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Research Article

Regularized Mixed Variational-Like Inequalities

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We use auxiliary principle technique coupled with iterative regularization method to suggest and analyze some new iterative methods for solving mixed variational-like inequalities. The convergence analysis of these new iterative schemes is considered under some suitable conditions. Some special cases are also discussed. Our method of proofs is very simple as compared with other methods. Our results represent a significant refinement of the previously known results.

1. Introduction

Variational inequalities are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of variational inequalities have been considered in different directions using a novel and innovative technique. A useful and important generalization of the variational inequalities is called the variational-like inequality, which has been studied and investigated extensively. It has been shown [1–3] that the minimum of the differentiable preinvex (invex) functions on the preinvex sets can be characterized by the variational-like inequalities. Note that the preinvex functions may not be convex functions and the invex sets may not be convex sets. This implies that the concept ofinvxesity plays same roles in the variational-like inequalities as the convexity plays the role in the variational inequalities. We would like to point out that the variational-like inequalities are quite different then variational inequalities in several aspects. For example, one can prove that the variational inequalities are equivalent to the fixed point problems, whereas variational-like inequalities are not equivalent to the fixed point problems. However, if the invex set is equivalent to the convex set, then variational-like inequalities collapse to the variational inequalities. This shows that variational-like inequalities include variational inequalities as a special case. Authors are advised to see the delicate difference between these two different problems. For

other kind of variational inequalities involving two and three operators, see Noor [4–7] and Noor et al. [8–13].

There is a substantial number of numerical methods including the projection technique and its variant forms including the Wiener-Hopf equations, auxiliary principle, and resolvent equations methods for solving variational inequalities and related optimization problems. However, it is known that the projection method, Wiener-Hopf equations, and resolvent equations techniques cannot be extended to suggest and analyze similar iterative methods for solving variational-like inequalities due to the presence of the bifunction $\eta(\cdot,\cdot)$. This fact motivated us to use the auxiliary principle technique of Glowinski et al. [14]. In this technique, one consider an auxiliary problem associated with the original problem. This way, one defines a mapping and shows that this mapping has a fixed point, which is a solution of the original problem. This fact enables us to suggest and analyze some iterative methods for solving the original problem. This technique has been used to suggest and analyze several iterative methods for solving various classes of variational inequalities and their generalizations, see [1, 2, 4–34] and the references therein.

The principle of iterative regularization is also used for solving variational inequalities. It was introduced by Bakušinskiĭ [16] in connection with variational inequalities in 1979. An important extension of this approach is presented by Alber and Ryazantseva [15]. In this approach, the regularized parameter is changed at each iteration which is in contrast with the common practice for parameter identification of using a fixed regularization parameter throughout the minimization process. One can combine these two different techniques for solving the variational inequalities and related optimization problems. This approach was used by Khan and Rouhani [22] and Noor et al. [9, 10] for solving the mixed variational inequalities.

Motivated and inspired by the these activities, we suggest and analyze some iterative algorithms based on auxiliary principle and principle of iterative regularization for solving a class of mixed variational-like inequalities. For the convergence analysis of the explicit version of this iterative algorithm, we use partially relaxed strongly monotone operator which is a weaker condition than strongly monotonicity used by Khan and Rouhani [22]. We also suggest a new implicit iterative algorithm, the convergence of which requires only the monotonicity, which is weaker condition than strongly monotonicity. Results proved in this paper represent a significant improvement of the previously known results. The comparison of these methods with other methods is an interesting problem for future research.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed set in H. Let $f : K \to R$ and $\eta(\cdot,\cdot) : K \times K \to H$ be mappings. First of all, we recall the following well-known results and concepts; see [1–3, 21, 33].

Definition 2.1. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(\cdot,\cdot)$, if

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K,\ t \in [0,1]. \tag{2.1}$$

K is said to be an invex set with respect to $\eta(\cdot,\cdot)$, if K is invex at each $u \in K$. The invex set K is also called η-connected set. Clearly, every convex set is an invex set with $\eta(v,u) = v - u$, for all $u, v \in K$, but the converse is not true; see [3, 33].

From now onwards, $K$is a nonempty closed and invex set in H with respect to $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 2.2. A function $f : K \to R$ is said to be preinvex with respect to $\eta(\cdot,\cdot)$, if

$$f\big(u + t\eta(v,u)\big) \le (1-t)f(u) + tf(v), \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.2}$$

Note that every convex function is a preinvex function, but the converse is not true; see [3, 33].

Definition 2.3. A function f is said to be a strongly preinvex function on K with respect to the function $\eta(\cdot,\cdot)$ with modulus μ, if

$$f\big(u + t\eta(v,u)\big) \le (1-t)f(u) + tf(v) - t(1-t)\mu\big\|\eta(v,u)\big\|^2, \quad \forall v,u \in K,\ t \in [0,1]. \tag{2.3}$$

Clearly, a differentiable strongly preinvex function f is a strongly invex function with constant $\mu > 0$, that is,

$$f(v) - f(u) \ge \big\langle f'(u), \eta(v,u)\big\rangle + \mu\big\|\eta(v,u)\big\|^2, \quad \forall v,u \in K, \tag{2.4}$$

and the converse is also true under certain conditions.

We remark that if $t = 1$, then Definitions 2.2 and 2.3 reduce to

$$f\big(u + \eta(v,u)\big) \le f(v), \quad \forall u,v \in K. \tag{2.5}$$

One can easily show that the minimum of the differentiable preinvex function on the invex set K is equivalent to finding $u \in K$ such that

$$\big\langle f'(u), \eta(v,u)\big\rangle \ge 0, \quad \forall v \in K, \tag{2.6}$$

which is known as the variational-like inequality. This shows that the preinvex functions play the same role in the study of variational-like inequalities as the convex functions play in the theory of variational inequalities. For other properties of preinvex functions, see [3, 30, 33] and the references therein.

Let K be a nonempty closed and invex set in H. For given nonlinear operator $T : K \to H$ and a continuous function $\phi(\cdot)$, we consider the problem of finding $u \in K$ such that

$$\big\langle Tu, \eta(v,u)\big\rangle + \phi(v) - \phi(u) \ge 0, \quad \forall v \in K, \tag{2.7}$$

which is called the *mixed variational-like inequality* introduced and studied by [1]. It has been shown in [1–3] that a minimum of differentiable preinvex functions $f(u)$ on the invex sets in the normed spaces can be characterized by a class of variational-like inequalities (2.7) with $Tu = f'(u)$ where $f'(u)$ is the differential of a preinvex function $f(u)$. This shows that the concept of variational-like inequalities is closely related to the concept of invexity. For the

applications, numerical methods, and other aspects of the mixed variational-like inequalities, see [1, 2, 29] and the references therein.

We note that if $\eta(v,u) = v - u$, then the invex set K becomes the convex set K and problem (2.7) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in K, \tag{2.8}$$

which is known as a *mixed variational inequality*. It has been shown [1–14, 17–35] that a wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems (2.7) and (2.8).

If $\phi(\cdot) = 0$, then problem (2.7) is equivalent to finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K, \tag{2.9}$$

which is known as the variational-like inequality and has been studied extensively in recent years. For $\eta(v,u) = v - u$, the variational-like inequality (2.9) reduces to the original variational inequality, which was introduced and studied by Stampacchi [32] in 1964. For the applications, numerical methods, dynamical system, and other aspects of variational inequalities and related optimization problems, see [1–35] and the references therein.

Definition 2.4. An operator $T : K \to K$ is said to be

(i) η-Monotone, if and only if, $\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0$, for all $u, v \in K$.

(ii) Partially relaxed strongly η-monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu, \eta(v,u)\rangle + \langle Tz, \eta(u,v)\rangle \leq \alpha\|\eta(z,u)\|^2, \quad \forall u, v, z \in K. \tag{2.10}$$

Note that for $z = v$, partially relaxed strong η-monotonicity reduces to η-monotonicity of the operator T. For $\eta(v,u) = v - u$, the invex set K becomes the convex set and consequently Definition 2.4 collapses to the well concept of monotonicity and partial relaxed strongly monotonicity of the operator.

Assumption 2.5. Assume that the bifunction $\eta : K \times K \to H$ satisfies the condition

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u, v, z \in K. \tag{2.11}$$

In particular, it follows that $\eta(u,u) = 0$ and

$$\eta(u,v) + \eta(u,v) = 0, \quad \forall u, v \in H. \tag{2.12}$$

Assumption 2.5 has been used to suggest and analyze some iterative methods for various classes of variational-like inequalities.

3. Auxiliary Principle Technique/Principle of Iterative Regularization

In this section, we will discuss the solution of mixed variational-like inequality (2.7) using its regularized version. We will use auxiliary principle technique [14] coupled with principle of iterative regularization for solving the mixed variational-like inequalities.

For a given $u \in K$ satisfying (2.7), we consider the problem of finding $z \in K$ such that

$$\langle Tw + E'(w) - E'(u),\ \eta(v,w)\rangle + \phi(v) - \phi(w) \geq 0, \quad \forall v \in K. \tag{3.1}$$

Note that, if $w = u$, then (3.1) reduces to (2.7). Using (3.1), we suggest an iterative scheme for solving (2.7). For a given $u \in K$, consider the problem of finding a solution $z \in K$ satisfying the auxiliary variational-like inequality

$$\rho_n\langle Tw + \varepsilon_n w + E'(w) - E'(u), \eta(v,w)\rangle + \rho_n\phi(v) - \rho_n\phi(w) \geq 0, \quad \forall v \in K, \tag{3.2}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive real, and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real such that $\varepsilon_n \to 0$ as $n \to \infty$. Clearly, if $w = u$ and $\varepsilon_n \to 0$ as $n \to \infty$, then w is a solution of (2.7).

Now, we consider the regularized version of (2.7). For a fixed but arbitrary $n \in N$ and for $\varepsilon_n > 0$, find $u_{\varepsilon_n} \in K$ such that

$$\langle Tu_{\varepsilon_n} + \varepsilon_n u_{\varepsilon_n}, \eta(v, u_{\varepsilon_n})\rangle + \phi(v) - \phi(u_{\varepsilon_n}) \geq 0, \quad \forall v \in K. \tag{3.3}$$

Algorithm 3.1. For a given $u_0 \in K$, compute $u_{n+1} \in K$ from the iterative scheme

$$\langle \rho_n(T_n u_{n+1} + \varepsilon_n u_{n+1}) + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle + \rho_n\phi(v) - \rho_n\phi(u_{n+1}) \geq 0, \quad \forall v \in K, \tag{3.4}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive real and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive reals such that $\varepsilon_n \to 0$ as $n \to \infty$.

We now study the convergence analysis of Algorithm 3.1.

Theorem 3.2. *Let T be a monotone operator. For the approximation T_n of T, assume that there exists $\{\delta_n\}$ such that $\delta_n > 0$ such that*

$$\|T_n(u) - T(v)\| \leq c\delta_n(1 + \|\eta(u,v)\|), \quad \forall u \in K, \text{ where } c \text{ is a constant}. \tag{3.5}$$

Also for the sequences $\{\varepsilon_n\}$, $\{\delta_n\}$, and $\{\rho_n\}$, one has

$$\sum_{n=0}^{\infty}\delta_n^2 < \infty, \qquad \sum_{n=0}^{\infty}\left(\rho_n^2 + \delta_n^2\right) < \infty, \qquad \sum_{n=0}^{\infty}\varepsilon_n\rho_n < \infty, \qquad \sum_{n=0}^{\infty}\alpha_n\rho_n < \infty. \tag{3.6}$$

Then the approximate solution u_{n+1} obtained fromAlgorithm 3.1 converges to an exact solution $u \in K$ satisfying (2.7).

Proof. Let $u_{\varepsilon_n} \in K$ satisfying the regularized mixed variational-like inequality (3.3). Then replacing v by u_{n+1} in (3.3), we have

$$\left\langle \rho_n(Tu_{\varepsilon_n} + \varepsilon_n u_{\varepsilon_n}), \eta(u_{n+1}, u_{\varepsilon_n}) \right\rangle + \rho_n \phi(u_{n+1}) - \rho_n \phi(u_{\varepsilon_n}) \geq 0. \tag{3.7}$$

Let $u_{n+1} \in K$ be the approximate solution obtained from (3.4). Replacing v by u_{ε_n}, we have

$$\left\langle \rho_n(T_n u_{n+1} + \varepsilon_n u_{n+1}) + E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle + \rho_n \phi(u_{\varepsilon_n}) - \rho_n \phi(u_{n+1}) \geq 0. \tag{3.8}$$

For the sake of simplicity, we have $T + \varepsilon_n = F_n$ and $F_n + \varepsilon_n = \widetilde{F}_n$ in (3.7) and (3.8), respectively, and then adding the resultant inequalities, we have

$$\left\langle E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \geq \left\langle \rho_n F_n u_{\varepsilon_n} - \rho_n \widetilde{F}_n u_{n+1}, \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle. \tag{3.9}$$

We consider the Bregman function:

$$B(u, w) = E(u) - E(w) - \left\langle E'(u), \eta(w, u) \right\rangle \geq \mu \left\| \eta(w, u) \right\|^2. \tag{3.10}$$

Now

$$\begin{aligned}
B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &= E(u_{\varepsilon_{n-1}}) - E(u_n) - \left\langle E'(u_n), \eta(u_{\varepsilon_{n-1}}, u_n) \right\rangle \\
&\quad - E(u_{\varepsilon_n}) + E(u_{n+1}) + \left\langle E'(u_{n+1}), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\
&= E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\
&\quad + E(u_{n+1}) - E(u_n) - \left\langle E'(u_n), \eta(u_{\varepsilon_{n-1}}, u_{n+1}) \right\rangle - \left\langle E'(u_n), \eta(u_{n+1}, u_n) \right\rangle \\
&\geq E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\
&\quad - \left\langle E'(u_n), \eta(u_{\varepsilon_{n-1}}, u_{n+1}) \right\rangle + \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 \\
&\geq E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\
&\quad + \left\langle E'(u_n), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}}) \right\rangle + \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 \\
&\geq E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \rho_n \left\langle F_n u_{\varepsilon_n} - \widetilde{F}_n u_{n+1}, \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\
&\quad + \left\langle E'(u_n), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}}) \right\rangle + \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 \\
&\geq \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + \mu \left\| \eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n}) \right\|^2 \\
&\quad + \left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}}) \right\rangle \\
&\quad + \rho_n \left\langle F_n u_{\varepsilon_n} - \widetilde{F}_n u_{n+1}, \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle.
\end{aligned} \tag{3.11}$$

Since T is a monotone operator, $F_n = T + \varepsilon_n$ is strongly monotone with constant $(\alpha + \varepsilon_n) = \alpha_n$ (say), we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq{}& \mu\left\|\eta(u_{n+1}, u_n)\right\|^2 + \mu\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2 \\ &+ \left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}})\right\rangle \\ &+ \rho_n\alpha_n\left\|\eta(u_n, u_{\varepsilon_n})\right\|^2 - \rho_n\left\langle \widetilde{F}_n u_n - F_n u_{n+1}, \eta(u_{\varepsilon_n}, u_{n+1})\right\rangle, \end{aligned} \tag{3.12}$$

from which, we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq{}& \mu\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2 + \mu\left\|\eta(u_{n+1}, u_n)\right\|^2 \\ &+ \rho_n\alpha_n\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 + \tau_1 + \tau_2, \end{aligned} \tag{3.13}$$

where

$$\tau_1 = \left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}})\right\rangle. \tag{3.14}$$

Using Lemma 2.1 and Lipschitz continuity of operator E', we have

$$\geq -\frac{\beta^2\varepsilon^2}{2}\left\|\eta(u_n, u_{\varepsilon_n})\right\|^2 - \frac{1}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2. \tag{3.15}$$

Thus

$$\tau_1 \geq -\frac{\varepsilon_n\rho_n}{2}\left\|\eta(u_n, u_{\varepsilon_n})\right\|^2 - \frac{\beta^2}{2\varepsilon_n\rho_n}\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2. \tag{3.16}$$

Solving for τ_2, where

$$\begin{aligned} \tau_2 &\geq -\rho_n\left\langle \widetilde{F}_n u_{n+1} - F_n u_{n+1}, \eta(u_{\varepsilon_n}, u_{n+1})\right\rangle \\ &\geq -\frac{\varepsilon^2\rho_n}{2}\left\|\widetilde{F}_n u_{n+1} - F_n u_n\right\|^2 - \frac{\varepsilon^2\rho_n}{2}\left\|F_n u_n - F_n u_{n+1}\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2. \end{aligned} \tag{3.17}$$

Using (3.5), we obtain

$$\tau_2 \geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[1 + \left\|\eta(u_n, u_{n+1})\right\|\right]^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2, \tag{3.18}$$

where we have used the Lipschitz continuity of $F_n(= T + \varepsilon_n)$ with constant $\gamma_n(= \gamma + \varepsilon_n)$.

Now using Assumption 2.5, we have

$$\begin{aligned}
\tau_2 &\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[1+\left\|\eta(u_n,u_{\varepsilon_n})+\eta(u_{\varepsilon_n},u_{n+1})\right\|\right]^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n,u_{n+1})\right\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2\\
&\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[t+\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|\right]^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n,u_{n+1})\right\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2.\\
&= -c^2\delta_n^2\varepsilon^2\rho_n t^2-c^2\delta_n^2\varepsilon^2\rho_n\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2, \qquad t\geq 1+\left\|\eta(u_n,u_{\varepsilon_n})\right\|\\
&\quad -\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n,u_{n+1})\right\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2.
\end{aligned} \tag{3.19}$$

From (3.13), (3.16) and (3.19), we have

$$\begin{aligned}
B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1}) &\geq \mu\left\|\eta(u_{n+1},u_n)\right\|^2+\mu\left\|\eta(u_{\varepsilon_{n-1}},u_{\varepsilon_n})\right\|^2+\rho_n\alpha_n\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2\\
&\quad -\frac{\varepsilon_n\rho_n}{2}\left\|\eta(u_{n+1},u_{\varepsilon_n})\right\|^2-\frac{\beta^2}{2\varepsilon_n\rho_n}\left\|\eta(u_{\varepsilon_{n-1}},u_{\varepsilon_n})\right\|^2-c^2\delta_n^2\varepsilon^2\rho_n t^2\\
&\quad -c^2\delta_n^2\varepsilon^2\rho_n\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n,u_{n+1})\right\|^2\\
&\quad -\frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2\\
&\geq \left(\mu-\frac{\gamma_n}{2}\right)\left\|\eta(u_{n+1},u_n)\right\|^2+C_1\varepsilon_n\rho_n\left\|\eta(u_{\varepsilon_{n-1}},u_{\varepsilon_n})\right\|^2\\
&\quad +\rho_n\alpha_n\left\|\eta(u_n,u_{\varepsilon_n})\right\|^2\\
&\quad -\frac{\varepsilon_n\rho_n}{2}\left\|\eta(u_n,u_{\varepsilon_n})\right\|^2-\frac{c^2\delta_n^2t^2}{\gamma_n}-C_2\left(\delta_n^2+\rho_n{}^2\right)\left\|\eta(u_{\varepsilon_n},u_{n+1})\right\|^2.
\end{aligned} \tag{3.20}$$

Using conditions (3.6), we have

$$B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1})\geq\left(\mu-\frac{\gamma_n}{2}\right)\left\|\eta(u_{n+1},u_n)\right\|^2. \tag{3.21}$$

If $u_{n+1}=u_n$, it is easily shown that u_n is a solution of the variational-like inequality (2.7).

Otherwise, the assumption $\gamma_n > 2\mu$ implies that $B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\left\|\eta(u_{n+1},u_n)\right\|=0. \tag{3.22}$$

From (3.22), it follows that the sequence $\{u_n\}$ is bounded. Let $\hat{u}\in K$ be a cluster point of the sequence $\{u_n\}$and let the subsequence $\{u_{n_i}\}$ of this sequence converges to $\hat{u}\in K$. Now essentially using the technique of Zhu and Marcotte [35], it can be shown that the entire

sequence $\{u_n\}$ converges to the cluster point $\hat{u} \in K$ satisfying the variational-like inequality (2.7). □

To implement the proximal method, one has to calculate the solution implicitly, which is itself a difficult problem. We again use the auxiliary principle technique to suggest another iterative method, the convergence of which requires only the partially relaxed strongly monotonicity of the operator. For this, we rewrite (3.1) as follows.

For a given $u \in K$, consider the problem of finding $z \in K$such that

$$\langle Tu + E'(z) - E'(u), \eta(v,z)\rangle + \phi(v) - \phi(z) \geq 0, \quad \forall v \in K. \tag{3.23}$$

Note that if $z = u$, then (3.23) reduces to (2.7). Using (3.23), we develop an iterative scheme for solving (2.7).

For a given $u \in K$, consider the problem of finding a solution $z \in K$ satisfying the auxiliary variational-like inequality

$$\rho_n\langle Tu + \varepsilon_n u + E'(z) - E'(u), \eta(v,z)\rangle + \rho_n\phi(v) - \rho_n\phi(z) \geq 0, \quad \forall v \in K, \tag{3.24}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive reals, and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive reals such that $\varepsilon_n \to 0$ as $n \to \infty$.

Note that if $z = u$ and $\varepsilon_n \to 0$ as $n \to \infty$, then z is a solution of (2.7).

Algorithm 3.3. For a given $u_0 \in K$, compute $u_{n+1} \in K$ from the iterative scheme

$$\begin{aligned} \langle \rho_n(T_n u_n + \varepsilon_n u_n) + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle + \rho_n\phi(v) - \rho_n\phi(u_{n+1}) \geq 0, \\ \forall v \in K, \quad n = 0,1,2,\ldots, \end{aligned} \tag{3.25}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive such that $\varepsilon_n \to 0$ as $n \to \infty$.

Using the technique of Theorem 3.2, one can prove the convergence of Algorithm 3.3. We include its proof for the sake of completeness.

Theorem 3.4. *Let T be a partially relaxed strongly monotone operator with constant $\alpha > 0$. For the approximation T_n of T, let* (3.5) *holds. Also for the sequences $\{\varepsilon_n\}$, $\{\delta_n\}$ and $\{\rho_n\}$,* (3.6) *is satisfied. Then the approximate solution u_{n+1} obtained from Algorithm 3.3 converges to an exact solution $u \in K$ satisfying* (2.7).

Proof. Let $u_{\varepsilon_n} \in K$ satisfying the regularized mixed variational-like inequality (3.3), then replacing v by u_{n+1}, we have

$$\langle \rho_n(Tu_{\varepsilon_n} + \varepsilon_n u_{\varepsilon_n}), \eta(u_{n+1}, u_{\varepsilon_n})\rangle + \rho_n\phi(u_{n+1}) - \rho_n\phi(u_{\varepsilon_n}) \geq 0. \tag{3.26}$$

Let $u_{n+1} \in K$ be the approximate solution obtained from (3.25). Replacing v by u_{ε_n}, we have

$$\langle \rho_n(T_n u_n + \varepsilon_n u_n) + E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1})\rangle + \rho_n\phi(u_{\varepsilon_n}) - \rho_n\phi(u_{n+1}) \geq 0. \tag{3.27}$$

For the sake of simplicity, we have $T+\varepsilon_n = F_n$ and $T_n+\varepsilon_n = \widetilde{F}_n$ in (3.26) and (3.27), respectively, and then adding the resultant inequalities, we have

$$\left\langle \rho_n \widetilde{T}_n u_n - \rho_n T_n u_n + E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \geq 0, \tag{3.28}$$

from which, we have

$$\left\langle E'(u_{n+1}) - E'(u_n), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \geq \left\langle \rho_n T_n u_{\varepsilon_n} - \rho_n \widetilde{T}_n u_n, \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle. \tag{3.29}$$

We consider the Bregman function:

$$B(u, w) = E(u) - E(w) - \left\langle E'(u), w - u \right\rangle. \tag{3.30}$$

Now, we investigate the difference. Using the strongly preinvexity of E, we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &= E(u_{\varepsilon_{n-1}}) - E(u_n) - \left\langle E'(u_n), \eta(u_{\varepsilon_{n-1}}, u_n) \right\rangle \\ &\quad - E(u_{\varepsilon_n}) + E(u_{n+1}) + \left\langle E'(u_{n+1}), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\ &\geq E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}), \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle \\ &\quad - \left\langle E'(u_n), \eta(u_{\varepsilon_{n-1}}, u_{n+1}) \right\rangle + \mu \left\| \eta(u_{n+1}, u_n) \right\|^2. \end{aligned} \tag{3.31}$$

Since T is partially relaxed strongly monotone with constant $\alpha > 0$, $T_n = T + \varepsilon_n$ is partially relaxed strongly monotone with constant $(\alpha + \varepsilon_n/4) = \alpha_n$ (say), we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &\geq \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + \mu \left\| \eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n}) \right\|^2 \\ &\quad + \left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}}) \right\rangle \\ &\quad + \rho_n \alpha_n \left\| \eta(u_n, u_{\varepsilon_n}) \right\|^2 - \rho_n \left\langle \widetilde{T_n} u_n - T_n u_n, \eta(u_{\varepsilon_n}, u_{n+1}) \right\rangle. \end{aligned} \tag{3.32}$$

From which, we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &\geq \mu \left\| \eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n}) \right\|^2 + \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 \\ &\quad + \rho_n \alpha_n \left\| \eta(u_n, u_{\varepsilon_n}) \right\|^2 + \tau_1 + \tau_2, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} \tau_1 &= \left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_n}, u_{\varepsilon_{n-1}}) \right\rangle, \\ &= -\left\langle E'(u_n) - E'(u_{\varepsilon_n}), \eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n}) \right\rangle \\ &\geq -\frac{\varepsilon^2}{2} \left\| E'(u_n) - E'(u_{\varepsilon_n}) \right\|^2 - \frac{1}{2\varepsilon^2} \left\| \eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n}) \right\|^2. \end{aligned} \tag{3.34}$$

Using Lipschitz continuity of operator E', we have

$$\tau_1 \geq -\frac{\beta^2\varepsilon^2}{2}\left\|\eta(u_n, u_{\varepsilon_n})\right\|^2 - \frac{1}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2. \tag{3.35}$$

Put $\varepsilon = \sqrt{\varepsilon_n\rho_n/\beta^2}$, we have

$$\tau_1 \geq -\frac{\varepsilon_n\rho_n}{2}\left\|\eta(u_n, u_{\varepsilon_n})\right\|^2 - \frac{\beta^2}{2\varepsilon_n\rho_n}\left\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\right\|^2. \tag{3.36}$$

Solving for τ_2, where

$$\begin{aligned}
\tau_2 &\geq -\rho_n\left\langle \widetilde{T_n}u_n - T_nu_n, \eta(u_{\varepsilon_n}, u_{n+1})\right\rangle \\
&\geq -\frac{\varepsilon^2\rho_n}{2}\left\|\widetilde{T_n}u_n - T_nu_n\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 \\
&\geq -\frac{\varepsilon^2\rho_n}{2}\left\|\widetilde{T_n}u_n - T_nu_{n+1}\right\|^2 - \frac{\varepsilon^2\rho_n}{2}\left\|T_nu_{n+1} - T_nu_n\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2.
\end{aligned} \tag{3.37}$$

Using (3.5), we obtain

$$\tau_2 \geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[1 + \left\|\eta(u_n, u_{n+1})\right\|\right]^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2, \tag{3.38}$$

where we have used the Lipschitz continuity of $T_n(= T + \varepsilon_n)$ with constant $\gamma_n(= \gamma + \varepsilon_n)$.

Now using Assumption 2.5, we have, for any $t \geq 1 + \|\eta(u_n, u_{\varepsilon_n})\|$,

$$\begin{aligned}
\tau_2 &\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[1 + \left\|\eta(u_n, u_{\varepsilon_n}) + \eta(u_{\varepsilon_n}, u_{n+1})\right\|\right]^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 \\
&\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[t + \left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|\right]^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 \\
&= \frac{-c^2\delta_n^2\varepsilon^2\rho_nt^2}{2} - \frac{c^2\delta_n^2\varepsilon^2\rho_n}{2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 + \frac{c^2\delta_n^2\varepsilon^2\rho_n}{2}\left[-\frac{1}{2}\|t\|^2 - \frac{1}{2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2\right] \\
&\quad - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 \\
&= -c^2\delta_n^2\varepsilon^2\rho_nt^2 - c^2\delta_n^2\varepsilon^2\rho_n\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\left\|\eta(u_n, u_{n+1})\right\|^2 - \frac{\rho_n}{2\varepsilon^2}\left\|\eta(u_{\varepsilon_n}, u_{n+1})\right\|^2.
\end{aligned} \tag{3.39}$$

Combining all the results above, we have

$$\begin{aligned}
B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq{}& \mu\|\eta(u_{n+1}, u_n)\|^2 + \mu\|\eta(u_{n+1}, u_n)\|^2 + \rho_n\alpha_n\|\eta(u_n, u_{\varepsilon_n})\|^2 \\
&- \frac{\varepsilon_n\rho_n}{2}\|\eta(u_n, u_{\varepsilon_n})\|^2 - \frac{\beta^2}{2\varepsilon_n\rho_n}\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\|^2 - c^2\delta_n^2\varepsilon^2\rho_n t^2 \\
&- c^2\delta_n^2\varepsilon^2\rho_n\|\eta(u_{\varepsilon_n}, u_{n+1})\|^2 - \frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|\eta(u_n, u_{n+1})\|^2 \\
&- \frac{\rho_n}{2\varepsilon^2}\|\eta(u_{\varepsilon_n}, u_{n+1})\|^2.
\end{aligned} \tag{3.40}$$

Taking $\varepsilon^2 = 1/\gamma_n\rho_n$, we have

$$\begin{aligned}
B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq{}& \mu\|\eta(u_{n+1}, u_n)\|^2 + \mu\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\|^2 + \rho_n\alpha_n\|\eta(u_n, u_{\varepsilon_n})\|^2 \\
&- \frac{\varepsilon_n\rho_n}{2}\|\eta(u_n, u_{\varepsilon_n})\|^2 - \frac{\beta^2}{2\varepsilon_n\rho_n}\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\|^2 - \frac{c^2\delta_n^2 t^2}{\gamma_n} \\
&- \frac{c^2\delta_n^2}{\gamma_n}\|\eta(u_{\varepsilon_n}, u_{n+1})\|^2 - \frac{\gamma_n}{2}\|\eta(u_n, u_{n+1})\|^2 - \frac{{\rho_n}^2\gamma_n}{2}\|\eta(u_{\varepsilon_n}, u_{n+1})\|^2 \\
\geq{}& \left(\mu - \frac{\gamma_n}{2}\right)\|\eta(u_{n+1}, u_n)\|^2 + C_1\varepsilon_n\rho_n\|\eta(u_{\varepsilon_{n-1}}, u_{\varepsilon_n})\|^2 \\
&+ \rho_n\alpha_n\|\eta(u_n, u_{\varepsilon_n})\|^2 \\
&- \frac{\varepsilon_n\rho_n}{2}\|\eta(u_n, u_{\varepsilon_n})\|^2 - \frac{c^2\delta_n^2 t^2}{\gamma_n} - C_2\left(\delta_n^2 + {\rho_n}^2\right)\|\eta(u_{\varepsilon_n}, u_{n+1})\|^2.
\end{aligned} \tag{3.41}$$

Using conditions (3.6), we have

$$B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq \left(\mu - \frac{\gamma_n}{2}\right)\|\eta(u_{n+1}, u_n)\|^2. \tag{3.42}$$

If $u_{n+1} = u_n$, then it can easily shown that u_n is a solution of the variational-like inequality (2.7). Otherwise, the assumption $\gamma_n > 2\mu$ implies that $B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|\eta(u_{n+1}, u_n)\| = 0. \tag{3.43}$$

From (3.43), it follows that the sequence $\{u_n\}$ is bounded. Let $\hat{u} \in K$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_{n_i}\}$ of this sequence converges to $\hat{u} \in K$. Now essentially using the technique of Zhu and Marcotte [35], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\hat{u} \in K$ satisfying the variational-like inequality (2.7). □

4. Conclusion

In this paper, we have suggested and analyzed some new iterative methods for solving the regularized mixed variational-like inequalities. We have also discussed the convergence analysis of the suggested iterative methods under some suitable and weak conditions. Results proved in this are new and original ones. We hope to extend the idea and technique of this paper for solving invex equilibrium problems and this is the subject of another paper.

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