1} + \beta u = 0, \tag{4.1}$$

obtained upon using the wave variable $\xi = kx + \omega t + \xi_0$.

Balancing the term u'' with u^{2n+1}, we find

$$m = \frac{1}{n}. \tag{4.2}$$

To obtain a closed-form solution, we use the transformation:

$$u(x,t) = v^{1/n}(x,t), \tag{4.3}$$

which will carry out (4.1) into the ODE

$$n\omega v v' + \alpha k^2(n-1)(v')^2 - \alpha k^2 n v v'' + n^2 v^2 (v-1)(v-\beta) = 0. \tag{4.4}$$

Balancing vv'' with v^4 gives $m = 1$. Using the extended hyperbolic function method, we set

$$v(\xi) = c + df(\xi) + eg(\xi) \tag{4.5}$$

in the case of $\varepsilon = \pm 1$, and

$$v(\xi) = c + dg(\xi), \tag{4.6}$$

in the case of $\varepsilon = 0$, where $\xi = kx + \omega t + \xi_0$, and $c, d, e, k, \omega, \xi_0$ are constants to be determined.

Substituting (4.5) (or (4.6), resp.) into (4.4), and proceeding as before, we obtain the twenty sets of solutions.

(a) In the case of $\varepsilon = 1$, there are thirteen sets of solutions:

(1)

$$k = \frac{\sqrt{\alpha(1+n)}n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{(n\beta+\beta-1)n}{1+n}, \qquad c = \frac{1}{2}, \qquad d = \frac{1}{2}, \qquad e = \frac{-1}{2}, \tag{4.7}$$

(2)

$$k = \frac{\sqrt{\alpha(1+n)}n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{(n\beta+\beta-1)n}{1+n}, \qquad c = \frac{1}{2}, \qquad d = \frac{-1}{2}, \qquad e = \frac{-1}{2}, \tag{4.8}$$

(3)

$$k = \frac{\sqrt{\alpha(1+n)}n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{(-n\beta-\beta+1)n}{1+n}, \qquad c = \frac{1}{2}, \qquad d = \frac{1}{2}, \qquad e = \frac{1}{2}, \tag{4.9}$$

(4)

$$k = \frac{\sqrt{\alpha(1+n)}n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{(-n\beta-\beta+1)n}{1+n}, \qquad c = \frac{1}{2}, \qquad d = \frac{-1}{2}, \qquad e = \frac{1}{2}, \tag{4.10}$$

(5)

$$k = \frac{\sqrt{\alpha(1+n)}\beta n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{n\beta(n+1-\beta)}{1+n}, \qquad c = \frac{\beta}{2}, \qquad d = \frac{\beta}{2}, \qquad e = -\frac{\beta}{2}, \tag{4.11}$$

(6)

$$k = \frac{\sqrt{\alpha(1+n)}\beta n}{\alpha(1+n)}, \qquad r = r, \qquad \omega = \frac{n\beta(n+1-\beta)}{1+n}, \qquad c = \frac{\beta}{2}, \qquad d = -\frac{\beta}{2}, \qquad e = -\frac{\beta}{2}, \tag{4.12}$$

(7)

$$k=\frac{\sqrt{\alpha(1+n)}\beta n}{\alpha(1+n)},\qquad r=r,\qquad \omega=\frac{(-n-1+\beta)n\beta}{1+n},\qquad c=\frac{\beta}{2},\qquad d=\frac{\beta}{2},\qquad e=\frac{\beta}{2}, \tag{4.13}$$

(8)

$$k=\frac{\sqrt{\alpha(1+n)}\beta n}{\alpha(1+n)},\qquad r=r,\qquad \omega=\frac{(-n-1+\beta)n\beta}{1+n},\qquad c=\frac{\beta}{2},\qquad d=-\frac{\beta}{2},\qquad e=\frac{\beta}{2}, \tag{4.14}$$

(9)

$$k=\frac{\sqrt{\beta\alpha}n}{\alpha},\qquad r=\frac{n\beta+n+\beta+1}{\sqrt{\beta(1+n)}(n+2)},\qquad \omega=0,\qquad c=0,\qquad d=\sqrt{\beta(1+n)},\qquad e=0, \tag{4.15}$$

(10)

$$k=\frac{\sqrt{\alpha(1+n)}n}{2\alpha(1+n)},\qquad r=0,\qquad \omega=\frac{(n\beta+\beta-1)n}{2(1+n)},\qquad c=\frac{1}{2},\qquad d=0,\qquad e=\frac{-1}{2}, \tag{4.16}$$

(11)

$$k=\frac{\sqrt{\alpha(1+n)}n}{2\alpha(1+n)},\qquad r=0,\qquad \omega=-\frac{(n\beta+\beta-1)n}{2(1+n)},\qquad c=\frac{1}{2},\qquad d=0,\qquad e=\frac{1}{2}, \tag{4.17}$$

(12)

$$k=\frac{\sqrt{\alpha(1+n)}\beta n}{2\alpha(1+n)},\qquad r=0,\qquad \omega=\frac{n\beta(n+1-\beta)}{2(1+n)},\qquad c=\frac{\beta}{2},\qquad d=0,\qquad e=-\frac{\beta}{2}, \tag{4.18}$$

(13)

$$k=\frac{\sqrt{\alpha(1+n)}\beta n}{2\alpha(1+n)},\qquad r=0,\qquad \omega=-\frac{n\beta(n+1-\beta)}{2(1+n)},\qquad c=\frac{\beta}{2},\qquad d=0,\qquad e=\frac{\beta}{2}. \tag{4.19}$$

(b) In the case of $\varepsilon = -1$, there are seven sets of solutions:

(14)

$$k = \frac{\sqrt{-\alpha(n+1)n}}{\alpha(n+1)}, \qquad r = r, \qquad \omega = \frac{-(n\beta - 1 + \beta)n}{n+1}i, \qquad c = \frac{1}{2}, \qquad d = \frac{1}{2}i, \qquad e = \frac{1}{2}i, \tag{4.20}$$

(15)

$$k = \frac{\sqrt{-\alpha(n+1)n}}{\alpha(n+1)}, \qquad r = r, \qquad \omega = \frac{-(n\beta - 1 + \beta)n}{n+1}i, \qquad c = \frac{1}{2}, \qquad d = \frac{-1}{2}i, \qquad e = \frac{1}{2}i, \tag{4.21}$$

(16)

$$k = \frac{\sqrt{-\alpha(n+1)\beta n}}{\alpha(n+1)}, \qquad r = r, \qquad \omega = \frac{-n\beta(n+1-\beta)}{n+1}i, \qquad c = \frac{\beta}{2}, \qquad d = \frac{1}{2}\beta i, \qquad e = \frac{1}{2}\beta i, \tag{4.22}$$

(17)

$$k = \frac{\sqrt{-\alpha(n+1)\beta n}}{\alpha(n+1)}, \qquad r = r, \qquad \omega = \frac{-n\beta(n+1-\beta)}{n+1}i, \qquad c = \frac{\beta}{2}, \qquad d = \frac{-1}{2}\beta i, \qquad e = \frac{1}{2}\beta i, \tag{4.23}$$

(18)

$$k = \frac{\sqrt{-\beta\alpha}n}{\alpha}, \qquad r = \frac{-1-\beta-n-n\beta}{\sqrt{-\beta(n+1)}(2+n)}, \qquad \omega = 0, \qquad c = 0, \qquad d = \sqrt{-\beta(n+1)}, \qquad e = 0, \tag{4.24}$$

(19)

$$k = \frac{\sqrt{-\alpha(n+1)n}}{2\alpha(n+1)}, \qquad r = 0, \qquad \omega = \frac{-(n\beta - 1 + \beta)n}{2(n+1)}i, \qquad c = \frac{1}{2}, \qquad d = 0, \qquad e = \frac{1}{2}i, \tag{4.25}$$

(20)

$$k = \frac{\sqrt{-\alpha(n+1)\beta n}}{2\alpha(n+1)}, \qquad r = 0, \qquad \omega = \frac{-n\beta(n+1-\beta)}{2(n+1)}i, \qquad c = \frac{\beta}{2}, \qquad d = 0, \qquad e = \frac{1}{2}\beta i. \tag{4.26}$$

(c) In the case of $\varepsilon = 0$, there is no solution.

Owing to $u(x,t) = v^{1/n}(x,t)$, we obtain the following thirteen sets of solutions from (4.5), (4.7)–(4.19):

$$u_1(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a-b)(\cosh(\xi)-\sinh(\xi))+r+1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.27}$$

where $\xi := (\sqrt{\alpha(1+n)}n/\alpha(1+n))x + ((n\beta+\beta-1)n/(1+n))t + \xi_0$;

$$u_2(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a-b)(\cosh(\xi)-\sinh(\xi))+r-1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.28}$$

where $\xi := (\sqrt{\alpha(1+n)}n/\alpha(1+n))x + ((n\beta+\beta-1)n/(1+n))t + \xi_0$;

$$u_3(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+b)(\cosh(\xi)+\sinh(\xi))+r+1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.29}$$

where $\xi := (\sqrt{\alpha(1+n)}n/\alpha(1+n))x + ((-n\beta-\beta+1)n/(1+n))t + \xi_0$;

$$u_4(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+b)(\cosh(\xi)+\sinh(\xi))+r-1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.30}$$

where $\xi := (\sqrt{\alpha(1+n)}n/\alpha(1+n))x + ((-n\beta-\beta+1)n/(1+n))t + \xi_0$;

$$u_5(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a-b)(\cosh(\xi)-\sinh(\xi))+r+1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.31}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/\alpha(1+n))x + (n\beta(n+1-\beta)/(1+n))t + \xi_0$;

$$u_6(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a-b)(\cosh(\xi)-\sinh(\xi))+r-1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.32}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/\alpha(1+n))x + (n\beta(n+1-\beta)/(1+n))t + \xi_0$;

$$u_7(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a+b)(\cosh(\xi)+\sinh(\xi))+r+1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.33}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/\alpha(1+n))x + ((-n-1+\beta)n\beta/(1+n))t + \xi_0$;

$$u_8(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a+b)(\cosh(\xi)+\sinh(\xi))+r-1}{a\cosh(\xi)+b\sinh(\xi)+r}}, \tag{4.34}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/\alpha(1+n))x + ((-n-1+\beta)n\beta/(1+n))t + \xi_0$;

$$u_9(x,t) = \sqrt[n]{\frac{\sqrt{\beta(n+1)}}{(a\cosh(\xi) + b\sinh(\xi)) + (1+n)(1+\beta)/\left(\sqrt{\beta(n+1)}(n+2)\right)}}, \tag{4.35}$$

where $\xi := (\sqrt{\beta\alpha}n/\alpha)x + \xi_0$;

$$u_{10}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a-b)(\cosh(\xi) - \sinh(\xi))}{a\cosh(\xi) + b\sinh(\xi)}}, \tag{4.36}$$

where $\xi := (\sqrt{\alpha(1+n)}n/2\alpha(1+n))x + ((n\beta+\beta-1)n/2(1+n))t + \xi_0$;

$$u_{11}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+b)(\cosh(\xi) + \sinh(\xi))}{a\cosh(\xi) + b\sinh(\xi)}}, \tag{4.37}$$

where $\xi := (\sqrt{\alpha(1+n)}n/2\alpha(1+n))x - ((n\beta+\beta-1)n/2(1+n))t + \xi_0$;

$$u_{12}(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a-b)(\cosh(\xi) - \sinh(\xi))}{\mathrm{a}\cosh(\xi) + b sinh(\xi)}}, \tag{4.38}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/2\alpha(1+n))x + (n\beta(n+1-\beta)/2(1+n))t + \xi_0$;

$$u_{13}(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a+b)(\cosh(\xi) + \sinh(\xi))}{a\cosh(\xi) + b sinh(\xi)}}, \tag{4.39}$$

where $\xi := (\sqrt{\alpha(1+n)}\beta n/2\alpha(1+n))x - (n\beta(n+1-\beta)/2(1+n))t + \xi_0$.

Combining (4.3), (4.5) with (4.20)–(4.26), we find the seven sets of complex solutions

$$u_{14}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+bi)(\cosh(\xi) + \sinh(\xi)) + r + i}{a\cosh(\xi) + bi\sinh(\xi) + r}}, \tag{4.40}$$

where $\xi := (\sqrt{\alpha(\mathrm{n}+1)}n/\alpha(n+1))x - ((n\beta-1+\beta)n/(n+1))t + \xi_0$;

$$u_{15}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+bi)(\cosh(\xi) + \sinh(\xi)) + r - i}{a\cosh(\xi) + bi\ \sinh(\xi) + r}}, \tag{4.41}$$

where $\xi := (\sqrt{\alpha(n+1)}n/\alpha(n+1))x - ((n\beta-1+\beta)n/(n+1))t + \xi_0$;

$$u_{16}(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a+bi)(\cosh(\xi) + \sinh(\xi)) + r + i}{a\cosh(\xi) + bi\sinh(\xi) + r}}, \tag{4.42}$$

where $\xi := (\sqrt{\alpha(n+1)}\beta n/\alpha(n+1))x - (n\beta(n+1-\beta)/(n+1))t + \xi_0$;

$$u_{17}(x,t) = \sqrt[n]{\frac{\beta}{2}\frac{(a+bi)(\cosh(\xi)+\sinh(\xi)) + r - i}{a\cosh(\xi) + bi\sinh(\xi) + r}}, \tag{4.43}$$

where $\xi = (\sqrt{\alpha(n+1)}\beta n/\alpha(n+1))x - (n\beta(n+1-\beta)/(n+1))t + \xi_0$;

$$u_{18}(x,t) = \sqrt[n]{\frac{\sqrt{\beta(n+1)}i}{(a\cosh(\xi) + bi\sinh(\xi)) + (1+n)(1+\beta)i/\left(\sqrt{-\beta(n+1)}(n+2)\right)}}, \tag{4.44}$$

where $\xi = (\sqrt{\beta\alpha}n/\alpha)x + \xi_0$;

$$u_{19}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+bi)(\cosh(\xi)+\sinh(\xi))}{a\cosh(\xi) + bi\sinh(\xi)}}, \tag{4.45}$$

where $\xi = (\sqrt{\alpha(n+1)}n/2\alpha(n+1))x - ((n\beta-1+\beta)n/2(n+1))t + \xi_0$;

$$u_{20}(x,t) = \sqrt[n]{\frac{1}{2}\frac{(a+bi)(\cosh(\xi)+\sin(\xi))}{a\cosh(\xi) + bi\sinh(\xi)}}, \tag{4.46}$$

where $\xi = (\sqrt{\alpha(n+1)}n/2\alpha(n+1))x - ((n\beta-1+\beta)n/2(n+1))t + \xi_0$;

Remark 4.1. Wazwaz obtained six sets of solutions of (1.2) in [27]. It is worth pointing out that the solutions (85) and (88) of [27] are not new solutions. We can reduce the solution (85) (and (88)) of [27] to the solutions (84) (and (87)) of [27] by using the formulae $\tanh x + \coth x = 2\coth 2x$. There is a mistake in the solution (87) of [27], that is, the first constant factor $1/2$ should be $k/2$. For $\alpha = 1$, $n = 1$, Wazwaz finds nine sets of solutions of (1.2) in [28]. The solutions (61)–(63) of [28] are also not new solutions. The solution (61) (and (62), (63), resp.) of [28] can be reduced to the solution (58) (and (59), (60), resp.) of [28] by using the formulae $\tanh x + \coth x = 2\coth 2x$. Therefore, Wazwaz actually finds six sets of solutions of (1.2).

Remark 4.2. Letting $a = 1$, $b = 0$, $\xi_0 = 0$ (or $a = 0$, $b = 1$, $\xi_0 = 0$, resp.) in (4.36), (4.37), we obtain the solutions (83) (or (84), resp.) of [27]. Setting $a = 1$, $b = 0$, $\xi_0 = 0$ (or $a = 0$, $b = 1$, $\xi_0 = 0$, resp.) in (4.38), (4.39), we obtain the solutions (86) (or (87), resp.) of [27]. Furthermore, as $\alpha = 1$, $n = 1$, we obtain the solutions (55), (58) of [28] from the solution (4.37) and the solutions (56), (59) of [28] from the solution (4.39). Therefore, the known solutions of (1.2) in previous works are some special cases of the solutions obtained in this paper. All other solutions are entirely new solutions reported in the present paper.

Remark 4.3. The solutions (4.35) and (4.44) are two static solutions of (1.2). All other solutions are traveling wave solutions.

5. Conclusions

In this paper, the extended hyperbolic function method is used to establish abundant traveling wave solutions, mostly kinks solutions. The balance parameter m plays a major role in the extended hyperbolic function method in that it should be a positive integer to derive a closed-form analytic solution. If m is not a positive integer, then an appropriate transformation should be used to overcome this difficulty. The extended hyperbolic function method is employed to develop many entirely solutions for generalized forms of nonlinear heat conduction and Huxley equations in addition to the solutions that exist in the previous works. Our method can also be regarded as an extension of the recent works by Wazwaz [24–28]. The results of [26–28] are supplemented and extended greatly.

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