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*Research Article*

# Dynamical Behavior of a Stochastic Ratio-Dependent Predator-Prey System

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This paper is concerned with a stochastic ratio-dependent predator-prey model with varible coefficients. By the comparison theorem of stochastic equations and the Itô formula, the global existence of a unique positive solution of the ratio-dependent model is obtained. Besides, some results are established such as the stochastically ultimate boundedness and stochastic permanence for this model.

## 1. Introduction

Ecological systems are mainly characterized by the interaction between species and their surrounding natural environment [1]. Especially, the dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology, due to its universal existence and importance [2–4]. The interaction mechanism of predators and their preys can be described as differential equations, such as Lotaka-Volterra models [5].

Recently, many researchers pay much attention to functional and numerical responses over typical ecological timescales, which depend on the densities of both predators and their preys (most likely and simply on their ration) [6–8]. Such a functional response is called a ratio-dependent response function, and these hypotheses have been strongly supported by numerous and laboratory experiments and observations [9–11].

It is worthy to note that, based on the Michaelis-Menten or Holling type II function, Arditi and Ginzburg [6] firstly proposed a ratio-dependent function of the form

$$P\left(\frac{x}{y}\right) = \frac{cx/y}{m + x/y} = \frac{cx}{my + x} \tag{1.1}$$

and a ratio-dependent predator-prey model of the form

$$\begin{aligned} \dot{x}(t) &= x(t)\left[a - bx(t) - \frac{cy(t)}{my(t) + x(t)}\right], \\ \dot{y}(t) &= y(t)\left[-d + \frac{fx(t)}{my(t) + x(t)}\right]. \end{aligned} \tag{1.2}$$

Here, $x(t)$ and $y(t)$ represent population densities of the prey and the predator at time $t$, respectively. Parameters $a, b, c, d, f$, and $m$ are positive constants in which $a/b$ is the carrying capacity of the prey, $a$, $c$, $m$, $f$, and $d$ stand for the prey intrinsic growth rate, capturing rate, half capturing saturation constant, conversion rate, and the predator death rate, respectively. In recent years, several authors have studied the ratio-dependent predator-prey model (1.2) and its extension, and they have obtained rich results [12–19].

It is well known that population systems are often affected by environmental noise. Hence, stochastic differential equation models play a significant role in various branches of applied sciences including biology and population dynamics as they provide some additional degree of realism compared to their deterministic counterpart [20, 21]. Recall that the parameters $a$ and $-d$ represent the intrinsic growth and death rate of $x(t)$ and $y(t)$, respectively. In practice we usually estimate them by an average value plus errors. In general, the errors follow normal distributions (by the well-known central limit theorem), but the standard deviations of the errors, known as the noise intensities, may depend on the population sizes. We may therefore replace the rates $a$ and $-d$ by

$$a \longrightarrow a + \alpha\dot{B}_1(t), \qquad -d \longrightarrow -(d+\beta)\dot{B}_2(t), \tag{1.3}$$

respectively, where $B_1(t)$ and $B_2(t)$ are mutually independent Brownian motions and $\alpha$ and $\beta$ represent the intensities of the white noises. As a result, (1.2) becomes a stochastic differential equation (SDE, in short):

$$\begin{aligned} dx(t) &= x(t)\left[a - bx(t) - \frac{cy(t)}{my(t) + x(t)}\right]dt + \alpha x(t)dB_1(t), \\ dy(t) &= y(t)\left[-d + \frac{fx(t)}{my(t) + x(t)}\right]dt - \beta y(t)dB_2(t). \end{aligned} \tag{1.4}$$

By the Itô formula, Ji et al. [3] showed that (1.4) is persistent or extinct in some conditions.

The predator-prey model describes a prey population $x$ that serves as food for a predator $y$. However, due to the varying of the effects of environment and such as weather, temperature, food supply, the prey intrinsic growth rate, capturing rate, half capturing saturation constant, conversion rate, and predator death rate are functions of time $t$ [22–26].

Therefore, Zhang and Hou [27] studied the following general ratio-dependent predator-prey model of the form:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{m(t)y(t) + x(t)}\right], \\ \dot{y}(t) &= y(t)\left[-d(t) + \frac{f(t)x(t)}{m(t)y(t) + x(t)}\right], \end{aligned} \tag{1.5}$$

which is more realistic. Motivated by [3, 27], this paper is concerned with a stochastic ratio-dependent predator-prey model of the following form:

$$\begin{aligned} dx(t) &= x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{m(t)y(t) + x(t)}\right]dt + \alpha(t)x(t)dB_1(t), \\ dy(t) &= y(t)\left[-d(t) + \frac{f(t)x(t)}{m(t)y(t) + x(t)}\right]dt - \beta(t)y(t)dB_2(t), \end{aligned} \tag{1.6}$$

where $a(t), b(t), c(t), d(t), f(t)$, and $m(t)$ are positive bounded continuous functions on $[0, \infty)$ and $\alpha(t)$, $\beta(t)$ are bounded continuous functions on $[0, \infty)$, and $B_1(t)$ and $B_2(t)$ are defined in (1.4). There would be some difficulties in studying this model since the parameters are changed by time $t$. Under some suitable conditions, we obtain some results such as the stochastic permanence of (1.6).

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $P$-null sets). Let $B_1(t)$ and $B_2(t)$ be mutually independent Brownian motions, $R_+^2$ the positive cone in $R^2$, $X(t) = (x(t), y(t))$, and $|X(t)| = (x^2(t) + y^2(t))^{1/2}$.

For convenience and simplicity in the following discussion, we use the notation

$$\varphi^u = \sup_{t\in[0,\infty)} \varphi(t), \qquad \varphi_l = \inf_{t\in[0,\infty)} \varphi(t), \tag{1.7}$$

where $\varphi(t)$ is a bounded continuous function on $[0, \infty)$.

This paper is organized as follows. In Section 2, by the Itô formula and the comparison theorem of stochastic equations, the existence and uniqueness of the global positive solution are established for any given positive initial value. In Section 3, we find that both the prey population and predator population of (1.6) are bounded in mean. Finally, we give some conditions that guarantee that (1.6) is stochastically permanent.

## 2. Global Positive Solution

As $x(t)$ and $y(t)$ in (1.6) are population densities of the prey and the predator at time $t$, respectively, we are only interested in the positive solutions. Moreover, in order for a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial value, the coefficients of equation are generally required to satisfy the linear growth condition and local Lipschitz condition [28]. However, the

coefficients of (1.6) satisfy neither the linear growth condition nor the local Lipschitz continuous. In this section, by making the change of variables and the comparison theorem of stochastic equations [29], we will show that there is a unique positive solution with positive initial value of system (1.6).

**Lemma 2.1.** *For any given initial value $X_0 \in R_+^2$, there is a unique positive local solution $X(t)$ to (1.6) on $t \in [0, \tau_e)$ a.s.*

*Proof.* We first consider the equation

$$\begin{aligned} du(t) &= \left[a(t) - \frac{\alpha^2(t)}{2} - b(t)e^{u(t)} - \frac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}}\right]dt + \alpha(t)dB_1(t), \\ dv(t) &= \left[-d(t) - \frac{\beta^2(t)}{2} + \frac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}}\right]dt - \beta(t)dB_2(t) \end{aligned} \tag{2.1}$$

on $t \geq 0$ with initial value $u(0) = \ln x_0$, $v(0) = \ln y_0$. Since the coefficients of system (2.1) satisfy the local Lipschitz condition, there is a unique local solution $(u(t), v(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time [28]. Therefore, by the Itô formula, it is easy to see that $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$ is the unique positive local solution of system (2.1) with initial value $X_0 = (x_0, y_0) \in R_+^2$. Lemma 2.1 is finally proved. □

Lemma 2.1 only tells us that there is a unique positive local solution of system (1.6). Next, we show that this solution is global, that is, $\tau_e = \infty$.

Since the solution is positive, we have

$$dx(t) \leq x(t)[a(t) - b(t)x(t)]dt + \alpha(t)x(t)dB_1(t). \tag{2.2}$$

Let

$$\Phi(t) = \frac{\exp\left\{\int_0^t [a(s) - (\alpha^2(s)/2)]ds + \int_0^t \alpha(s)dB_1(s)\right\}}{x_0^{-1} + \int_0^t b(s)\exp\left\{\int_0^s [a(\tau) - (\alpha^2(\tau)/2)]d\tau + \int_0^s \alpha(\tau)dB_1(\tau)\right\}ds}. \tag{2.3}$$

Then, $\Phi(t)$ is the unique solution of equation

$$\begin{aligned} d\Phi(t) &= \Phi(t)[a(t) - b(t)\Phi(t)]dt + \alpha(t)\Phi(t)dB_1(t), \\ \Phi(0) &= x_0, \end{aligned} \tag{2.4}$$

$$x(t) \leq \Phi(t) \quad \text{a.s. } t \in [0, \tau_e) \tag{2.5}$$

by the comparison theorem of stochastic equations. On the other hand, we have

$$dx(t) \geq x(t)\left[a(t) - \frac{c(t)}{m(t)} - b(t)x(t)\right]dt + \alpha(t)x(t)dB_1(t). \tag{2.6}$$

Similarly,

$$\phi(t) = \frac{\exp\left\{\int_0^t \left[a(s) - (c(s)/m(s)) - \left(\alpha^2(s)/2\right)\right] ds + \int_0^t \alpha(s) dB_1(s)\right\}}{x_0^{-1} + \int_0^t b(s) \exp\left\{\int_0^s [a(\tau) - (c(\tau)/m(\tau)) - (\alpha^2(\tau)/2)] d\tau + \int_0^s \alpha(\tau) dB_1(\tau)\right\} ds} \tag{2.7}$$

is the unique solution of equation

$$\begin{gathered} d\phi(t) = \phi(t)\left[a(t) - \frac{c(t)}{m(t)} b(t) - \phi(t)\right] dt + \alpha(t)\phi(t) dB_1(t), \\ \phi(0) = x_0, \\ x(t) \geq \phi(t) \quad \text{a.s. } t \in [0, \tau_e). \end{gathered} \tag{2.8}$$

Consequently,

$$\phi(t) \leq x(t) \leq \Phi(t) \quad \text{a.s. } t \in [0, \tau_e). \tag{2.9}$$

Next, we consider the predator population $y(t)$. As the arguments above, we can get

$$\begin{gathered} dy(t) \leq y(t)\left[-d(t) + f(t)\right] dt - \beta(t) y(t) dB_2(t), \\ dy(t) \geq -d(t) y(t) dt - \beta(t) y(t) dB_2(t). \end{gathered} \tag{2.10}$$

Let

$$\begin{gathered} \underline{y}(t) := y_0 \exp\left\{-\int_0^t \left[d(s) + \frac{\beta^2(s)}{2}\right] ds - \int_0^t \beta(s) dB_2(s)\right\}, \\ \overline{y}(t) := y_0 \exp\left\{\int_0^t \left[-d(s) + f(s) - \frac{\beta^2(s)}{2}\right] ds - \int_0^t \beta(s) dB_2(s)\right\}. \end{gathered} \tag{2.11}$$

By using the comparison theorem of stochastic equations again, we obtain

$$\underline{y}(t) \leq y(t) \leq \overline{y}(t) \quad \text{a.s. } t \in [0, \tau_e). \tag{2.12}$$

From the representation of solutions $\phi(t)$, $\Phi(t)$, $\underline{y}(t)$, and $\overline{y}(t)$, we can easily see that they exist on $t \in [0, \infty)$, that is, $\tau_e = \infty$. Therefore, we get the following theorem.

**Theorem 2.2.** *For any initial value $X_0 \in R_+^2$, there is a unique positive solution $X(t)$ to (1.6) on $t \geq 0$ and the solution will remain in $R_+^2$ with probability 1, namely, $X(t) \in R_+^2$ for all $t \geq 0$ a.s. Moreover, there exist functions $\phi(t)$, $\Phi(t)$, $\underline{y}(t)$, and $\overline{y}(t)$ defined as above such that*

$$\phi(t) \leq x(t) \leq \Phi(t), \quad \underline{y}(t) \leq y(t) \leq \overline{y}(t), \quad \text{a.s. } t \geq 0. \tag{2.13}$$

## 3. Asymptotic Bounded Properties

In Section 2, we have shown that the solution of (1.6) is positive, which will not explode in any finite time. This nice positive property allows to further discuss asymptotic bounded properties for the solution of (1.6) in this section.

**Lemma 3.1** (see [30]). *Let $\Phi(t)$ be a solution of system* (2.4). *If $b_l > 0$, then*

$$\limsup_{t\to\infty} E[\Phi(t)] \le \frac{a^u}{b_l}. \tag{3.1}$$

Now we show that the solution of system (1.6) with any positive initial value is uniformly bounded in mean.

**Theorem 3.2.** *If $b_l > 0$ and $d_l > 0$, then the solution $X(t)$ of system* (1.6) *with any positive initial value has the following properties:*

$$\limsup_{t\to\infty} E[x(t)] \le \frac{a^u}{b_l}, \quad \limsup_{t\to\infty} E\left[x(t) + \frac{c_l}{f^u} y(t)\right] \le \frac{(a^u + d^u)^2}{4b_l d_l}, \tag{3.2}$$

*that is, it is uniformly bounded in mean. Furthermore, if $c_l > 0$, then*

$$\limsup_{t\to\infty} E\left[y(t)\right] \le \frac{f^u (a^u + d^u)^2}{4b_l c_l d_l}. \tag{3.3}$$

*Proof.* Combining $x(t) \le \Phi(t)$ a.s. with (3.1), it is easy to see that

$$\limsup_{t\to\infty} E[x(t)] \le \frac{a^u}{b_l}. \tag{3.4}$$

Next, we will show that $y(t)$ is bounded in mean. Denote

$$G(t) = x(t) + \frac{c_l}{f^u} y(t). \tag{3.5}$$

Calculating the time derivative of $G(t)$ along system (1.6), we get

$$\begin{aligned}
dG(t) &= x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{m(s)y(t) + x(t)}\right]dt + \alpha(t)x(t)dB_1(t) \\
&\quad + y(t)\left[-\frac{c_l}{f^u}d(t) + \frac{c_l}{f^u}\frac{f(t)x(t)}{m(s)y(t) + x(t)}\right]dt - \frac{c_l}{f^u}\beta(t)y(t)dB_2(t) \\
&= \left\{[a(t) + d(t)]x(t) - b(t)x^2(t) - d(t)G(t) + \left[-c(t) + \frac{c_l}{f^u}f(t)\right]\frac{x(t)y(t)}{m(s)y(t) + x(t)}\right\}dt \\
&\quad + \alpha(t)x(t)dB_1(t) - \frac{c_l}{f^u}\beta(t)y(t)dB_2(t).
\end{aligned} \tag{3.6}$$

Integrating it from 0 to $t$ yields

$$\begin{aligned}
G(t) = G(0) &+ \int_0^t \left\{[a(s) + d(s)]x(s) - b(s)x^2(s) - d(s)G(s)\right. \\
&\qquad \left. + \left[-c(s) + \frac{c_l}{f^u}f(s)\right]\frac{x(s)y(s)}{m(s)y(s) + x(s)}\right\}ds \\
&+ \int_0^t \alpha(s)x(s)dB_1(s) - \int_0^t \frac{c_l}{f^u}\beta(s)dB_2(s),
\end{aligned} \tag{3.7}$$

which implies

$$\begin{aligned}
E[G(t)] &= G(0) + E\int_0^t \left\{[a(s) + d(s)]x(s) - b(s)x^2(s) - d(s)G(s)\right. \\
&\qquad \left. + \left[-c(s) + \frac{c_l}{f^u}f(s)\right]\frac{x(s)y(s)}{m(s)y(s) + x(s)}\right\}ds, \\
\frac{dE[G(t)]}{dt} &= [a(t) + d(t)]E[x(t)] - b(t)E\left[x^2(t)\right] - d(t)E[G(t)] \\
&\quad + \left[-c(t) + \frac{c_l}{f^u}f(t)\right]E\left[\frac{x(t)y(t)}{m(t)y(t) + x(t)}\right] \\
&\le [a(t) + d(t)]E[x(t)] - b(t)E\left[x^2(t)\right] - d(t)E[G(t)] \\
&\le (a^u + d^u)E[x(t)] - b_l(E[x(t)])^2 - d_lE[G(t)].
\end{aligned} \tag{3.8}$$

Obviously, the maximum value of $(a^u + d^u)E[x(t)] - b_lE^2[x(t)]$ is $(a^u + d^u)^2/4b_l$, so

$$\frac{dE[G(t)]}{dt} \le \frac{(a^u + d^u)^2}{4b_l} - d_lE[G(t)]. \tag{3.9}$$

Thus, we get by the comparison theorem that

$$0 \le \limsup_{t\to\infty} E[G(t)] \le \frac{(a^u + d^u)^2}{4b_l d_l}. \tag{3.10}$$

Since the solution of system (1.6) is positive, it is clear that

$$\limsup_{t\to\infty} E\left[y(t)\right] \le \frac{f^u(a^u + d^u)^2}{4b_l c_l d_l}. \tag{3.11}$$

□

*Remark 3.3.* Theorem 3.2 tells us that the solution of (1.6) is uniformly bounded in mean.

*Remark 3.4.* If $a$, $b$, $c$, $d$, and $f$ are positive constant numbers, we will get Theorem 2.1 in [3].

## 4. Stochastic Permanence of (1.6)

For population systems, permanence is one of the most important and interesting characteristics, which mean that the population system will survive in the future. In this section, we firstly give two related definitions and some conditions that guarantee that (1.6) is stochastically permanent.

*Definition 4.1.* Equation (1.6) is said to be stochastically permanent if, for any $\varepsilon \in (0,1)$, there exist positive constants $H = H(\varepsilon)$, $\delta = \delta(\varepsilon)$ such that

$$\liminf_{t\to+\infty} P\{|X(t)| \le H\} \ge 1-\varepsilon, \quad \liminf_{t\to+\infty} P\{|X(t)| \ge \delta\} \ge 1-\varepsilon, \tag{4.1}$$

where $X(t) = (x(t), y(t))$ is the solution of (1.6) with any positive initial value.

*Definition 4.2.* The solutions of (1.6) are called stochastically ultimately bounded, if, for any $\varepsilon \in (0,1)$, there exists a positive constant $H = H(\varepsilon)$ such that the solutions of (1.6) with any positive initial value have the property

$$\limsup_{t\to+\infty} P\{|X(t)| > H\} < \varepsilon. \tag{4.2}$$

It is obvious that if a stochastic equation is stochastically permanent, its solutions must be stochastically ultimately bounded.

**Lemma 4.3** (see [30]). *One has*

$$E\left[\exp\left\{\int_{t_0}^{t} \alpha(s)dB(s)\right\}\right] = \exp\left\{\frac{1}{2}\int_{t_0}^{t} \alpha^2(s)ds\right\}, \quad 0 \le t_0 \le t. \tag{4.3}$$

**Theorem 4.4.** *If $b_l > 0, c_l > 0$, and $d_l > 0$, then solutions of* (1.6) *are stochastically ultimately bounded.*

*Proof.* Let $X(t) = (x(t), y(t))$ be an arbitrary solution of the equation with positive initial. By Theorem 3.2, we know that

$$\limsup_{t\to\infty} E[x(t)] \le \frac{a^u}{b_l}, \quad \limsup_{t\to\infty} E[y(t)] \le \frac{f_l(a^u + d^u)^2}{4b_l c_l d_l}. \tag{4.4}$$

Now, for any $\varepsilon > 0$, let $H_1 > a^u/b_l\varepsilon$ and $H_2 > (a^u + d^u)^2 f^u/4b_l c_l d_l \varepsilon$. Then, by Chebyshev's inequality, it follows that

$$\begin{aligned} P\{x(t) > H_1\} &\le \frac{E[x(t)]}{H_1} < \varepsilon, \\ P\{y(t) > H_2\} &\le \frac{E[y(t)]}{H_2} < \varepsilon. \end{aligned} \tag{4.5}$$

Taking $H = 3\max\{H_1, H_2\}$, we have

$$P\{|X(t)| > H\} \le P\{x(t) + y(t) > H\} \le \frac{E[x(t) + y(t)]}{H} < \frac{2}{3}\varepsilon. \tag{4.6}$$

Hence,

$$\limsup_{t\to\infty} P\{|X(t)| > H\} < \varepsilon. \tag{4.7}$$

This completes the proof of Theorem 4.4. □

**Lemma 4.5.** *Let $X(t)$ be the solution of* (1.6) *with any initial value $X_0 \in R_+^2$. If $r_l > 0$, then*

$$\limsup_{t\to+\infty} E\left[\frac{1}{x(t)}\right] \le \frac{b^u}{r_l}, \tag{4.8}$$

*where $r(t) = a(t) - c(t)/m(t) - \alpha^2(t)$.*

*Proof.* Combing (2.7) with Lemma 4.3, we have

$$\begin{aligned}
E\left[\frac{1}{\phi(t)}\right] &= x_0^{-1}E\left[\exp\left\{-\int_0^t\left[a(s)-\frac{c(s)}{m(s)}-\frac{\alpha^2(s)}{2}\right]ds-\int_0^t\alpha(s)dB_1(s)\right\}\right] \\
&\quad + E\left[\int_0^t b(s)\exp\left\{-\int_s^t\left[a(\tau)-\frac{c(\tau)}{m(\tau)}-\frac{\alpha^2(\tau)}{2}\right]d\tau-\int_s^t\alpha(\tau)dB_1(\tau)\right\}ds\right] \\
&= x_0^{-1}\exp\left\{-\int_0^t\left[a(s)-\frac{c(s)}{m(s)}-\frac{\alpha^2(s)}{2}\right]ds\right\}E\left[\exp\left\{-\int_0^t\alpha(s)dB_1(s)\right\}\right] \\
&\quad + \int_0^t b(s)\exp\left\{-\int_s^t\left[a(\tau)-\frac{c(\tau)}{m(\tau)}-\frac{\alpha^2(\tau)}{2}\right]d\tau\right\}E\left[\exp\left\{-\int_s^t\alpha(\tau)dB_1(\tau)\right\}\right]ds \\
&= x_0^{-1}\exp\left\{-\int_0^t r(s)ds\right\}+\int_0^t b(s)\exp\left\{-\int_s^t r(\tau)d\tau\right\}ds \\
&\le x_0^{-1}e^{-r_l t}+b^u\int_0^t e^{-r_l(t-s)}ds \le x_0^{-1}e^{-r_l t}+\frac{b^u}{r_l}.
\end{aligned} \tag{4.9}$$

From (2.9), it has

$$E\left[\frac{1}{x(t)}\right] \le E\left[\frac{1}{\phi(t)}\right] \le x_0^{-1}e^{-r_l t.}+\frac{b^u}{r_l}. \tag{4.10}$$

This completes the proof of Lemma 4.3. □

**Theorem 4.6.** *Let $X(t)$ be the solution of* (1.6) *with any initial value $X_0 \in R_+^2$. If $b_l > 0$ and $r_l > 0$, then, for any $\varepsilon > 0$, there exist positive constants $\delta = \delta(\varepsilon)$ and $H = H(\varepsilon)$ such that*

$$\liminf_{t\to+\infty} P\{x(t) \le H\} \ge 1-\varepsilon, \quad \liminf_{t\to+\infty} P\{x(t) \ge \delta\} \ge 1-\varepsilon. \tag{4.11}$$

*Proof.* By Theorem 3.2, there exists a positive constant $M$ such that $E[x(t)] \le M$. Now, for any $\varepsilon > 0$, let $H = M/\varepsilon$. Then, by Chebyshev's inequality, we obtain

$$P\{x(t) > H\} \le \frac{E[x(t)]}{H} \le \varepsilon, \tag{4.12}$$

which implies

$$P\{x(t) \le H\} \ge 1-\varepsilon. \tag{4.13}$$

By Lemma 4.3, we know that

$$\limsup_{t\to+\infty} E\left[\frac{1}{x(t)}\right] \le \frac{b^u}{r_l}. \tag{4.14}$$

For any $\varepsilon > 0$, let $\delta = \varepsilon r_l / b^u$; then

$$P\{x(t) < \delta\} = P\left\{\frac{1}{x(t)} > \frac{1}{\delta}\right\} \le \frac{E[1/x(t)]}{1/\delta} \le \delta E[1/x(t)], \tag{4.15}$$

which yields

$$\limsup_{t\to+\infty} P[x(t) < \delta] \le \frac{\delta b^u}{r_l} = \varepsilon. \tag{4.16}$$

This implies

$$\liminf_{t\to+\infty} P[x(t) \ge \delta] \ge 1 - \varepsilon. \tag{4.17}$$

This completes the proof of Theorem 4.6 □

*Remark 4.7.* Theorem 4.6 shows that if we guarantee $b_l > 0$ and $r_l > 0$, then the prey species $x$ must be permanent. Otherwise, the prey species $x$ may be extinct. Thus the predator species $y$ will be extinct too whose survival is absolutely dependent on $x$. However, if $y$ becomes extinct, then $x$ will not turn to extinct when the noise intensities $\alpha(t)$ are sufficiently small in the sense that $b_l > 0$ and $r_l > 0$.

**Theorem 4.8.** *If $b_l > 0$, $c_l > 0$, $d_l > 0$, and $r_l > 0$, then* (1.6) *is stochastically permanent.*

*Proof.* Assume that $X(t)$ is an arbitrary solution of the equation with initial value $X_0 \in R_+^2$. By Theorem 4.6, for any $\varepsilon > 0$, there exists a positive constant $\delta$ such that

$$\liminf_{t\to+\infty} P\{x(t) \ge \delta\} \ge 1 - \varepsilon. \tag{4.18}$$

Hence,

$$\liminf_{t\to+\infty} P\{|X(t)| \ge \delta\} \ge \liminf_{t\to+\infty} P\{x(t) \ge \delta\} \ge 1 - \varepsilon. \tag{4.19}$$

For any $\varepsilon > 0$, we have by Theorem 4.4 that

$$\liminf_{t\to+\infty} P\{|X(t)| \le H\} \ge 1 - \varepsilon. \tag{4.20}$$

This completes the proof of Theorem 4.8 □

*Remark 4.9.* Theorem 4.8 shows that if we guarantee $b_l > 0$, $c_l > 0$, $d_l > 0$, and $r_l > 0$, (1.6) is permanent in probability, that is, the total number of predators and their preys is bounded in probability.

**Lemma 4.10.** *Assume that $X(t)$ is the solution of* (1.6) *with any initial value $X_0 \in R_+^2$. If $\rho_l > 0$ and $\sigma_l > 0$, then*

$$\limsup_{t\to+\infty} E\left[\frac{1}{y(t)}\right] \le y_0^{-1} + f^u m^u \left[2x_0^{-1} + 2(b^u)^2 \rho_l^{-2}\right]^{1/2}, \tag{4.21}$$

*where $\rho(t) = a(t) - c(t)/m(t) - 3/2\alpha^2(t)$, $\sigma(t) = f(t) - d(t) - 3/2\beta^2(t)$.*

*Proof.* By (2.9), it is easy to have

$$\begin{aligned} dy(t) &= y(t)\left(-d(t) + f(t) - \frac{f(t)m(t)y(t)}{m(t)y(t) + x(t)}\right)dt - \beta(t)y(t)dB_2(t) \\ &\ge y(t)\left(-d(t) + f(t) - \frac{f(t)m(t)y(t)}{x(t)}\right)dt - \beta(t)y(t)dB_2(t) \\ &\ge y(t)\left(-d(t) + f(t) - \frac{f(t)m(t)y(t)}{\phi(t)}\right)dt - \beta(t)y(t)dB_2(t). \end{aligned} \tag{4.22}$$

Let $\Psi(t)$ be the unique solution of equation

$$\begin{gathered} d\Psi(t) = \Psi(t)\left(-d(t) + f(t) - \frac{f(t)m(t)}{\phi(t)}\Psi(t)\right)dt - \beta(t)\Psi(t)dB_2(t), \\ \Psi(0) = y_0. \end{gathered} \tag{4.23}$$

Then, by the comparison theorem of stochastic equations, we have

$$y(t) \ge \Psi(t), \tag{4.24}$$

$$\Psi(t) = \frac{\exp\left\{\int_0^t \left[-d(s) + f(s) - (1/2)\beta^2(s)\right]ds - \int_0^t \beta(s)dB_2(s)\right\}}{y_0^{-1} + \int_0^t \left(f(s)m(s)/\phi(s)\right)\exp\left\{\int_0^s \left[-d(\tau) + f(\tau) - (1/2)\beta^2(\tau)\right]d\tau - \int_0^s \beta(\tau)dB_2(\tau)\right\}ds}. \tag{4.25}$$

So,

$$\begin{aligned} \Psi^{-1}(t) &= y_0^{-1}\exp\left\{\int_0^t \left[d(s) - f(s) + \frac{1}{2}\int_0^t \beta^2(s)\right]ds + \beta(s)dB_2(s)\right\} \\ &\quad + \int_0^t \frac{f(s)m(s)}{\phi(s)}\exp\left\{\int_s^t \left[d(\tau) - f(\tau) + \frac{1}{2}\beta^2(\tau)\right]d\tau + \int_s^t \beta(\tau)dB_2(\tau)\right\}ds. \end{aligned} \tag{4.26}$$

Denote

$$\lambda(t) = d(t) - f(t) + \frac{1}{2}\beta^2(t), \qquad \nu(t) = a(t) - \frac{\alpha^2(t)}{2} - \frac{c(t)}{m(t)}. \tag{4.27}$$

By Lemma 4.3 and Hölder's inequality, it is easy to get that

$$\begin{aligned}
E\left[\Psi^{-1}(t)\right] &= y_0^{-1}\exp\left\{\int_0^t\left[d(s) - f(s) + \beta^2(s)\right]ds\right\} \\
&\quad + \int_0^t f(s)m(s)\exp\left\{\int_s^t \lambda(\tau)d\tau\right\}E\left[\phi^{-1}(s)\exp\left\{\int_s^t \beta(\tau)dB_2(\tau)\right\}\right]ds \\
&\le y_0^{-1}\exp\left\{\int_0^t\left[d(s) - f(s) + \beta^2(s)\right]ds\right\} \\
&\quad + \int_0^t f(s)m(s)\exp\left\{\int_s^t \lambda(\tau)d\tau\right\}\left\{E\left[\phi^{-2}(s)\right]E\left[\exp\left\{2\int_s^t \beta(\tau)dB_2(\tau)\right\}\right]\right\}^{1/2}ds \\
&\le y_0^{-1}\exp\left\{\int_0^t\left[d(s) - f(s) + \beta^2(s)\right]ds\right\} \\
&\quad + \int_0^t f(s)m(s)\exp\left\{\int_s^t\left[d(\tau) - f(\tau) + \frac{3}{2}\beta^2(\tau)\right]d\tau\right\}\left\{E\left[\phi^{-2}(s)\right]\right\}^{1/2}ds.
\end{aligned} \tag{4.28}$$

Combing $(a+b)^2 \le 2(a^2+b^2)$ with (2.7), it follows that

$$\begin{aligned}
E\left[\phi^{-2}(t)\right] &= E\left\{x_0^{-1}\exp\left\{-\int_0^t \nu(s)ds - \int_0^t \alpha(s)dB_1(s)\right\}\right. \\
&\quad \left. + \int_0^t b(s)\exp\left\{-\int_s^t \nu(\tau)d\tau - \int_s^t \alpha(\tau)dB_1(\tau)\right\}ds\right\}^2 \\
&\le 2x_0^{-2}E\left\{\exp\left\{-2\int_0^t \nu(s)ds - 2\int_0^t \alpha(s)dB_1(s)\right\}\right\} \\
&\quad + 2E\left\{\int_0^t b(s)\exp\left\{-\int_s^t \nu(\tau)d\tau - \int_s^t \alpha(\tau)dB_1(\tau)\right\}ds\right\}^2.
\end{aligned} \tag{4.29}$$

It is easy to compute that

$$
\begin{aligned}
&E\left\{\int_0^t b(s)\exp\left\{-\int_s^t \nu(\tau)d\tau-\int_s^t \alpha(\tau)dB_1(\tau)\right\}ds\right\}^2\\
&\quad= E\left[\int\int_0^t b(s)b(u)\exp\left\{-\int_s^t \nu(\tau)d\tau-\int_s^t \alpha(\tau)dB_1(\tau)\right\}\right.\\
&\qquad\left.\cdot\exp\left\{-\int_u^t \nu(\tau)d\tau-\int_u^t \alpha(\tau)dB_1(\tau)\right\}duds\right]\\
&\quad= \int\int_0^t b(s)b(u)\exp\left\{-\int_s^t \nu(\tau)d\tau\right\}\exp\left\{-\int_u^t \nu(\tau)d\tau\right\}\\
&\qquad\cdot E\left[\exp\left\{-\int_s^t \alpha(\tau)dB_1(\tau)\right\}\exp\left\{-\int_u^t \alpha(\tau)dB_1(\tau)\right\}\right]du\,ds.
\end{aligned}
\tag{4.30}
$$

By Hölder's inequality again,

$$
\begin{aligned}
&E\left[\exp\left\{-\int_s^t \alpha(\tau)dB_1(\tau)\right\}\exp\left\{-\int_u^t \alpha(\tau)dB_1(\tau)\right\}\right]\\
&\quad\le\left\{E\left[\exp\left\{-2\int_s^t \alpha(\tau)dB_1(\tau)\right\}\right]E\left[\exp\left\{-2\int_u^t \alpha(\tau)dB_1(\tau)\right\}\right]\right\}^{1/2}\\
&\quad= \exp\left\{\int_s^t \alpha^2(\tau)d\tau\right\}\exp\left\{\int_u^t \alpha^2(\tau)d\tau\right\}.
\end{aligned}
\tag{4.31}
$$

Substituting (4.31) into (4.30) yields

$$
\begin{aligned}
&E\left\{\int_0^t b(s)\exp\left\{-\int_s^t \nu(\tau)d\tau-\int_s^t \alpha(\tau)dB_1(\tau)\right\}ds\right\}^2\\
&\quad= \left\{\int_0^t b(s)\exp\left\{-\int_s^t \nu(\tau)d\tau+\int_s^t \alpha^2(\tau)d\tau\right\}ds\right\}^2\\
&\quad= \left\{\int_0^t b(s)\exp\left\{-\int_s^t \left[a(\tau)-\frac{c(\tau)}{m(\tau)}-\frac{3}{2}\alpha^2(\tau)\right]d\tau\right\}ds\right\}^2.
\end{aligned}
\tag{4.32}
$$

On the other hand, by (4.29) and (4.32), we get

$$\begin{aligned} E\left[\phi^{-2}(t)\right] &\le 2x_0^{-2}\exp\left\{-2\int_0^t \rho(s)ds\right\} + 2\left\{\int_0^t b(s)\exp\left\{-\int_s^t \rho(\tau)d\tau\right\}ds\right\}^2 \\ &\le 2x_0^{-1}\exp\{-2\rho_l t\} + 2\left(\frac{b^u}{\rho_l}\right)^2 \\ &\le 2x_0^{-1} + 2\left(\frac{b^u}{\rho_l}\right)^2. \end{aligned} \tag{4.33}$$

Finally, substituting (4.33) into (4.28) and noting from (4.24), we obtain the required assertion (4.21). □

By Theorem 3.2 and Lemma 4.10, similar to the proof of Theorem 4.6, we obtain the following result.

**Theorem 4.11.** *Let $X(t)$ be the solution of* (1.6) *with any initial value $X_0 \in R_+^2$. If $b_l > 0$, $c_l > 0$, $d_l > 0$, $\rho_l > 0$, and $\sigma_l > 0$, then, for any $\varepsilon > 0$, there exist positive constants $\delta = \delta(\varepsilon)$, $H = H(\varepsilon)$ such that*

$$\liminf_{t\to+\infty} P\{y(t) \le H\} \ge 1-\varepsilon, \qquad \liminf_{t\to+\infty} P\{y(t) \ge \delta\} \ge 1-\varepsilon. \tag{4.34}$$

*Remark 4.12.* Theorem 4.11 shows that if $b_l > 0$, $c_l > 0$, $d_l > 0$, $\rho_l > 0$, and $\sigma_l > 0$, then the predator species $y$ must be permanent in probability. This implies that species prey $x$ and (1.6) are permanent in probability. In other words, the predator species $y$ and species prey $x$ in (1.6) are both permanent in probability.

*Remark 4.13.* Obviously, system (1.4) is a special case of system (1.6). If $a-(3/2)\alpha^2-(c/m) > 0$ and $f-d-(3/2)\beta^2 > 0$, then, by Theorem 3.3 in [3], (1.4) is persistent in mean, but, by our Theorem 4.11, the predator species $y$ and species prey $x$ in (1.4) are both stochastically permanent.

## 5. Conclusions

In this paper, by the comparison theorem of stochastic equations and the Itô formula, some results are established such as the stochastically ultimate boundedness and stochastic permanence for a stochastic ratio-dependent predator-prey model with variable coefficients. It is seen that several results in this paper extend and improve the earlier publications (see Remark 3.4).

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## References

[1] I. Miranda Cabrera, H. L. Baños Díaz, and M. L. Martínez, "Dynamical system and nonlinear regression for estimate host-parasitoid relationship," *Journal of Applied Mathematics*, Article ID 851037, 10 pages, 2010.

[2] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, vol. 57, Marcel Dekker, New York, NY, USA, 1980.

[3] C. Ji, D. Jiang, and X. Li, "Qualitative analysis of a stochastic ratio-dependent predator-prey system," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1326–1341, 2011.

[4] S. A. Hadley and L. K. Forbes, "Dynamical systems analysis of a five-dimensional trophic foodweb model in the Southern Oceans," *Journal of Applied Mathematics*, vol. 2009, Article ID 575047, 17 pages, 2009.

[5] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, vol. 191, Academic Press, Boston, Mass, USA, 1993.

[6] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.

[7] R. Arditi, L. R. Ginzburg, and H. R. Akcakaya, "Variation in plankton densities among lakes: a case for ratio-dependent models," *The American Naturalist*, vol. 138, no. 5, pp. 1287–1296, 1991.

[8] A. P. Gutierrez, "The physiological basis of ratio-dependent predator-prey theory: ametabolic pool model of Nicholson's blowies as an example," *Ecology*, vol. 73, no. 5, pp. 1552–1563, 1992.

[9] R. Arditi, N. Perrin, and H. Saiah, "Functional responses and heterogeneities: an experimental test with cladocerans," *Oikos*, vol. 60, no. 1, pp. 69–75, 1991.

[10] R. Arditi and H. Saiah, "Empirical evidence of the role of heterogeneity in ratio-dependent consumption," *Ecology*, vol. 73, no. 5, pp. 1544–1551, 1992.

[11] I. Hanski, "The functional response of predators: worries about scale," *Trends in Ecology and Evolution*, vol. 6, no. 5, pp. 141–142, 1991.

[12] E. Beretta and Y. Kuang, "Global analyses in some delayed ratio-dependent predator-prey systems," *Nonlinear Analysis*, vol. 32, no. 3, pp. 381–408, 1998.

[13] F. Berezovskaya, G. Karev, and R. Arditi, "Parametric analysis of the ratio-dependent predator-prey model," *Journal of Mathematical Biology*, vol. 43, no. 3, pp. 221–246, 2001.

[14] M. Fan and K. Wang, "Periodicity in a delayed ratio-dependent predator-prey system," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 1, pp. 179–190, 2001.

[15] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, "Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 42, no. 6, pp. 489–506, 2001.

[16] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, "Rich dynamics of a ratio-dependent one-prey two-predators model," *Journal of Mathematical Biology*, vol. 43, no. 5, pp. 377–396, 2001.

[17] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 36, no. 4, pp. 389–406, 1998.

[18] R. Xu and L. Chen, "Persistence and stability for a two-species ratio-dependent predator-prey system with time delay in a two-patch environment," *Computers & Mathematics with Applications*, vol. 40, no. 4-5, pp. 577–588, 2000.

[19] R. Xu and L. Chen, "Persistence and global stability for $n$-species ratio-dependent predator-prey system with time delays," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 1, pp. 27–43, 2002.

[20] M. Carletti, K. Burrage, and P. M. Burrage, "Numerical simulation of stochastic ordinary differential equations in biomathematical modelling," *Mathematics and Computers in Simulation*, vol. 64, no. 2, pp. 271–277, 2004.

[21] E. Renshaw, *Modelling Biological Populations in Space and Time*, vol. 11, Cambridge University Press, Cambridge, UK, 1991.

[22] B. Dai, H. Su, and D. Hu, "Periodic solution of a delayed ratio-dependent predator-prey model with monotonic functional response and impulse," *Nonlinear Analysis*, vol. 70, no. 1, pp. 126–134, 2009.
[23] X. Song and Y. Li, "Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect," *Nonlinear Analysis*, vol. 9, no. 1, pp. 64–79, 2008.
[24] L.-L. Wang and Y.-H. Fan, "Note on permanence and global stability in delayed ratio-dependent predator-prey models with monotonic functional responses," *Journal of Computational and Applied Mathematics*, vol. 234, no. 2, pp. 477–487, 2010.
[25] F. Zhang and C. Zheng, "Positive periodic solutions for the neutral ratio-dependent predator-prey model," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2221–2226, 2011.
[26] D.-S. Tian and X.-W. Zeng, "Existence of at least two periodic solutions of a ratio-dependent predator-prey model with exploited term," *Acta Mathematicae Applicatae Sinica*, vol. 21, no. 3, pp. 489–494, 2005.
[27] Z. Zhang and Z. Hou, "Existence of four positive periodic solutions for a ratio-dependent predator-prey system with multiple exploited (or harvesting) terms," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1560–1571, 2010.
[28] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Wiley & Sons, New York, NY, USA, 1974.
[29] N. Ikeda, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, The Netherlands, 1981.
[30] D. Jiang and N. Shi, "A note on nonautonomous logistic equation with random perturbation," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 1, pp. 164–172, 2005.