Research Article

Some Fixed Point Theorems for Nonlinear Set-Valued Contractive Mappings

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Four fixed point theorems for nonlinear set-valued contractive mappings in complete metric spaces are proved. The results presented in this paper are extensions of a few well-known fixed point theorems. Two examples are also provided to illustrate our results.

1. Introduction and Preliminaries

The existence of fixed points for various set-valued contractive mappings had been researched by many authors under different conditions, see, for example, [1–9] and the references cited therein. In 1969, Nadler [7] proved a well-known fixed point theorem for the set-valued contraction mapping (1.1) below.

Theorem 1.1 (see [7]). *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a set-valued mapping such that*

$$H(Tx, Ty) \leq rd(x, y), \quad \forall x, y \in X, \quad (1.1)$$

where $r \in (0, 1)$ is a constant. Then T has a fixed point.

In 1972, Reich [8] extended Nadler's result and established an interesting fixed point theorem for the set-valued contraction mapping (1.2) below.

Theorem 1.2 (see [8]). *Let (X, d) be a complete metric space and $T : X \rightarrow C(X)$ satisfy that*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \quad \forall x, y \in X, \quad (1.2)$$

where

$$\varphi : (0, +\infty) \longrightarrow [0, 1) \text{ with } \limsup_{r \rightarrow t^+} \varphi(r) < 1, \quad \forall t \in (0, +\infty). \quad (1.3)$$

Then T has a fixed point.

In [8] Reich posed the question whether Theorem 1.2 is also true for the set-valued contractive mapping $T : X \rightarrow CB(X)$ with (1.2). The affirmative answer under the hypothesis of $\limsup_{r \rightarrow t^+} \varphi(r) < 1$, for all $t \in [0, +\infty)$ was given by Mizoguchi and Takahashi in [6]. They deduced the following fixed point theorem which is a generalization of the Nadler fixed point theorem.

Theorem 1.3 (see [6]). *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ satisfy (1.2), where*

$$\varphi : (0, +\infty) \longrightarrow [0, 1) \text{ with } \limsup_{r \rightarrow t^+} \varphi(r) < 1, \quad \forall t \in [0, +\infty). \quad (1.4)$$

Then T has a fixed point.

Remark 1.4. It is clear that the mappings T in Theorems 1.1–1.3 are continuous on X .

Remark 1.5. Each of Theorems 1.2 and 1.3 ensures that T has a fixed point $a \in Ta \subseteq X$, which together with (1.2) implies that $\varphi(0) = \varphi(d(a, a))$, that is, φ is defined at 0. Thus the domain of φ in each of (1.3) and (1.4) should be $[0, +\infty)$ but not $(0, +\infty)$.

The aim of this paper is to present four fixed point theorems for some nonlinear set-valued contractive mappings. Our results extend, improve, and unify the corresponding results in [6–8]. Two nontrivial examples are given to show that our results are genuine generalizations or different from these results in [6–8].

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of all positive integers and nonnegative integers, respectively, and

$$\Theta = \{\theta : \theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ satisfies (a)–(d)}\}, \quad (1.5)$$

where

- (a) θ is nondecreasing on $\mathbb{R}^+$;
- (b) $\theta(t) > 0$, for all $t \in (0, +\infty)$;
- (c) θ is subadditive in $(0, +\infty)$, that is,

$$\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2), \quad \forall t_1, t_2 \in (0, +\infty); \quad (1.6)$$

- (d) $\theta(\mathbb{R}^+) = \mathbb{R}^+$.

Clearly (a)–(d) imply that

- (e) θ is strictly inverse on $\mathbb{R}^+$, that is, if there exist $t, s \in \mathbb{R}^+$ satisfying $\theta(t) < \theta(s)$, then $t < s$.

Let (X, d) be a metric space, $CL(X)$, $CB(X)$, and $C(X)$ denote the families of all non-empty closed, all nonempty bounded closed, and all nonempty compact subsets of X . For $x \in X$ and $A, B \in CL(X)$, put $d(x, A) = \inf\{d(x, y) : y \in A\}$ and

$$H(A, B) = \begin{cases} \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, & \text{if the maximum exists} \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.7)$$

Such a mapping H is called a *generalized Hausdorff metric induced by d* in $CL(X)$. It is well known that H is a metric on $CB(X)$. Let $T : X \rightarrow CL(X)$ be a set-valued mapping, $x_0 \in X$ and $f : X \rightarrow \mathbb{R}^+$ be defined by

$$f(x) = d(x, Tx), \quad \forall x \in X. \quad (1.8)$$

A sequence $\{x_n\}_{n \in \mathbb{N}_0}$ is said to be an *orbit* of T if it satisfies that $\{x_n\}_{n \in \mathbb{N}_0} \subset X$ and $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}_0$. The function $f : X \rightarrow \mathbb{R}^+$ is said to be *T -orbitally lower semicontinuous at $z \in X$* if for each orbit $\{x_n\}_{n \in \mathbb{N}_0} \subset X$ of T with $\lim_{n \rightarrow \infty} x_n = z$, we have that $f(z) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

2. Main Results

The following lemmas play important roles in this paper.

Lemma 2.1. *Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ and $\varepsilon > 0$ there exists $b \in B$ satisfying $d(x, b) \leq d(x, B) + \varepsilon$.*

Proof. Suppose that there exist $x_0 \in X$ and $\varepsilon_0 > 0$ such that

$$d(x_0, b) > d(x_0, B) + \varepsilon_0, \quad \forall b \in B, \quad (2.1)$$

which yields that

$$d(x_0, B) = \inf_{b \in B} d(x_0, b) \geq d(x_0, B) + \varepsilon_0 > d(x_0, B), \quad (2.2)$$

which is a contradiction. This completes the proof. $\square$

Lemma 2.2. *Let (X, d) be a metric space, $B \in CL(X)$ and $\theta \in \Theta$. Then for each $x \in X$ and $q > 1$ there exists $b \in B$ such that*

$$\theta(d(x, b)) \leq q\theta(d(x, B)). \quad (2.3)$$

Proof. Let $x \in X$ and $q > 1$. Now we consider two possible cases as follows.

Case 1. Suppose that $\theta(d(x, B)) = 0$. It follows from (b) and (d) that $d(x, B) = 0$. Since B is a closed subset of X , it follows that $x \in B$. Put $b = x$. Clearly (2.3) holds.

Case 2. Suppose that $\theta(d(x, B)) > 0$. Note that (b) and (d) mean that

$$(q-1)\theta(d(x, B)) \in \mathbb{R}^+ \setminus \{0\} = \theta(\mathbb{R}^+ \setminus \{0\}). \quad (2.4)$$

Choose $p \in \theta^{-1}((q-1)\theta(d(x, B)))$ and $\varepsilon = p/2 > 0$. Lemma 2.1 ensures that there exists $b \in B$ satisfying $d(x, b) \leq d(x, B) + \varepsilon$, which together with (a) and (c) gives that

$$\begin{aligned} \theta(d(x, b)) &\leq \theta(d(x, B) + \varepsilon) \leq \theta(d(x, B)) + \theta(\varepsilon) \\ &\leq \theta(d(x, B)) + \theta\left(\theta^{-1}((q-1)\theta(d(x, B)))\right) = q\theta(d(x, B)). \end{aligned} \quad (2.5)$$

That is, (2.3) holds. This completes the proof. $\square$

Now we prove four fixed point theorems for the nonlinear set-valued contractive mappings (2.6), (2.25), (2.26), and (2.36) below in complete metric spaces.

Theorem 2.3. Let (X, d) be a complete metric space and $T : X \rightarrow CL(X)$ satisfy that

$$\theta(d(y, Ty)) \leq \varphi(d(x, y))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \quad (2.6)$$

where $\theta \in \Theta$ and

$$\varphi : \mathbb{R}^+ \longrightarrow [0, 1) \text{ with } \limsup_{r \rightarrow t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+. \quad (2.7)$$

Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n \in \mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) is T orbitally lower semicontinuous at z .

Proof. Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.6), (2.7) and Lemma 2.2 that for $q_1 = 1/\max\{\sqrt{\varphi(d(x_0, x_1))}, 1/2\} > 1$ there exists $x_2 \in Tx_1$ satisfying

$$\begin{aligned} \theta(d(x_1, x_2)) &\leq \frac{\theta(d(x_1, Tx_1))}{\max\{\sqrt{\varphi(d(x_0, x_1))}, 1/2\}} \leq \frac{\varphi(d(x_0, x_1))\theta(d(x_0, x_1))}{\max\{\sqrt{\varphi(d(x_0, x_1))}, 1/2\}} \\ &\leq \sqrt{\varphi(d(x_0, x_1))}\theta(d(x_0, x_1)), \end{aligned} \quad (2.8)$$

and for $q_2 = 1/\max\{\sqrt{\varphi(d(x_1, x_2))}, 1/3\} > 1$ there exists $x_3 \in Tx_2$ satisfying

$$\begin{aligned} \theta(d(x_2, x_3)) &\leq \frac{\theta(d(x_2, Tx_2))}{\max\{\sqrt{\varphi(d(x_1, x_2))}, 1/3\}} \leq \frac{\varphi(d(x_1, x_2))\theta(d(x_1, x_2))}{\max\{\sqrt{\varphi(d(x_1, x_2))}, 1/3\}} \\ &\leq \sqrt{\varphi(d(x_1, x_2))}\theta(d(x_1, x_2)). \end{aligned} \quad (2.9)$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n \in \mathbb{N}_0} \subset X$ such that $x_k \in Tx_{k-1}$ for $1 \leq k \leq n$ and for $q_n = 1/\max\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\} > 1$, there exists $x_{n+1} \in Tx_n$ satisfying

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \frac{\theta(d(x_n, Tx_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\}} \\ &\leq \frac{\varphi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\}} \\ &\leq \sqrt{\varphi(d(x_{n-1}, x_n))}\theta(d(x_{n-1}, x_n)), \quad \forall n \geq 1. \end{aligned} \quad (2.10)$$

Suppose that there exists some $n_0 \in \mathbb{N}_0$ satisfying $x_{n_0} = x_{n_0+1} \in Tx_{n_0}$. It follows from (a), (b), and (2.10) that $x_n = x_{n_0}$ for all $n \geq n_0 + 1$. It is clear the conclusion of Theorem 2.3 holds.

Suppose that $x_{n+1} \in Tx_n \setminus \{x_n\}$ for any $n \in \mathbb{N}_0$. It follows that $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}_0$. Note that (b), (2.7), and (2.10) give that $\{\theta(d(x_n, x_{n+1}))\}_{n \in \mathbb{N}_0}$ is a positive and decreasing sequence. It follows from (e) that $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}$ is decreasing. Therefore, there exist constants p and q satisfying

$$\lim_{n \rightarrow \infty} \theta(d(x_n, x_{n+1})) = p \geq 0, \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = q \geq 0. \quad (2.11)$$

Notice that (2.7) implies that there exists a constant r satisfying

$$\limsup_{n \rightarrow \infty} \varphi(d(x_n, x_{n+1})) \leq \limsup_{t \rightarrow q^+} \varphi(t) = r \in [0, 1). \quad (2.12)$$

Taking upper limits in (2.10) and by (2.11) and (2.12) we get that

$$p \leq \sqrt{\limsup_{n \rightarrow \infty} \varphi(d(x_{n-1}, x_n))} \limsup_{n \rightarrow \infty} \theta(d(x_{n-1}, x_n)) \leq \sqrt{r}p, \quad (2.13)$$

which implies that $p = 0$.

Next we assert that $q = 0$. Since $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}$ is a decreasing sequence, it follows from (a) and (2.11) that

$$0 \leq \theta(q) < \theta(d(x_n, x_{n+1})) \longrightarrow p = 0 \quad \text{as } n \longrightarrow \infty, \quad (2.14)$$

that is, $\theta(q) = 0$, which together with (b) and (d) yields that $q = 0$.

Put $c = (1 + r)/2$. It follows from (2.12) that $c \in (r, 1) \subset [0, 1)$, which gives that $c^2 \in (r, 1)$. Notice that (2.11), (2.12), and $q = 0$ ensure that there exist $\delta > 0$ and $N \in \mathbb{N}$ satisfying

$$\varphi(t) < c^2, \quad \forall t \in (0, \delta), \quad d(x_n, x_{n+1}) < \delta, \quad \forall n \geq N, \quad (2.15)$$

which implies that

$$\varphi(d(x_n, x_{n+1})) < c^2, \quad \forall n \geq N. \quad (2.16)$$

Note that (2.10) and (2.16) mean that

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \prod_{k=N}^{n-1} \sqrt{\varphi(d(x_k, x_{k+1}))} \theta(d(x_N, x_{N+1})) \\ &\leq c^{n-N} \theta(d(x_N, x_{N+1})), \quad \forall n \geq N. \end{aligned} \quad (2.17)$$

Given $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} c^{n-N} \theta(d(x_N, x_{N+1})) = 0$, it follows from (b) that there exists $N_1 > N$ satisfying

$$\frac{c^{n-N}}{1-c} \theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall n \geq N_1, \quad (2.18)$$

which together with (2.17), (a), and (c) gives that

$$\begin{aligned} \theta(d(x_n, x_m)) &\leq \theta\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq \sum_{k=n}^{m-1} \theta(d(x_k, x_{k+1})) \\ &\leq \sum_{k=n}^{m-1} c^{k-N} \theta(d(x_N, x_{N+1})) \\ &\leq \frac{c^{n-N}}{1-c} \theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall m > n \geq N_1. \end{aligned} \quad (2.19)$$

In view of (e) and (2.19), we deduce that $d(x_n, x_m) < \varepsilon$, for all $m > n \geq N_1$, which means that $\{x_n\}_{n \in \mathbb{N}_0}$ is a Cauchy sequence. Hence there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$ by completeness of X .

Suppose that f is T orbitally lower semicontinuous at z . Since $\{x_n\}_{n \geq 0}$ is an orbit of T with $\lim_{n \rightarrow \infty} x_n = z$, it follows that

$$f(z) \leq \liminf_{n \rightarrow \infty} f(x_n). \quad (2.20)$$

Using (2.6) and (2.7), we infer that

$$\theta(d(x_n, Tx_n)) \leq \varphi(d(x_{n-1}, x_n)) \theta(d(x_{n-1}, x_n)) < \theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \quad (2.21)$$

which together with (e), (2.11), and $q = 0$ implies that

$$0 < d(x_n, Tx_n) < d(x_{n-1}, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (2.22)$$

that is, $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, which together with (2.20) yields that

$$0 \leq d(z, Tz) = f(z) \leq \liminf_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0, \quad (2.23)$$

which gives that $d(z, Tz) = 0$, that is, $z \in Tz$.

Conversely, suppose that $z \in X$ is a fixed point of T . Let $\{y_n\}_{n \in \mathbb{N}_0} \subset X$ be an arbitrarily orbit of T with $\lim_{n \rightarrow \infty} y_n = z$. It is clear that

$$f(z) = d(z, Tz) = 0 \leq \liminf_{n \rightarrow \infty} f(y_n), \quad (2.24)$$

which implies that f is T orbitally lower semicontinuous at z . This completes the proof. $\square$

Notice that $d(y, Ty) \leq H(Tx, Ty)$ for each $y \in Tx$. In light of Theorem 2.3, we have

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \rightarrow CL(X)$ satisfy that*

$$\theta(H(Tx, Ty)) \leq \varphi(d(x, y))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \quad (2.25)$$

where $\theta \in \Theta$ and φ satisfies (2.7). Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n \in \mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) is T orbitally lower semicontinuous at z .

If $\varphi(d(x, y))$ in (2.6) is replaced by $\varphi(d(x, Tx))$, one has

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \rightarrow CL(X)$ satisfy that*

$$\theta(d(y, Ty)) \leq \varphi(d(x, Tx))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \quad (2.26)$$

where $\theta \in \Theta$ and φ satisfies (2.7). Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n \in \mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) is T orbitally lower semicontinuous at z .

Proof. Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.7), (2.26), and Lemma 2.2 that for $q = 1/\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\} > 1$ there exists $x_2 \in Tx_1$ such that

$$\begin{aligned}
 \theta(d(x_1, x_2)) &\leq \frac{\theta(d(x_1, Tx_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\}} \\
 &\leq \frac{\varphi(d(x_0, Tx_0))\theta(d(x_0, x_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\}} \\
 &\leq \sqrt{\varphi(d(x_0, Tx_0))}\theta(d(x_0, x_1)), \\
 \theta(d(x_2, Tx_2)) &\leq \varphi(d(x_1, Tx_1))\theta(d(x_1, x_2)) \\
 &\leq \frac{\varphi(d(x_1, Tx_1))\theta(d(x_1, Tx_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\}} \\
 &\leq \sqrt{\varphi(d(x_1, Tx_1))}\theta(d(x_1, Tx_1)).
 \end{aligned} \tag{2.27}$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n \in \mathbb{N}_0} \subset X$ satisfying $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
 \theta(d(x_n, x_{n+1})) &\leq \frac{\theta(d(x_n, Tx_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\}} \\
 &\leq \frac{\varphi(d(x_{n-1}, Tx_{n-1}))\theta(d(x_{n-1}, x_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\}} \\
 &\leq \sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}\theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N},
 \end{aligned} \tag{2.28}$$

$$\begin{aligned}
 \theta(d(x_{n+1}, Tx_{n+1})) &\leq \varphi(d(x_n, Tx_n))\theta(d(x_n, x_{n+1})) \\
 &\leq \frac{\varphi(d(x_n, Tx_n))\theta(d(x_n, Tx_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\}} \\
 &\leq \sqrt{\varphi(d(x_n, Tx_n))}\theta(d(x_n, Tx_n)), \quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{2.29}$$

Suppose that $x_{n_0} \in Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$. It is easy to verify that $x_n = x_{n_0}$ for all $n \geq n_0$ and the conclusion of Theorem 2.5 holds.

Suppose that $x_n \notin Tx_n$ for each $n \in \mathbb{N}_0$. It follows that $\{d(x_n, Tx_n)\}_{n \in \mathbb{N}_0}$ and $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}$ are positive sequences. Combining (2.7), (2.28), (2.29), (b) and (e), we infer that $\{\theta(d(x_n, x_{n+1}))\}_{n \in \mathbb{N}_0}$ and $\{\theta(d(x_n, Tx_n))\}_{n \in \mathbb{N}_0}$ are both positive and decreasing, so do $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}$ and $\{d(x_n, Tx_n)\}_{n \in \mathbb{N}_0}$. It follows that there exist constants α, β, s and t satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \theta(d(x_n, x_{n+1})) &= \alpha \geq 0, & \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) &= \beta \geq 0, \\ \lim_{n \rightarrow \infty} \theta(d(x_n, Tx_n)) &= s \geq 0, & \lim_{n \rightarrow \infty} d(x_n, Tx_n) &= t \geq 0. \end{aligned} \quad (2.30)$$

Notice that (2.7) implies that there exists a constant r such that

$$\limsup_{n \rightarrow \infty} \varphi(d(x_n, Tx_n)) \leq \limsup_{l \rightarrow t^+} \varphi(l) = r \in [0, 1). \quad (2.31)$$

Taking upper limits in (2.29) and by (2.30) and (2.31) we get that

$$s \leq \sqrt{\limsup_{n \rightarrow \infty} \varphi(d(x_n, Tx_n))} \limsup_{n \rightarrow \infty} \theta(d(x_n, Tx_n)) \leq \sqrt{r}s, \quad (2.32)$$

which implies that $s = 0$, which together with (2.30) and (a) ensures that

$$0 \leq \theta(t) < \theta(d(x_n, Tx_n)) \longrightarrow 0, \quad n \longrightarrow \infty, \quad (2.33)$$

that is, $\theta(t) = 0$, which gives that $t = 0$ by (b) and (d). It follows from (2.28), (2.30), and (2.31) that

$$\alpha \leq \sqrt{\limsup_{n \rightarrow \infty} \varphi(d(x_n, Tx_n))} \limsup_{n \rightarrow \infty} \theta(d(x_{n-1}, x_n)) \leq \sqrt{r}\alpha, \quad (2.34)$$

which yields that $\alpha = 0$. Notice that (2.30) and (a) guarantee that

$$0 \leq \theta(\beta) < \theta(d(x_n, x_{n+1})) \longrightarrow 0, \quad n \longrightarrow \infty, \quad (2.35)$$

which together with (b) and (d) yields that $\beta = 0$. The rest of the proof is similar to that of Theorem 2.3 and is omitted. This completes the proof. $\square$

The result below follows from Theorem 2.5.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \rightarrow CL(X)$ satisfy that*

$$\theta(H(Tx, Ty)) \leq \varphi(d(x, Tx))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \quad (2.36)$$

where $\theta \in \Theta$ and φ satisfies (2.7). Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n \in \mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) is T orbitally lower semicontinuous at z .

3. Comparisons and Examples

Now we construct two examples to compare the results in Section 2 with the corresponding results in [6–8].

Remark 3.1. Theorems 2.3 and 2.4 extend Theorems 1.1–1.3, and Theorems 2.5 and 2.6 are different from Theorems 1.1–1.3, respectively, in the following ways:

- (1) the ranges $CL(X)$ of the nonlinear set-valued contractive mappings T in Theorems 2.3–2.6 are more general than the ranges $C(X)$ and $CB(X)$ of the set-valued contraction mappings T in Theorems 1.1–1.3, respectively;
- (2) the T orbit lower semicontinuity at some $z \in X$ of the functions $f(x) = d(x, Tx)$ in Theorems 2.3 and 2.4 is weaker than the continuity of the set-valued contraction mappings T in X in Theorems 1.1–1.3, respectively;
- (3) the set-valued contraction mappings (1.1) and (1.2) are special cases of the nonlinear set-valued contractive mapping (2.6) with $\theta \equiv 1$ because

$$d(y, Ty) \leq H(Tx, Ty), \quad \forall (x, y) \in X \times Tx. \quad (3.1)$$

Example 3.2 below shows that Theorems 2.3 and 2.4 extend substantively Theorems 1.1–1.3, respectively.

Example 3.2. Let $X = (-\infty, 3/10]$ and d be the standard metric in X . Let $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$ and $T : X \rightarrow CL(X)$ be defined by

$$\theta(t) = t^{1/2}, \quad \varphi(t) = \frac{2\sqrt{6}}{5}, \quad \forall t \in \mathbb{R}^+, \quad Tx = \begin{cases} \left(-\infty, \frac{1}{4}x\right], & \forall x \in (-\infty, 0), \\ [0, 2x^2], & \forall x \in \left[0, \frac{3}{10}\right], \end{cases} \quad (3.2)$$

respectively. It is clear that $\theta \in \Theta$, φ satisfies (2.7) and

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (-\infty, 0) \\ x - 2x^2, & \forall x \in \left[0, \frac{3}{10}\right] \end{cases} \quad (3.3)$$

is T orbitally lower semicontinuous in X . In order to prove (2.6) holds, we consider two possible cases.

Case 1. Let $x \in (-\infty, 0)$ and $y \in Tx = (-\infty, (1/4)x]$. It is clear that

$$\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) = \frac{1}{2}\theta(d(x, y)) \leq \varphi(d(x, y))\theta(d(x, y)). \quad (3.4)$$

Case 2. Let $x \in [0, 3/10]$ and $y \in Tx = [0, 2x^2]$. It follows that

$$\begin{aligned}\theta(d(y, Ty)) &\leq \theta(H(Tx, Ty)) = \sqrt{2}|x + y|^{1/2}\theta(d(x, y)) \\ &\leq \sqrt{2}\left(\frac{3}{10} + \frac{9}{50}\right)^{1/2}\theta(d(x, y)) = \varphi(d(x, y))\theta(d(x, y)),\end{aligned}\quad (3.5)$$

that is, (2.6) holds. Therefore all assumptions of Theorems 2.3 and 2.4 are satisfied. It follows from each of Theorems 2.3 and 2.4 that T has a fixed point in X . However, we cannot invoke any one of Theorems 1.1–1.3 to show the existence of fixed points for the mapping T in X . Indeed, taking $x_0 = 3/10$ and $y_0 = 1/5$, we get that

$$H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \not\leq \frac{r}{10} = rd(x_0, y_0), \quad (3.6)$$

for any $r \in (0, 1)$ and

$$H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \not\leq \frac{1}{10}\varphi\left(\frac{1}{10}\right) = \varphi(d(x_0, y_0))d(x_0, y_0), \quad (3.7)$$

for any mapping $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$ with each of (1.3) and (1.4).

Next we construct an example to explain Theorems 2.5 and 2.6.

Example 3.3. Let $X = [-3/10, +\infty)$ and d be the standard metric in X . Define $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$ and $T : X \rightarrow CL(X)$ by

$$\begin{aligned}\theta(t) &= t^{1/2}, \quad \forall t \in \mathbb{R}^+, \quad \varphi(t) = \begin{cases} 2\sqrt{2}t^{1/2}, & \forall t \in \left(0, \frac{1}{8}\right), \\ \frac{2\sqrt{6}}{5}, & \forall t \in \{0\} \cup \left[\frac{1}{8}, +\infty\right), \end{cases} \\ Tx &= \begin{cases} \left[\frac{x}{4(1+x)}, +\infty\right), & \forall x \in (0, +\infty), \\ [-2x^2, 0], & \forall x \in \left[-\frac{3}{10}, 0\right], \end{cases}\end{aligned}\quad (3.8)$$

respectively. It is easy to see that (2.7) holds and

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (0, +\infty), \\ -2x^2 - x, & \forall x \in \left[-\frac{3}{10}, 0\right] \end{cases} \quad (3.9)$$

is T orbitally lower semicontinuous in X . In order to check (2.26), we have to consider two cases as follows.

Case 1. Let $x \in (0, +\infty)$ and $y \in Tx = [x/4(1+x), +\infty)$. It is clear that

$$\begin{aligned}\theta(d(y, Ty)) = 0 &\leq \theta(H(Tx, Ty)) = \left| \frac{x}{4(1+x)} - \frac{y}{4(1+y)} \right|^{1/2} \\ &= \frac{\theta(d(x, y))}{2(1+x)^{1/2}(1+y)^{1/2}} \leq \frac{\theta(d(x, y))}{2(1+x)^{1/2}(1+x/4(1+x))^{1/2}} \\ &= \frac{\theta(d(x, y))}{(5x+4)^{1/2}} \leq \frac{\theta(d(x, y))}{2} \leq \frac{2\sqrt{6}}{5}\theta(d(x, y)) \\ &= \varphi(0)\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).\end{aligned}\quad (3.10)$$

Case 2. Let $x \in [-3/10, 0]$ and $y \in Tx = [-2x^2, 0]$. It follows that

$$\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) = \sqrt{2}|x+y|^{1/2}\theta(d(x, y)) \leq \sqrt{2}|x-2x^2|^{1/2}\theta(d(x, y)).\quad (3.11)$$

For $x = 0$, we have

$$\sqrt{2}|x-2x^2|^{1/2}\theta(d(x, y)) = 0 \leq \varphi(d(x, Tx))\theta(d(x, y)).\quad (3.12)$$

For $x \in [-3/10, -1/4) \cup (-1/4, 0)$, we infer that

$$\sqrt{2}|x-2x^2|^{1/2}\theta(d(x, y)) \leq 2\sqrt{2}(-2x^2-x)^{1/2}\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).\quad (3.13)$$

For $x = -1/4$, we get that

$$\sqrt{2}|x-2x^2|^{1/2}\theta(d(x, y)) = \frac{\sqrt{3}}{2}\theta(d(x, y)) \leq \varphi\left(\frac{1}{8}\right)\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).\quad (3.14)$$

Hence (2.26) holds. Thus all assumptions of Theorems 2.5 and 2.6 are satisfied. It follows from each of Theorems 2.5 and 2.6 that T has a fixed point in X .

Taking $x_0 = 1$ and $y_0 = -3/10$, we deduce that

$$H(Tx_0, Ty_0) = H\left(\left[\frac{1}{8}, +\infty\right), \left[-\frac{9}{50}, 0\right]\right) = +\infty \not\leq \frac{13r}{10} = rd(x_0, y_0),\quad (3.15)$$

for any $r \in (0, 1)$, and

$$H(Tx_0, Ty_0) = +\infty \not\leq \frac{2\sqrt{6}}{5} \cdot \frac{13}{10} = \varphi(d(x_0, y_0))d(x_0, y_0),\quad (3.16)$$

for any mapping $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$ with each of (1.3) and (1.4). That is, Theorems 1.1–1.3 are inapplicable in proving the existence of fixed points for the nonlinear set-valued contractive mapping T .

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