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Research Article

Dual Hesitant Fuzzy Sets

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In recent decades, several types of sets, such as fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, type 2 fuzzy sets, type n fuzzy sets, and hesitant fuzzy sets, have been introduced and investigated widely. In this paper, we propose dual hesitant fuzzy sets (DHFSs), which encompass fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and fuzzy multisets as special cases. Then we investigate the basic operations and properties of DHFSs. We also discuss the relationships among the sets mentioned above, use a notion of nested interval to reflect their common ground, then propose an extension principle of DHFSs. Additionally, we give an example to illustrate the application of DHFSs in group forecasting.

1. Introduction

Since Zadeh [1] introduced fuzzy sets (FSs) and gave intensive research [2–5], several famous extensions have been developed, such as intuitionistic fuzzy sets (IFSs) [6], type 2 fuzzy sets (T2FSs) [3, 7], type n fuzzy sets (TnFSs), fuzzy multisets (FMSs) [8–14], interval-valued fuzzy sets (IVFSs) [3, 15], interval-valued intuitionistic fuzzy sets (IVIFSs) [16], and hesitant fuzzy sets (HFSs) [17–20]. Actually, these sets have given various ways to assign the membership degree or the nonmembership degree of an element to a given set characterized by different properties.

IFSs, also known as IVFSs from a mathematical point of view, can be modeled with two functions that define an interval to reflect some uncertainty on the membership function of the elements. IVFSs are the generalization of FSs and can model uncertainty due to the lack of information, in which a closed subinterval of $[0, 1]$ is assigned to the membership degree. Atanassov and Gargov [16] proved that IFSs and IVFSs are equipollent generalizations of FSs, and proposed the notion of IVIFS, which has been studied and used extensively [21–25].

T2FSs, described by membership functions that are characterized by more parameters, permit the fuzzy membership as a fuzzy set improving the modeling capability than the

original one. Mathematically, IFSs can be seen as a particular case of T2FSs, where the membership function returns a set of crisp intervals. Despite the wide applications of T2FSs [26–29], they have difficulties in establishing the secondary membership functions and difficulties in manipulation [30–32].

FMSs are another generalization of FSs that permit multiple occurrences of an element, and correspond to the case where the membership degrees to the multisets are not Boolean but fuzzy. Note that although the features of FMSs allow the application to information retrieval on the world wide web, where a search engine retrieves multiple occurrences of the same subjects with possible different degrees of relevance [10], they have problems with the basic operations, such as the definitions for union and intersection, which do not generalize the ones for FSs. Miyamoto [13] gave an alternative definition that emphasizes the usefulness of a commutative property between a set operation and an α-*cut*, resolving this problem [8–13].

HFSs were originally introduced by Torra [17, 18]. The motivation to propose the HFSs is that when defining the membership of an element, the difficulty of establishing the membership degree is not a margin of error (as in IFSs), or some possibility distribution (as in T2FSs) on the possible values, but a set of possible values. Torra [17] reviewed IFSs and FMSs, drew comparisons, and created inherent connections among them. He pointed out that the operations for FMSs do not apply correctly to HFSs, although in some situations we can use FMSs as a model. HFSs were deemed IFSs when the HFS is a nonempty closed interval. Based on the relationships between IFSs and HFSs, Torra [17] gave a definition corresponding to the envelope of HFS. Xu and Xia [19, 20] investigated the aggregation operators, distance, and similarity measures for HFSs and applied them to decision making.

In this paper, we introduce dual hesitant fuzzy set (DHFS), which is a new extension of FS. As we know, in natural language, many categories cannot be distinguished clearly, but can be represented by a matter of degree in the notion of fuzziness. For example, when we talk about fish and monkeys, clear separation can be recognized between them. However, the borderline may not be easy to be distinguished with respect to starfish or bacteria. Although sometimes humans cannot recognize an object clearly, this class of fuzzy recognition against preciseness plays a vital role in human thinking, pattern recognition, and communication of information. The FS, which is stuck into the transition between the membership and the nonmembership, is the gradualness of predication. Zadeh's original intuition [1] is to show the objectivity of truth as "gradual rather than abrupt." Atanassov's IFS [6] used two functions to handle the membership and the nonmembership separately, as it seems to be the case in the human brain, which is limited by the perception of shades. The membership and the nonmembership represent the opposite epistemic degrees, apparently, the membership comes to grips with epistemic certainty, and the nonmembership comes to grips with epistemic uncertainty, they can reflect the gradual epistemic degrees respectively to be the bipolar notions. Similar to HFSs, we can also use the nonmembership to deal with a set of possible values manifesting either a precise gradual composite entity or an epistemic construction refereeing to an ill-known object.

Furthermore, DHFSs consist of two parts, that is, the membership hesitancy function and the nonmembership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of hesitancy in this situation. The existing sets, including FSs, IFSs, HFSs, and FMSs, can be regarded as special cases of DHFSs; we do not confront an interval of possibilities (as in IVFSs or IVIFSs), or some possibility distributions (as in T2FSs) on the possible values, or multiple occurrences of an element (as in FMSs), but several different possible values indicate the epistemic

degrees whether certainty or uncertainty. For example, in a multicriteria decision-making problem, some decision makers consider as possible values for the membership degree of x into the set A a few different values 0.1, 0.2, and 0.3, and for the nonmembership degrees 0.4, 0.5 and 0.6 replacing just one number or a tuple. So, the certainty and uncertainty on the possible values are somehow limited, respectively, which can reflect the original information given by the decision makers as much as possible. Utilizing DHFSs can take much more information into account, the more values we obtain from the decision makers, the greater epistemic certainty we have, and thus, compared to the existing sets mentioned above, DHFS can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments.

We organize the remainder of the paper as follows. In Section 2, we review some basic knowledge of the existing sets. Section 3 proposes DHFSs and investigates some of their basic operations and properties. Then, in Section 4, we present an extension principle of DHFSs, and give some examples to illustrate our results. Section 5 ends the paper with the concluding remarks.

2. Preliminaries

2.1. FSs, T2FSs, TnFSs, and FMSs

In this section, we review some basic definitions and operations, necessary to understand the proposal of the DHFS and its use.

Definition 2.1 (see [1]). Given a reference set X, a fuzzy set (FS) A on X is in terms of the function $\mu\colon X \to [0,1]$.

Definition 2.2 (see [18]). Let M be the set of all fuzzy sets on $\mu\colon [0,1] \to [0,1]$.

Definition 2.3 (see [18]). Let M^i be the set of all fuzzy sets of type i on $[0,1]$. That is, the set of all $\mu^i\colon [0,1] \to M^{i-1}$, where M^1 is defined as M.

Apparently, if we use the functions μ^2 or μ^n to replace the membership function μ, and returns an FS, then we obtain the notions of type 2 fuzzy sets (T2FSs) and type n fuzzy sets (TnFSs).

Yager [14] and Miyamoto [8–13] first studied FMSs and defined several basic operations. FMSs generalize the multisets, which are also known as bags allowing multiple occurrences of elements, associating with the membership degrees.

Definition 2.4 (see [14]). Let A and B be two multisets, and a the element in the reference set, then

(1) addition: $\mathrm{count}_{A\oplus B}(a) = \mathrm{count}_A(a) + \mathrm{count}_B(a)$;

(2) union: $\mathrm{count}_{A\cup B}(a) = \max(\mathrm{count}_A(a), \mathrm{count}_B(a))$;

(3) intersection: $\mathrm{count}_{A\cap B}(a) = \min(\mathrm{count}_A(a), \mathrm{count}_B(a))$.

However, this definition comes into conflict with FSs. Miyamoto [8–13] gave the corresponding solutions. He proposed an alternative definition for FMSs. Using the membership sequence $\mathrm{Seq}_A(a)$ (the membership values a in a fuzzy multiset A) in decreasing order to define the union and intersection operators.

2.2. IFSs

Atanassov [6] gave the definition of IFSs as follows.

Definition 2.5 (see [6]). Let X be a fixed set, an intuitionistic fuzzy set (IFS) A on X is represented in terms of two functions μ: $X \to [0,1]$ and ν: $X \to [0,1]$, with the condition $0 \le \mu(x) + \nu(x) \le 1$, for all $x \in X$.

We use $\langle x, \mu_A, \nu_A \rangle$ for all $x \in X$ to represent IFSs considered in the rest of the paper without explicitly mentioning it.

Furthermore, $\pi(x) = 1 - \mu(x) - \nu(x)$ is called a hesitancy degree or an intuitionistic index of x in A. In the special case $\pi(x) = 0$, that is, $\mu(x) + \nu(x) = 1$, the IFS A reduces to an FS.

Atanassov [6] and De et al. [33] gave some basic operations on IFSs, which ensure that the operational results are also IFSs.

Definition 2.6 (see [6]). Let a set X be fixed, and let A (represented by the functions μ_A and ν_A), A_1 (μ_{A_1} and ν_{A_1}), A_2 (μ_{A_2} and ν_{A_2}), be three IFSs. Then the following operations are valid:

(1) complement: $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$;

(2) union: $A_1 \cap A_2 = \{\langle x, \min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \max\{\nu_{A_1}(x), \nu_{A_2}(x)\} \rangle \mid x \in X\}$;

(3) intersection: $A_1 \cup A_2 = \{\langle x, \max\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\nu_{A_1}(x), \nu_{A_2}(x)\} \rangle \mid x \in X\}$;

(4) $\oplus$-union: $A_1 \oplus A_2 = \{\langle x, \mu_{A_1}(x) + \mu_{A_2}(x) - \mu_{A_1}(x)\mu_{A_2}(x), \nu_{A_1}(x)\nu_{A_2}(x) \rangle \mid x \in X\}$;

(5) $\otimes$-intersection: $A_1 \otimes A_2 = \{\langle x, \mu_{A_1}(x)\mu_{A_2}(x), \nu_{A_1}(x) + \nu_{A_2}(x) - \nu_{A_1}(x)\nu_{A_2}(x) \rangle \mid x \in X\}$.

De et al. [33] further gave another two operations of IFSs:

(6) $nA = \{\langle x, 1 - (1 - \mu_A(x))^n, (\nu_A(x))^n \rangle \mid x \in X\}$;

(7) $A^n = \{\langle x, (\mu_A(x))^n, 1 - (1 - \nu_A(x))^n \rangle \mid x \in X\}$, where n is a positive integer,

Atanassov and Gargov [16] used the following:

(1) the map f assigns to every IVFS $A(=[\mu_{AL}(x), \mu_{AU}(x)])$ an IFS, $B = f(A)$ given by $\mu_B(x) = \mu_{AL}(x)$, $\nu_A(x) = 1 - \mu_{AU}(x)$;

(2) the map g assigns to every IFS $B(=\langle \mu_A(x), \nu_A(x) \rangle)$ an IVFS $A = f(B)$ given by $\mu_A(x) = [\mu_A(x), 1 - \nu_A(x)]$, to prove that IFSs and IVFSs are equipollent generalizations of the notion of FSs.

Xu and Yager [34] called each pair $(\mu_A(x), \nu_A(x))$ an intuitionistic fuzzy number (IFN), and, for convenience, denoted an IFN by $\alpha = (\mu_\alpha, \nu_\alpha)$. Moreover, they gave a simple method to rank any two IFNs, and introduced some of their operational laws as follows.

Definition 2.7 (see [34]). Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1,2)$ be any two IFNs, $s_{\alpha_i} = \mu_{\alpha_i} - \nu_{\alpha_i}(i = 1,2)$ the scores of α_i $(i = 1,2)$, respectively, and $h_{\alpha_i} = \mu_{\alpha_i} + \nu_{\alpha_i}(i = 1,2)$ the accuracy degrees of α_i $(i = 1,2)$, respectively, then

(i) if $s_{\alpha_1} > s_{\alpha_2}$, then α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$;

(ii) if $s_{\alpha_1} = s_{\alpha_2}$, then

(1) if $h_{\alpha_1} = h_{\alpha_2}$, then α_1 and α_2 represent the same information, that is, $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, denoted by $\alpha_1 = \alpha_2$;

(2) if $h_{\alpha_1} > h_{\alpha_2}$, then α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$.

2.3. HFSs

Torra [17] defined the HFS in terms of a function that returns a set of membership values for each element in the domain and in terms of the union of their memberships.

Definition 2.8 (see [17]). Let X be a fixed set, then we define hesitant fuzzy set (HFS) on X is in terms of a function h applied to X returns a subset of $[0,1]$, and $h(x)$ a hesitant fuzzy element (HFE).

Then, Torra [17] gave an example to show several special sets for all x in X: (1) empty set: $h(x) = \{0\}$; (2) full set: $h(x) = \{1\}$; (3) complete ignorance: $h(x) = [0,1]$; (4) nonsense set: $h(x) = \emptyset$.

Apparently, this definition encompasses IFSs as a particular case in the form of a nonempty closed interval, and also a particular case of T2FSs from a mathematical point of view.

Torra and Narukawa [18] and Torra [17] showed that the envelop of a HFE is an IFN, expressed in the following definition.

Definition 2.9 (see [17, 18]). Given an HFE h, the pair of functions $h^- = \mu(h^- = \min\{\gamma \mid \gamma \in h\})$ and $1 - h^+ = \nu(h^+ = \max\{\gamma \mid \gamma \in h\})$ define an intuitionistic fuzzy set $(\langle x, \mu, \nu\rangle)$, denoted by $A_{\text{env}}(h)$.

According to this definition, IFSs can also be represented by HFSs, that is, for a given IFS $\{\langle x, \mu_A(x), \nu_A(x)\rangle\}$, the corresponding HFS is $h(x) = [\mu_A(x), 1 - \nu_A(x)]$, if $\mu_A(x) \neq 1 - \nu_A(x)$.

Torra [17] gave the complement of a HFS as the following.

Definition 2.10 (see [17]). Given an HFS represented by its membership function h, we define its complement as $h^c(x) = \cup_{\gamma \in h(x)}\{1 - \gamma\}$.

Additionally, Torra [17] considered the relationships between HFSs and fuzzy multisets (FMSs). He proved that a HFS can be represented a FMS.

Definition 2.11 (see [17]). Given a HFS A on X, and $h(x)$ for all x in X, then the HFS can be defined as a FMS: $\text{FMS}_A = \oplus_{x \in X} \oplus_{\gamma \in h(x)} \{(x, \gamma)\}$.

Torra [17] also proved that the union and the intersection of two corresponding FMSs do not correspond to the union and the intersection of two HFSs.

Xia and Xu [19] gave a method to rank any two HFEs as the following.

Definition 2.12 (see [19]). For a HFE h, $s(h) = (1/\#h) \sum_{\gamma \in h} \gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. Moreover, for two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

3. DHFSs

3.1. The Notion of DHFS

We now define dual hesitant fuzzy set in terms of two functions that return two sets of membership values and nonmembership values, respectively, for each element in the domain as follows.

Definition 3.1. Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) D on X is described as:

$$D = \{\langle x, h(x), g(x)\rangle x \in X\}, \tag{3.1}$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set D, respectively, with the conditions:

$$0 \le \gamma,\ \eta \le 1, \quad 0 \le \gamma^+ + \eta^+ \le 1, \tag{3.2}$$

where $\gamma \in h(x), \eta \in g(x), \gamma^+ \in h^+(x) = \cup_{\gamma\in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta\in g(x)} \max\{\eta\}$ for all $x \in X$. For convenience, the pair $d(x) = (h(x), g(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$, with the conditions: $\gamma \in h$, $\eta \in g$, $\gamma^+ \in h^+ = \cup_{\gamma\in h} \max\{\gamma\}$, $\eta^+ \in g^+ = \cup_{\eta\in g} \max\{\eta\}$, $0 \le \gamma, \eta \le 1$, and $0 \le \gamma^+ + \eta^+ \le 1$.

First we define some special DHFEs. Given a DHFE, d, then we have

(1) complete uncertainty: $d = \{\{0\}, \{1\}\}$;

(2) complete certainty: $d = \{\{1\}, \{0\}\}$;

(3) complete ill-known (all is possible): $d = [0, 1]$;

(4) nonsensical element: $d = \emptyset (h = \emptyset, g = \emptyset)$.

Based on the background knowledge introduced in Section 2, we can obtain some results in special cases. For a given $d \neq \emptyset$, if h and g have only one value γ and η, respectively, and $\gamma + \eta < 1$, then the DHFS reduces to an IFS. If h and g have only one value γ and η respectively, and $\gamma + \eta = 1$, or h owns one value, and $g = \emptyset$, then the DHFS reduces to an FS (also can be regarded as HFSs). If $g = \emptyset$ and $h \neq \emptyset$, then the DHFS reduces to a HFS, and according to Definition 2.11, DHFSs can be defined as FMSs. Thus the definition of DHFSs encompasses these fuzzy sets above. Next we will discuss the DHFS in detail and use $\gamma^-(\gamma^- \in h^- = \cup_{\gamma\in h(x)} \min\{\gamma\})$, γ^+, $\eta^-(\eta^- \in g^- = \cup_{\eta\in g(x)} \min\{\eta\})$, η^+ in the rest of the paper without explicitly mentioning it.

Actually, for a typical DHFS, h and g can be represented by two intervals as:

$$h = [\gamma^-, \gamma^+], \qquad g = [\eta^-, \eta^+]. \tag{3.3}$$

Based on Definition 2.9, there is a transformation between IFSs and HFSs, we can also transform g to h^2, that is, the number 2 HFE $h^2(x) = [1 - \eta^+, 1 - \eta^-]$ denoting the possible

membership degrees of the element $x \in X$. Thus, both h and h^2 indicate the membership degrees, we can use a "nested interval" to represent $d(x)$ as:

$$d = [[\gamma^-, \gamma^+], [1-\eta^+, 1-\eta^-]]. \tag{3.4}$$

The common ground of these sets is to reflect fuzzy degrees to an object, according to either fuzzy numbers or interval fuzzy numbers. Therefore, we use nonempty closed interval as a uniform framework to indicate a DHFE d, which is divided into different cases as follows:

$$d = \begin{cases} \emptyset, & \text{if } g = \emptyset,\ h = \emptyset, \\ (\gamma), & \begin{cases} \text{if } g = \emptyset,\ h \neq \emptyset,\ \gamma^- = \gamma^+ = \gamma, \\ \text{if } g \neq \emptyset,\ h \neq \emptyset,\ \gamma^- = \gamma^+ = \gamma = 1-\eta^- = 1-\eta^+ = 1-\eta, \end{cases} \\ (1-\eta), & \text{if } g \neq \emptyset,\ h = \emptyset,\ \eta^- = \eta^+ = \eta, \\ [\gamma^-, \gamma^+], & \text{if } g = \emptyset,\ h \neq \emptyset,\ \gamma^- \neq \gamma^+, \\ [1-\eta^+, 1-\eta^-], & \text{if } g \neq \emptyset,\ h = \emptyset,\ \eta^- \neq \eta^+, \\ [\gamma, [1-\eta^+, 1-\eta^-]], & \text{if } g \neq \emptyset,\ h \neq \emptyset,\ \eta^- \neq \eta^+,\ \gamma^- = \gamma^+ = \gamma, \\ [[\gamma^-, \gamma^+], \eta], & \text{if } g \neq \emptyset,\ h \neq \emptyset,\ \gamma^- \neq \gamma^+,\ \eta^- = \eta^+ = \eta, \\ [[\gamma^-, \gamma^+], [1-\eta^+, 1-\eta^-]], & \text{if } g \neq \emptyset,\ h \neq \emptyset,\ \eta^- \neq \eta^+,\ \gamma^- \neq \gamma^+, \end{cases} \tag{3.5}$$

which reflects the connections among all the sets mentioned above, and the merit of DHFS is more flexible to be valued in multifold ways according to the practical demands than the existing sets, taking much more information given by decision makers into account.

3.2. Basic Operations and Properties of DHFSs

Atanassov [6] and Torra [17] gave the complements of the IFSs and the HFSs, respectively, according to Definitions 2.6 and 2.10. In the following, we define the complement of the DHFS depending on different situations.

Definition 3.2. Given a DHFE represented by the function d, and $d \neq \emptyset$, its complement is defined as:

$$d^c = \begin{cases} \cup_{\gamma \in h, \eta \in g} \{\{\eta\}, \{\gamma\}\}, & \text{if } g \neq \emptyset,\ h \neq \emptyset, \\ \cup_{\gamma \in h} \{\{1-\gamma\}, \{\emptyset\}\}, & \text{if } g = \emptyset,\ h \neq \emptyset, \\ \cup_{\eta \in g} \{\{\emptyset\}, \{1-\eta\}\}, & \text{if } h = \emptyset,\ g \neq \emptyset. \end{cases} \tag{3.6}$$

Apparently, the complement is involutive represented as $(d^c)^c = d$.

We now define the union and the intersection of DHFSs. For two DHFSs d_1 and d_2, it is clear that the corresponding lower and upper bounds to h and g are h^-, h^+, g^- and

g^+, respectively, where $h^- = \cup_{\gamma\in h}\min\{\gamma\}$, $h^+ = \cup_{\gamma\in h}\max\{\gamma\}$, $g^- = \cup_{\eta\in g}\min\{\eta\}$, and $g^+ = \cup_{\eta\in g}\max\{\eta\}$ represent this group notations and no confusion will arise in the rest of this paper.

Definition 3.3. Let X be a fixed set, d_1 and d_2 two DHFEs, we define their union and intersection, respectively, as:

(1) $d_1 \cup d_2 = \{h \in (h_1 \cup h_2) \mid h \geq \max(h_1^-, h_2^-), g \in (g_1 \cap g_2) \mid g \leq \min(g_1^+, g_2^+)\}$;

(2) $d_1 \cap d_2 = \{h \in (h_1 \cap h_2) \mid h \leq \min(h_1^+, h_2^+), g \in (g_1 \cup g_2) \mid g \geq \max(g_1^-, g_2^-)\}$,

the following operations are valid:

(1) $\oplus$-union: $d_1 \oplus d_2 = \{h_{d_1} \oplus h_{d_2}, g_{d_1} \otimes g_{d_2}\} = \cup_{\gamma_{d_1}\in h_{d_1},\eta_{d_1}\in g_{d_1},\gamma_{d_2}\in h_{d_2},\eta_{d_2}\in g_{d_2}}\{\{\gamma_{d_1} + \gamma_{d_2} - \gamma_{d_1}\gamma_{d_2}\}, \{\eta_{d_1}\eta_{d_2}\}\}$;

(2) $\oplus$-intersection: $d_1 \otimes d_2 = \{h_{d_1} \otimes h_{d_2}, g_{d_1} \oplus g_{d_2}\} = \cup_{\gamma_{d_1}\in h_{d_1},\eta_{d_1}\in g_{d_1},\gamma_{d_2}\in h_{d_2},\eta_{d_2}\in g_{d_2}}\{\{\gamma_{d_1}\gamma_{d_2}\}, \{\eta_{d_1} + \eta_{d_2} - \eta_{d_1}\eta_{d_2}\}\}$;

(3) $nd = \cup_{\gamma_d\in h_d,\eta_d\in g_d}\{1-(1-\gamma_d)^n, (\eta_d)^n\}$;

(4) $d^n = \cup_{\gamma_d\in h_d,\eta_d\in g_d}\{(\gamma_d)^n, 1-(1-\eta_d)^n\}$, where n is a positive integral and all the results are also DHFEs.

Example 3.4. Let $d_1 = \{\{0.1, 0.3, 0.4\}, \{0.3, 0.5\}\}$ and $d_2 = \{\{0.2, 0.5\}, \{0.1, 0.2, 0.4\}\}$ be two DHFEs, then we have

(1) complement: $d_1^c = \{\{0.3, 0.5\}, \{0.1, 0.3, 0.4\}\}$;

(2) union: $d_1 \cup d_2 = \{\{0, 2, 0.3, 0.4, 0.5\}, \{0.1, 0.2, 0.3, 0.4\}\}$;

(3) intersection: $d_1 \cap d_2 = \{\{0, 1, 0.2, 0.3, 0.4\}, \{0.3, 0.4, 0.5\}\}$.

We can easily prove the following theorem according to Definition 3.3

Theorem 3.5. *Let d, d_1, and d_2 be any three DHFEs, $\lambda \geq 0$, then*

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$;

(2) $d_1 \otimes d_2 = d_2 \otimes d_1$;

(3) $\lambda(d_1 \otimes d_2) = \lambda d_1 \otimes \lambda d_2$;

(4) $(d_1 \otimes d_2)^\lambda = {d_1}^\lambda \otimes {d_2}^\lambda$.

To compare the DHFEs, and based on Definitions 2.7 and 2.12, we give the following comparison laws.

Definition 3.6. Let $d_i = \{h_{d_i}, g_{d_i}\}(i = 1, 2)$ be any two DHFEs, $s_{d_i} = (1/\#h)\sum_{\gamma\in h}\gamma - (1/\#g)\sum_{\eta\in g}\eta(i = 1, 2)$ the score function of $d_i(i = 1, 2)$, and $p_{d_i} = (1/\#h)\sum_{\gamma\in h}\gamma + (1/\#g)\sum_{\eta\in g}\eta(i = 1, 2)$ the accuracy function of $d_i(i = 1, 2)$, where $\#h$ and $\#g$ are the numbers of the elements in h and g, respectively, then

(i) if $s_{d_1} > s_{d_2}$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$;

(ii) if $s_{d_1} = s_{d_2}$, then

(1) if $p_{d_1} = p_{d_2}$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$;

(2) If $p_{d_1} > p_{d_2}$, then d_1 is superior than d_2, denoted by $d_1 \succ d_2$.

Example 3.7. Let $d_1 = \{\{0.1, 0.3\}, \{0.3, 0.5\}\}$ and $d_2 = \{\{0.2, 0.4\}, \{0.4, 0.6\}\}$ be two DHFEs, then based on Definition 3.1, we obtain $s_{d_1} = s_{d_2} = 0$, $p_{d_2}(0.8) > p_{d_1}(0.6)$, and thus, $d_2 \succ d_1$.

4. Extension Principle

Torra and Narukawa [18] introduced an extension principle applied to HFSs, which permits us to export operations on fuzzy sets to new types of sets. The extension of an operator O on a set of HFSs considers all the values in such sets and the application of O on them. The definition is as the following.

Definition 4.1 (see [18]). Let O be a function O: $[0,1]^N \rightarrow [0,1]$, and H a set of N HFSs on the reference set X (i.e., $H = \{h_1, h_2, \ldots, h_N\}$ is a HFS on X). Then, the extension of O on H is defined for each x in X by

$$O_H(x) = \cup_{\gamma \in \{h_1(x) \times h_2(x) \times \cdots \times h_N(x)\}} \{O(\gamma)\}. \tag{4.1}$$

Mesiar and Mesiarova-Zemankova [35] investigated the ordered modular average (OMA), which generalizes the ordered weighted average (OWA) operator, with the replacement of the additivity property by the modularity. The linear interpolating functions of the OWA operator were replaced by rather general nondecreasing functions in the OMA, which can be used to aggregate any finite number of input arguments.

Definition 4.2 (see [35]). Let A: $[0,1]^N \rightarrow [0,1]$ be a modular aggregation function (modular average), $A(x) = \sum_{i=1}^{n} f_i(x_i)$. Then, its symmetrization S_A is called the OMA, that is, $S_A =$ OMA is given by

$$\mathrm{OMA}(x) = \sum_{i=1}^{n} f_i(x_{\sigma(i)}), \tag{4.2}$$

where $f_1, f_2, \ldots, f_n$: $[0,1] \rightarrow [0,1]$ are the nondecreasing functions satisfying $\sum_{i=1}^{n} f_i = \mathrm{id}$ (identity on $[0,1]$).

Motivated by the extension principle of HFSs and the OMA, we propose an extension principle based on the OMA so as to develop basic operators and aggregation operations of DHFS. With respect to DHFSs, the new extension of an operator $\mathbb{Z}$ considers all the values in the DHFSs and the application of $\mathbb{Z}$ on them, which is defined as follows.

Definition 4.3. Let the functions $\mathbb{C}$: $[0,1]^N \rightarrow [0,1]$ and $\mathbb{N}$: $[0,1]^N \rightarrow [0,1]$, and let D be a set of $n$dual hesitant fuzzy sets on the reference set X represented as $D = \{d_1, d_2, \ldots, d_n\} = \{\langle x, \{h(x)\}, \{g(x)\}\rangle\}$. Then, the extension of $\mathbb{Z}$ on D is defined for each x in X by

$$\begin{aligned} \mathbb{Z}_D(x) &= \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x), \ldots, \gamma_n \in h_n(x); \eta_1 \in g_1(x), \eta_2 \in g_2(x), \ldots, \eta_n \in g_n(x)} \\ &= \{\mathbb{C}(f_1(\gamma_1), f_2(\gamma_2), \ldots, f_n(\gamma_n)), \mathbb{N}(f_1(\eta_1), f_2(\eta_2), \ldots, f_n(\eta_n))\}, \end{aligned} \tag{4.3}$$

where $f_i(x)$: $[0,1] \rightarrow [0,1]$, $i = 1, 2, \ldots n$.

If we let $f_i(x) = x (i = 1, 2, \ldots n)$, some basic operations can be obtained according to this extension principle in Example 4.4.

Example 4.4. Let d_1 and d_2 be two DHFEs such that

$$d_1 = \{\{0.1, 0.3, 0.4\}, \{0.3, 0.5\}\}, \qquad d_2 = \{\{0.2, 0.5\}, \{0.1, 0.2, 0.4\}\}. \tag{4.4}$$

If $\mathbb{C} = (\gamma_1, \gamma_2) \max(\gamma_1, \gamma_2)$ and $\mathbb{N}(\eta_1, \eta_2) = \min(\eta_1, \eta_2)$, then

$$\begin{aligned} d_1 \cup d_2 = \mathbb{Z}_{d_1 \cup d_2} &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\max(\gamma_1, \gamma_2), \min(\eta_1, \eta_2)\} \\ &= \{\{0, 2, 0.3, 0.4, 0.5\}, \{0.1, 0.2, 0.3, 0.4\}\}. \end{aligned} \tag{4.5}$$

Similarly, we have

$$\begin{aligned} d_1 \cap d_2 = \mathbb{Z}_{d_1 \cap d_2} &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\min(\gamma_1, \gamma_2), \max(\eta_1, \eta_2)\} \\ &= \{\{0, 1, 0.2, 0.3, 0.4\}, \{0.3, 0.4, 0.5\}\}. \end{aligned} \tag{4.6}$$

In particular, if $\gamma_1 = \gamma_2$, then $\min(\gamma_1, \gamma_2)$ is a real number; if $\eta_1 = \eta_2$, then $\max(\eta_1, \eta_2)$ is a real number:

$$\begin{aligned} d_1^c &= \mathbb{Z}_{d_1^c} = \cup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{\{\eta\}, \{\gamma\}\} \\ &= \{\{0.3, 0.5\}, \{0.1, 0.3, 0.4\}\}; \\ d_1 \oplus d_2 &= \mathbb{Z}_{d_1 \oplus d_2} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{(\gamma_1 \oplus \gamma_2), (\eta_1 \otimes \eta_2)\} \\ &= \{\{0.28, 0.55, 0.44, 0.65, 0.52, 0.70\}, \{0.05, 0.10, 0.20, 0.03, 0.06, 0.12\}\}; \\ d_1 \otimes d_2 &= \mathbb{Z}_{d_1 \otimes d_2} = \cup_{\gamma_1} \{(\gamma_1 \otimes \gamma_2), (\eta_1 \oplus \eta_2)\} \\ &= \{\{0.02, 0.05, 0.06, 0.15, 0.08, 0.20\}, \{0.37, 0.44, 0.05, 0.55, 0.03, 0.19\}\}. \end{aligned} \tag{4.7}$$

Furthermore, we can get lots of aggregation operators by this extension principle. For example, if we let $f_i(x_i) = \omega_i x_i$, $\sum_{i=1}^n \omega_i = 1$ and $\mathbb{C}(f_1(\gamma_1), f_2(\gamma_2), \ldots, f_n(\gamma_n)) = \oplus_{i=1}^n \omega_i \gamma_i$, then we can obtain the weighted dual hesitant fuzzy averaging (WDHFA) operator, which will not be discussed in this paper.

We now give another example, an application of DHFSs to group forecasting. In a traditional forecasting problem, the probability of uncertain event is often used to obtain expectations, however, it cannot reflect opinions from all decision makers, nor can it depict epistemic degrees of certainty and uncertainty in the same time. So, the DHFSs are employed to replace the probability in next example.

Example 4.5. Several directors of a pharmaceutical company need to decide the additional investment priorities to three subsidiaries in the next quarter based on the net income foresting of them. Assume that the epistemic degrees of three subsidiaries $y_i (i = 1, 2, 3)$ with respect to the predictive values of the net incomes $c_j (j = 1, 2, 3)$ are represented by the DHFEs $d_{ij} = \cup_{\gamma_{ij} \in h_{ij}, \eta_{ij} \in g_{ij}} \{\{\gamma_{ij}\}, \{\eta_{ij}\}\}$, where γ_{ij} indicates the degree that the alternative y_i satisfies the criterion c_j, η_{ij} indicates the degree that the alternative y_i does not satisfy the criterion c_j, such that $\gamma_{ij} \in [0, 1]$, $\eta_{ij} \in [0, 1]$, $\gamma_{ij}^+ + \eta_{ij}^+ \leq 1$, for details see Table 1.

Table 1: The DHFEs for group forecasting.

Subsidiaries	c_j	Predictive values (million)	d_{ij}
y_1	c_1	90	$\{\{0.3, 0.4\}, \{0.5\}\}$
	c_2	100	$\{\{0.3\}, \{0.4, 0.5, 0.6\}\}$
	c_3	50	$\{\{0.4\}, \{0.4\}\}$
y_2	c_1	100	$\{\{0.5\}, \{\emptyset\}\}$
	c_2	70	$\{\{0.3, 0.4\}, \{0.4, 0.5, 0.6\}\}$
	c_3	60	$\{\{0.7\}, \{0.1, 0, 2\}\}$
y_3	c_1	70	$\{\{\emptyset\}, \{0.5, 0.6\}\}$
	c_2	100	$\{\{0.5, 0.6, 0.7\}, \{\emptyset\}\}$
	c_3	80	$\{\{0.4\}, \{0.3, 0.4\}\}$

Table 2

b_{11}	b_{12}	b_{13}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
0.515	0.485	0	0.385	0.318	0.297	0.145	0.516	0.339

Table 3

e_1	e_2	e_3
94.85	78.58	88.87

In what follows, we give an approach for group foresting in terms of DHFEs as follows.

Step 1. Utilize the score function of DHFEs (Definition 3.6) to obtain the score of each DHFE, and transform the results into the normalizations by the method given as $f_i = \sum_{j=1}^{3}(s(d_{ij}) + 1)/2$, $b_{ij} = (s(d_{ij}) + 1)/2f_i$ where $b_{ij} \in [0, 1]$, $i, j = 1, 2, 3$as shown in Table 2.

Step 2. Use $e_i = c_{ij} \times b_{ij}$ $(i, j = 1, 2, 3)$ to obtain the expectations of net incomes as shown in Table 3.

Thus, $e_1 > e_3 > e_2$, and then $y_1 \succ y_3 \succ y_2$, that is, y_1 is the optimal choice of additional investment. Obviously, the DHFS is an effective and convenient tool applied to group forecasting. A transparent result can be obtained by utilizing the DHFS, which reflects the epistemic degree to the predictive values of net incomes. Comparing with other types of FSs, the DHFSs can take the information from the decision makers (directors) into account as much as possible, and it is more flexible in practical applications.

5. Concluding Remarks

In this paper, we have introduced the dual hesitant fuzzy set (DHFS), which is a comprehensive set encompassing several existing sets, and whose membership degrees and nonmembership degrees are represented by a set of possible values. The common ground on the existing sets and the DHFS has been found out. Although in special cases, the DHFS can be reduced to some existing ones, it has the desirable characteristics and advantages of its own and appears to be a more flexible method to be valued in multifold ways according to the practical demands than the existing fuzzy sets, taking much more information given by

decision makers into account. We have investigated some basic operations and properties of DHFSs, and an extension principle for DHFSs has also been developed for further study of basic operations and aggregation operators. Our results have been illustrated by a practical example of group forecasting.

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