Research Article

Approximation of Solutions of an Equilibrium Problem in a Banach Space

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An equilibrium problem is investigated based on a hybrid projection iterative algorithm. Strong convergence theorems for solutions of the equilibrium problem are established in a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property.

1. Introduction

Equilibrium problems which were introduced by Fan [1] and Blum and Oettli [2] have had a great impact and influence on the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. It has been shown [3–8] that equilibrium problems include variational inequalities, fixed point, the Nash equilibrium, and game theory as special cases. A number of iterative algorithms have recently been studying for fixed point and equilibrium problems, see [9–26] and the references therein. However, there were few results established in the framework of the Banach spaces. In this paper, we suggest and analyze a projection iterative algorithm for finding solutions of equilibrium in a Banach space.

2. Preliminaries

In what follows, we always assume that E is a Banach space with the dual space E^* . Let C be a nonempty, closed, and convex subset of E . We use the symbol J to stand for the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^* .

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in U_E$ with $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (2.2)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex; for details see [27] and the references therein.

Recall that a Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x \in C$, where $\rightharpoonup$ denotes the weak convergence, and $\|x_n\| \rightarrow \|x\|$, where $\rightarrow$ denotes the strong convergence, and then $x_n \rightarrow x$. It is known that if E is uniformly convex, then E enjoys the Kadec-Klee property; for details see [26] and the references therein.

E is said to be smooth provided $\lim_{t \rightarrow 0} (\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$.

It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive, and smooth, then J is single valued and demicontinuous; for more details see [27, 28] and the references therein.

It is also well known that if D is a nonempty, closed, and convex subset of a Hilbert space H , and $P_D : H \rightarrow D$ is the metric projection from H onto D , then P_D is nonexpansive. This fact actually characterizes the Hilbert spaces, and consequently, it is not available in more general Banach spaces. In this connection, Alber [29] introduced a generalized projection operator Π_D in the Banach spaces which is an analogue of the metric projection in the Hilbert spaces.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.3)$$

Notice that, in a Hilbert space H , (2.3) is reduced to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \rightarrow C$ is a mapping that is assigned to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x). \quad (2.4)$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and the strict monotonicity of the mapping J ; see, for example, [27, 28]. In the Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E, \quad (2.5)$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall x, y, z \in E. \quad (2.6)$$

Let $T : C \rightarrow C$ be a mapping. Recall that a point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\tilde{F}(T)$. T is said to be relatively nonexpansive if

$$\tilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in F(T). \quad (2.7)$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [27, 29, 30].

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we consider the following equilibrium problem. Find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \quad (2.8)$$

We use $EP(f)$ to denote the solution set of the equilibrium problem (2.3). That is,

$$EP(f) = \{p \in C : f(p, y) \geq 0, \forall y \in C\}. \quad (2.9)$$

Given a mapping $Q : C \rightarrow E^*$, let

$$f(x, y) = \langle Qx, y - x \rangle, \quad \forall x, y \in C. \quad (2.10)$$

Then $p \in EP(f)$ if and only if p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p \rangle \geq 0, \quad \forall y \in C. \quad (2.11)$$

To study the equilibrium problem (2.8), we may assume that f satisfies the following conditions:

- (A1) $f(x, x) = 0$, for all $x \in C$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;
- (A3)

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y), \quad \forall x, y, z \in C; \quad (2.12)$$

- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and weakly lower semicontinuous.

In this paper, we study the problem of approximating solutions of equilibrium problem (2.8) based on a hybrid projection iterative algorithm in a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property. To prove our main results, we need the following lemmas.

Lemma 2.1. *Let E be a strictly convex and uniformly smooth Banach space and C a nonempty, closed, and convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in E$. Then*

(a) (see [2]). *There exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.13)$$

(b) (see [31]). *Define a mapping $T_r^f : E \rightarrow C$ by*

$$T_r^f x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle, \quad \forall y \in C \right\}. \quad (2.14)$$

Then the following conclusions hold:

- (1) T_r^f is single valued;
- (2) T_r^f is a firmly nonexpansive-type mapping; that is, for all $x, y \in E$,

$$\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y \rangle \leq \langle T_r^f x - T_r^f y, Jx - Jy \rangle; \quad (2.15)$$

- (3) $F(T_r^f) = \text{EP}(f)$;
- (4) $\text{EP}(f)$ is closed and convex;
- (5) T_r^f is relatively nonexpansive.

Lemma 2.2 (see [29]). *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, closed, and convex subset of E . Let $x \in E$, and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \quad (2.16)$$

Lemma 2.3 (see [29]). *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, closed, and convex subset of E , and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \quad (2.17)$$

Lemma 2.4 (see [27]). *Let E be a reflexive, strictly convex, and smooth Banach space. Then one has the following*

$$\phi(x, y) = 0 \iff x = y, \quad \forall x, y \in E. \quad (2.18)$$

3. Main Results

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property and C a nonempty, closed, and convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $\text{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{aligned} x_0 &\in E \text{ chosen arbitrarily,} \\ C_1 &= C, \\ x_1 &= \Pi_{C_1} x_0, \\ y_n &\in C, \text{ such that } f(y_n, u) + \frac{1}{r_n} \langle u - y_n, Jy_n - Jx_n \rangle \geq 0, \quad \forall u \in C, \\ C_{n+1} &= \{u \in C_n : 2\langle x_n - u, Jx_n - Jy_n \rangle \geq \phi(x_n, y_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{aligned} \tag{3.1}$$

where $\{r_n\}$ is a real number sequence in $[r, \infty)$, where r is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{\text{EP}(f)} x_0$.

Proof. In view of Lemma 2.1, we see that $\text{EP}(f)$ is closed and convex. Next, we show that C_n is closed and convex. It is not hard to see that $C_1 = C$ is closed. Therefore, we only show that C_n is convex. It is obvious that $C_1 = C$ is convex. Suppose that C_h is convex for some $h \in \mathbb{N}$. Next, we show that C_{h+1} is also convex for the same h . Let $a, b \in C_{h+1}$ and $c = ta + (1-t)b$, where $t \in (0, 1)$. It follows that

$$\phi(x_h, y_h) \leq 2\langle x_h - a, Jx_h - Jy_h \rangle, \quad \phi(x_h, y_h) \leq 2\langle x_h - b, Jx_h - Jy_h \rangle, \tag{3.2}$$

where $a, b \in C_h$. From the above two inequalities, we can get that

$$\phi(x_h, y_h) \leq 2\langle x_h - c, Jx_h - Jy_h \rangle, \tag{3.3}$$

where $c \in C_h$. It follows that C_{h+1} is closed and convex. This completes the proof that C_n is closed, and convex.

Next, we show that $\text{EP}(f) \subset C_n$. It is obvious that $\text{EP}(f) \subset C = C_1$. Suppose that $\text{EP}(f) \subset C_h$ for some $h \in \mathbb{N}$. For any $z \in \text{EP}(f) \subset C_h$, we see from Lemma 2.1 that

$$\phi(z, y_h) \leq \phi(z, x_h). \tag{3.4}$$

On the other hand, we obtain from (2.6) that

$$\phi(z, y_h) = \phi(z, x_h) + \phi(x_h, y_h) + 2\langle z - x_h, Jx_h - Jy_h \rangle. \tag{3.5}$$

Combining (3.4) with (3.5), we arrive at

$$2\langle x_h - z, Jx_h - Jy_h \rangle \geq \phi(x_h, y_h) \quad (3.6)$$

which implies that $z \in C_{h+1}$. This shows that $\text{EP}(f) \subset C_{h+1}$. This completes the proof that $\text{EP}(f) \subset C_n$.

Next, we show that $\{x_n\}$ is a convergent sequence and strongly converges to $\bar{x}$, where $\bar{x} \in \text{EP}(f)$. Since $x_n = \Pi_{C_n} x_0$, we see from Lemma 2.2 that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n. \quad (3.7)$$

It follows from $\text{EP}(f) \subset C_n$ that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in \text{EP}(f). \quad (3.8)$$

By virtue of Lemma 2.3, we obtain that

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(\Pi_{\text{EP}(f)} x_0, x_0) - \phi(\Pi_{\text{EP}(f)} x_0, x_n) \\ &\leq \phi(\Pi_{\text{EP}(f)} x_0, x_0). \end{aligned} \quad (3.9)$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (2.5) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed and convex, we see that $\bar{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned} \phi(\bar{x}, x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_0 \rangle + \|x_0\|^2 \\ &\leq \liminf_{n \rightarrow \infty} (\|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2) \\ &= \liminf_{n \rightarrow \infty} \phi(x_n, x_0) \\ &\leq \limsup_{n \rightarrow \infty} \phi(x_n, x_0) \\ &\leq \phi(\bar{x}, x_0), \end{aligned} \quad (3.10)$$

which implies that $\phi(x_n, x_0) \rightarrow \phi(\bar{x}, x_0)$ as $n \rightarrow \infty$. Hence, $\|x_n\| \rightarrow \|\bar{x}\|$ as $n \rightarrow \infty$. In view of the Kadec-Klee property of E , we see that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Notice that $x_{n+1} = \Pi_{\text{EP}(f)} x_0 \in C_{n+1} \subset C_n$. It follows that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned} \quad (3.11)$$

Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we arrive at $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$. This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists. It follows that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \quad (3.12)$$

By virtue of $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we find that

$$\phi(x_n, y_n) \leq 2\langle x_n - x_{n+1}, Jx_n - Jy_n \rangle. \quad (3.13)$$

It follows that

$$\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0. \quad (3.14)$$

In view of (2.5), we see that

$$\lim_{n \rightarrow \infty} (\|x_n\| - \|y_n\|) = 0. \quad (3.15)$$

Since $x_n \rightarrow \bar{x}$, we find that

$$\lim_{n \rightarrow \infty} \|y_n\| = \|\bar{x}\|. \quad (3.16)$$

It follows that

$$\lim_{n \rightarrow \infty} \|Jy_n\| = \|J\bar{x}\|. \quad (3.17)$$

This implies that $\{Jy_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of the reflexivity of E , we see that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned} \phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n \rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jy_n \rangle + \|Jy_n\|^2. \end{aligned} \quad (3.18)$$

Taking $\liminf_{n \rightarrow \infty}$ on the both sides of the equality above yields that

$$\begin{aligned} 0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^* \rangle + \|y^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy \rangle + \|Jy\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy \rangle + \|y\|^2 \\ &= \phi(\bar{x}, y). \end{aligned} \quad (3.19)$$

That is, $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. It follows that $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (3.17) that $\lim_{n \rightarrow \infty} Jy_n = J\bar{x}$. Since $J^{-1} : E^* \rightarrow E$ is demicontinuous, we find that $y_n \rightharpoonup \bar{x}$. This implies from (3.16) and the Kadec-Klee property of E that $\lim_{n \rightarrow \infty} y_n = \bar{x}$. This in turn implies that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we find that

$$\lim_{n \rightarrow \infty} \|Jy_n - Jx_n\| = 0. \quad (3.20)$$

Next, we show that $\bar{x} \in EF(f)$. In view of Lemma 2.1, we find from $y_n = T_{r_n}^f x_n$ that

$$f(y_n, u) + \frac{1}{r_n} \langle u - y_n, Jy_n - Jx_n \rangle \geq 0, \quad \forall u \in C. \quad (3.21)$$

It follows from condition (A2) and (3.20) that

$$\frac{1}{r_n} \|u - y_n\| \|Jy_n - Jx_n\| \geq f(u, y_n), \quad \forall u \in C. \quad (3.22)$$

In view of condition (A4), we obtain from (3.17) that

$$f(u, \bar{x}) \leq 0, \quad \forall u \in C. \quad (3.23)$$

For $0 < t < 1$ and $u \in C$, define $u_t = tu + (1-t)\bar{x}$. It follows that $u_t \in C$, which yields that $f(u_t, \bar{x}) \leq 0$. It follows from conditions (A1) and (A4) that

$$0 = f(u_t, u_t) \leq tf(u_t, u) + (1-t)f(u_t, \bar{x}) \leq tf(u_t, u). \quad (3.24)$$

That is,

$$f(u_t, u) \geq 0. \quad (3.25)$$

Letting $t \downarrow 0$, we find from condition (A3) that $f(\bar{x}, u) \geq 0$, for all $u \in C$. This implies that $\bar{x} \in EP(f)$. This shows that $\bar{x} \in EP(f)$.

Finally, we prove that $\bar{x} = \Pi_{EP(f)} x_0$. Letting $n \rightarrow \infty$ in (3.8), we see that

$$\langle \bar{x} - w, Jx_0 - J\bar{x} \rangle \geq 0, \quad \forall w \in EP(f). \quad (3.26)$$

In view of Lemma 2.2, we can obtain that $\bar{x} = \Pi_{EP(f)} x_0$. This completes the proof. $\square$

In the framework of the Hilbert spaces, we have the following.

Corollary 3.2. Let E be a Hilbert space and C a nonempty, closed, and convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $\text{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{aligned}
 x_0 &\in E \text{ chosen arbitrarily,} \\
 C_1 &= C, \\
 x_1 &= P_{C_1} x_0, \\
 y_n &\in C, \text{ such that } f(y_n, u) + \frac{1}{r_n} \langle u - y_n, y_n - x_n \rangle \geq 0, \quad \forall u \in C, \\
 C_{n+1} &= \left\{ u \in C_n : 2 \langle x_n - u, x_n - y_n \rangle \geq \|x_n - y_n\|^2 \right\}, \\
 x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \geq 1,
 \end{aligned} \tag{3.27}$$

where $\{r_n\}$ is a real number sequence in $[r, \infty)$, where r is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{\text{EP}(f)} x_0$.

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