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Research Article

A Nonlocal Cauchy Problem for Fractional Integrodifferential Equations

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This paper is concerned with a nonlocal Cauchy problem for fractional integrodifferential equations in a separable Banach space X. We establish an existence theorem for mild solutions to the nonlocal Cauchy problem, by virtue of measure of noncompactness and the fixed point theorem for condensing maps. As an application, the existence of the mild solution to a nonlocal Cauchy problem for a concrete integrodifferential equation is obtained.

1. Introduction

Nonlocal Cauchy problem for equations is an initial problem for the corresponding equations with nonlocal initial data. Such a Cauchy problem has better effects than the normal Cauchy problem with the classical initial data when we deal with many concrete problem coming from engineering and physics (cf., e.g., [1–10] and references therein). Therefore, the study of this type of Cauchy problem is important and significant. Actually, as we have seen from the just mentioned literature, there have been many significant developments in this field.

On the other hand, fractional differential and integrodifferential equations arise from various real processes and phenomena appeared in physics, chemical technology, materials, earthquake analysis, robots, electric fractal network, statistical mechanics biotechnology, medicine, and economics. They have in recent years been an object of investigations with much increasing interest. For more information on this subject see for instance [9, 11–18] and references therein.

Throughout this paper, X is a separable Banach space; $L(X)$ is the Banach space of all linear bounded operators on X; A is the generator of an analytic and uniformly bounded semigroup $\{T(t)\}_{t\geq 0}$ on X with $\|T(t)\|_{L(X)} \leq M$ for a constant $M > 0$, and $C([a,b],\ X)$ is the space of all X-valued continuous functions on$[a,b]$ with the supremum norm as follows:

$$\|x\|_{[a,b]} := \max\{\|x(t)\| : t \in [a,b]\}, \quad \text{for any } x \in C([a,b],X). \tag{1.1}$$

Let $0 < q < 1$, $T > 0$, $\Delta = \{(t,s) \in [0,T] \times [0,T] \ : \ t \geq s\}$, $f \ : \ [0,T] \times C([0,T],X) \to X$, and $h \ : \ \Delta \times C\ ([0,T],\ X) \ \to \ X$. The nonlocal Cauchy problem for abstract fractional integrodifferential equations, with which we are concerned, is in the following form:

$$\begin{gathered} {}^{c}\mathrm{D}^{q}x(t) = Ax(t) + f(t,x(t)) + \int_0^t k(t,s)h(t,s,x(s))ds, \quad t \in [0,T], \\ x(0) = g(x) + x_0, \end{gathered} \tag{1.2}$$

where k and g are given functions to be specified later and the fractional derivative is understood in the Caputo sense, this means that, the fractional derivative is understood in the following sense:

$${}^{c}D^{q}x(t) \ := {}^{L}D^{q}(x(t) - x(0)), \quad t > 0,\ 0 < q < 1, \tag{1.3}$$

and where

$${}^{L}D^{q}x(t) \ := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)ds, \quad t > 0,\ 0 < q < 1 \tag{1.4}$$

is the Riemann-Liouville derivative of order q of $x(t)$, where $\Gamma(\cdot)$ is the Gamma function.

Our main purpose is to establish an existence theorem for the mild solutions to the nonlocal Cauchy problem based on a special measure of noncompactness under weak assumptions on the nonlinearity f and the semigroup $\{T(t)\}_{t\geq 0}$ generated by A.

2. Existence Result and Proof

As usual, we abbreviate $\|u\|_{L^p([0,T],\ \mathbf{R}^+)}$ with $\|u\|_{L^p}$, for any $u \in L^p([0,T],\mathbf{R}^+)$.

As in [16, 17], we define the fractional integral of order q with the lower limit zero for a function $f \in AC[0,\infty)$ as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad t > 0,\ 0 < q < 1, \tag{2.1}$$

provided the right side is point-wise defined on $[0,\infty)$.

Now we recall some very basic concepts in the theory of measures of noncompactness and condensing maps (see, e.g., [19, 20]).

Definition 2.1. Let E be a Banach space, 2^E the family of all nonempty subsets of E, $(\mathcal{A}, \geq)$ a partially ordered set, and $\alpha : 2^E \to \mathcal{A}$. If for every $\Omega \in 2^E$:

$$\alpha(\overline{\mathrm{co}}(\Omega)) = \alpha(\Omega), \tag{2.2}$$

then we say that α is a measure of noncompactness in E.

Definition 2.2. Let E be a Banach space, and $\mathfrak{F} : Y \subseteq E \to E$ is continuous. Let α be a measure of noncompactness in E such that

(i) for any $\Omega_0, \Omega_1 \in 2^E$ with $\Omega_0 \subset \Omega_1$,

$$\alpha(\Omega_0) \leq \alpha(\Omega_1); \tag{2.3}$$

(ii) for every $a_0 \in E$, $\Omega \in 2^E$,

$$\alpha(\{a_0\} \cup \Omega) = \alpha(\Omega). \tag{2.4}$$

If for every bounded set $\Omega \subseteq Y$ which is not relatively compact,

$$\alpha(\mathfrak{F}(\Omega)) < \alpha(\Omega), \tag{2.5}$$

then we say that $\mathfrak{F}$ is condensing with respect to the measure of noncompactness α (or α-condensing).

Definition 2.3. Let

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \sigma \in (0,\infty) \tag{2.6}$$

be a one-sided stable probability density, and

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-1/q}\varpi_q\left(\sigma^{-1/q}\right), \quad \sigma \in (0,\infty). \tag{2.7}$$

For any $z \in X$, we define operators $\{Y(t)\}_{t\geq 0}$ and $\{Z(t)\}_{t\geq 0}$ by

$$\begin{aligned} Y(t)z &= \int_0^{\infty}\xi_q(\sigma)T(t^q\sigma)z\,d\sigma, \\ Z(t)z &= q\int_0^{\infty}\sigma t^{q-1}\xi_q(\sigma)T(t^q\sigma)z\,d\sigma. \end{aligned} \tag{2.8}$$

If a continuous function $x : [0,T] \to X$ satisfies

$$x(t) = Y(t)\big(g(x) + x_0\big) + \int_0^t Z(t-s)\big[f(s,x(s)) + a(x)(s)\big]ds, \quad t \in [0,T], \tag{2.9}$$

then the function x is called a mild solution of (1.2).

Our main result is as follows.

Theorem 2.4. *Assume that*

(1) $f(\cdot,w)$ *and* $h(\cdot,\cdot,w)$ *are measurable for each* $w \in C([0,T],X)$; $k(t,\cdot)$ *is measurable for each* $t \in [0,T]$;

(2) $f(t,\cdot)$ *is continuous for a.e.* $t \in [0,T]$; g *is completely continuous;* $h(t,s,\cdot)$ *is continuous for a.e.* $(t,s) \in \Delta$; *the map* $t \to k_t := k(t,\cdot)$ *is continuous from* $[0,T]$ *to* $L^\infty([0,T],\mathbf{R})$;

(3) *there exist two positive functions* $\mu(\cdot),\eta(\cdot) \in L^p(0,T,\mathbf{R}^+)$ $(p > 1/q > 1)$ *and two positive functions* $m(\cdot,\cdot)$ *and* $\zeta(\cdot,\cdot)$ *on* Δ *with*

$$\sup_{t\in[0,T]} \int_0^t m(t,s)ds := m^* < \infty, \qquad \sup_{t\in[0,T]} \int_0^t \zeta(t,s)ds := \zeta^* < \infty, \tag{2.10}$$

such that

$$\begin{aligned} &\|f(t,w)\| \le \mu(t)\|w\| \quad (\text{a.e. } t \in [0,T]), \\ &\|h(t,s,w)\| \le m(t,s)\|w\| \quad (\text{a.e. } (t,s) \in \Delta), \end{aligned} \tag{2.11}$$

for all $w \in C([0,T],X)$, *and*

$$\begin{aligned} &\chi\big(f(t,D)\big) \le \eta(t)\chi(D), \quad (\text{a.e. } t \in [0,T]), \\ &\chi(h(t,s,D)) \le \zeta(t,s)\chi(D), \quad (\text{a.e. } (t,s) \in \Delta), \end{aligned} \tag{2.12}$$

for any bounded set $D \subset C([0,T],X)$, *where* χ *is the Hausdorff measure of noncompactness:*

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}. \tag{2.13}$$

(4) $g(\cdot)$ *satisfies*

$$\|g(x)\| \le b, \quad \forall\, x \in C([0,T],X), \tag{2.14}$$

for a positive constant b, *and*

$$k(t) := \text{ess sup}\{|k(t,s)|,\ 0 \le s \le t\} \tag{2.15}$$

is bounded on $[0,T]$.

Then the mild solutions set of problem (1.2) *is a nonempty compact subset of the space* $C([0,T],X)$, *in the case of*

$$\frac{qM}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p}T^{q-1/p}\|\mu\|_{L^p}<1. \tag{2.16}$$

Proof. First of all, let us prove our definition of the mild solution to problem (1.2) is well defined and reasonable. Actually, the proof is basic. We present it here for the completeness of the proof as well as the convenience of reading.

Write

$$\begin{aligned}
a(x)(t) &= \int_0^t k(t,s)h(t,s,x(s))ds,\\
\widehat{x}(\lambda) &= \int_0^\infty e^{-\lambda t}x(t)dt,\\
\widehat{f}(\lambda) &= \int_0^\infty e^{-\lambda t}f(t,x(t))dt,\\
\widehat{a}(\lambda) &= \int_0^\infty e^{-\lambda t}a(x)(t)dt.
\end{aligned} \tag{2.17}$$

Clearly, the nonlocal Cauchy problem (1.2) can be written as the following equivalent integral equation:

$$x(t)=g(x)+x_0+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[Ax(s)+f(s,x(s))+a(x)(s)\right]ds,\quad t\in[0,T], \tag{2.18}$$

provided that the integral in (2.18) exists. Formally taking the Laplace transform to (2.18), we have

$$\widehat{x}(\lambda)=\frac{1}{\lambda}\left(g(x)+x_0\right)+\frac{1}{\lambda^q}A\widehat{x}(\lambda)+\frac{1}{\lambda^q}\left[\widehat{f}(\lambda)+\widehat{a}(\lambda)\right]. \tag{2.19}$$

Therefore, if the related integrals exist, then we obtain

$$\begin{aligned}
\widehat{x}(\lambda) &= \lambda^{q-1}(\lambda^q-A)^{-1}\left(g(x)+x_0\right)+(\lambda^q-A)^{-1}\left[\widehat{f}(\lambda)+\widehat{a}(\lambda)\right]\\
&= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}T(s)\left(g(x)+x_0\right)ds+\int_0^\infty e^{-\lambda^q s}T(s)\left[\widehat{f}(\lambda)+\widehat{a}(\lambda)\right]ds\\
&= q\int_0^\infty (\lambda t)^{q-1}e^{-(\lambda t)^q}T(t^q)\left(g(x)+x_0\right)dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty e^{-(\lambda\tau)^q} q\tau^{q-1} T(\tau^q)\left(\int_0^\infty e^{-\lambda t}\big(f(t,x(t))+a(x)(t)\big)dt\right)d\tau \\
&= -\frac{1}{\lambda}\int_0^\infty \left(\frac{d}{dt}e^{-(\lambda t)^q}\right)T(t^q)\big(g(x)+x_0\big)dt \\
&\quad+\int_0^\infty\int_0^\infty e^{-\lambda\tau\sigma} q\tau^{q-1}\varpi_q(\sigma)T(\tau^q)\left(\int_0^\infty e^{-\lambda t}\big(f(t,x(t))+a(x)(t)\big)dt\right)d\sigma\, d\tau \\
&= \int_0^\infty\int_0^\infty e^{-\lambda t\sigma}\sigma\varpi_q(\sigma)T(t^q)\big(g(x)+x_0\big)d\sigma dt \\
&\quad+q\int_0^\infty\int_0^\infty e^{-\lambda\theta}\frac{\theta^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{\theta^q}{\sigma^q}\right)\left(\int_0^\infty e^{-\lambda t}\big(f(t,x(t))+a(x)(t)\big)dt\right)d\sigma\, d\theta \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \varpi_q(\sigma)T\left(\frac{t^q}{\sigma^q}\right)\big(g(x)+x_0\big)d\sigma\right]dt \\
&\quad+q\int_0^\infty\left(\int_0^\infty\int_t^\infty e^{-\lambda\tau}\frac{(\tau-t)^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{(\tau-t)^q}{\sigma^q}\right)\big(f(t,x(t))+a(x)(t)\big)d\tau\, dt\right)d\sigma \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \varpi_q(\sigma)T\left(\frac{t^q}{\sigma^q}\right)\big(g(x)+x_0\big)d\sigma\right]dt \\
&\quad+q\int_0^\infty\left(\int_0^\infty\int_0^\tau e^{-\lambda\tau}\frac{(\tau-t)^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{(\tau-t)^q}{\sigma^q}\right)\big(f(t,x(t))+a(x)(t)\big)dt\, d\tau\right)d\sigma \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \varpi_q(\sigma)T\left(\frac{t^q}{\sigma^q}\right)\big(g(x)+x_0\big)d\sigma\right]dt \\
&\quad+\int_0^\infty e^{-\lambda t}\left[q\int_0^t\int_0^\infty\frac{(t-s)^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{(t-s)^q}{\sigma^q}\right)\big(f(s,x(s))+a(x)(s)\big)d\sigma\, ds\right]dt.
\end{aligned}
\tag{2.20}
$$

Now using the uniqueness of the Laplace transform (cf., e.g., [21, Theorem 1.1.6]), we deduce that

$$
\begin{aligned}
x(t) &= \int_0^\infty \varpi_q(\sigma)T\left(\frac{t^q}{\sigma^q}\right)\big(g(x)+x_0\big)d\sigma \\
&\quad+q\int_0^t\int_0^\infty\frac{(t-s)^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{(t-s)^q}{\sigma^q}\right)f(s,x(s))d\sigma\, ds \\
&\quad+q\int_0^t\int_0^\infty\frac{(t-s)^{q-1}}{\sigma^q}\varpi_q(\sigma)T\left(\frac{(t-s)^q}{\sigma^q}\right)a(x)(s)d\sigma\, ds \\
&= \int_0^\infty \xi_q(\sigma)T(t^q\sigma)\big(g(x)+x_0\big)d\sigma
\end{aligned}
$$

$$+ q\int_0^t\int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T\big((t-s)^q\sigma\big)f(s,x(s))d\sigma\, ds$$
$$+ q\int_0^t\int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T\big((t-s)^q\sigma\big)a(x)(s)d\sigma\, ds. \tag{2.21}$$

Consequently, we see that the mild solution to problem (1.2) given by Definition 2.3 is well defined.

Next, we define the operator $\mathcal{F} : C([0,T],X) \to C([0,T],X)$ as follows:

$$(\mathcal{F}x)(t) = Y(t)\big(g(x)+x_0\big) + \int_0^t Z(t-s)\big[f(s,x(s)) + a(x)(s)\big]ds, \quad t\in[0,T]. \tag{2.22}$$

It is clear that the operator $\mathcal{F}$ is well defined.

The operator $\mathcal{F}$ can be written in the form $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where the operators $\mathcal{F}_i,\ i = 1,2$ are defined as follows:

$$\begin{aligned}(\mathcal{F}_1x)(t) &= Y(t)\big(g(x)+x_0\big), \quad t\in[0,T],\\ (\mathcal{F}_2x)(t) &= \int_0^t Z(t-s)\big[f(s,x(s)) + a(x)(s)\big]ds, \quad t\in[0,T].\end{aligned} \tag{2.23}$$

The following facts will be used in the proof.

(1)

$$\int_0^\infty \xi_q(\sigma)d\sigma = 1, \tag{2.24}$$

which implies that

$$\|Y(t)\| \le \text{Const}; \tag{2.25}$$

(2)

$$\int_0^\infty \sigma^\nu\xi_q(\sigma)d\sigma = \int_0^\infty \frac{1}{\sigma^{q\nu}}\varpi_q(\sigma)d\sigma = \frac{\Gamma(1+\nu)}{\Gamma(1+q\nu)}, \quad \nu\in(0,1], \tag{2.26}$$

which implies that

$$\|Z(t)\| \le \frac{qM}{\Gamma(1+q)}t^{q-1}, \quad t>0. \tag{2.27}$$

Let $\{x_n\}_{n\in\mathbf{N}} \subset C([0,T],X)$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{[0,T]} = 0, \tag{2.28}$$

for an $x \in C([0,T],X)$. Then by the assumptions, we know that for almost every $t \in [0,T]$ and $(t,s) \in \Delta$:

$$\begin{aligned} &\lim_{n\to\infty} f(t,x_n(t)) = f(t,x(t)), \\ &\lim_{n\to\infty} h(t,s,x_n(s)) = h(t,s,x(s)). \end{aligned} \tag{2.29}$$

Therefore, for sufficiently large n, we have

$$\begin{aligned} \left\| f(t,x_n(t)) - f(t,x(t)) \right\| &\le \mu(t)\left(1 + 2\|x\|_{[0,T]}\right), \\ \|h(t,s,x_n(s)) - h(t,s,x(s))\| &\le m(t,s)\left(1 + 2\|x\|_{[0,T]}\right), \\ \left\| \int_0^t k(t,s)h(t,s,x_n(s))ds - \int_0^t k(t,s)h(t,s,x(s))ds \right\| &\le m^* k^* \left(1 + 2\|x\|_{[0,T]}\right), \end{aligned} \tag{2.30}$$

where

$$k^* := \sup_{t\in[0,T]} k(t). \tag{2.31}$$

Hence,

$$\lim_{n\to\infty} \left\| \int_0^t k(t,s)h(t,s,x_n(s))ds - \int_0^t k(t,s)h(t,s,x(s))ds \right\| = 0. \tag{2.32}$$

Thus,

$$\|\mathcal{F}_2 x_n - \mathcal{F}_2 x\|_{[0,T]} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.33}$$

since (2.27) implies that

$$\begin{aligned} \Bigg\| \int_0^t Z(t-s) \Bigg\{ & f(s,x_n(s)) + \int_0^s k(s,\tau)h(s,\tau,x_n(\tau))d\tau \\ & - \left[f(s,x(s)) + \int_0^s k(s,\tau)h(s,\tau,x(\tau))d\tau \right] \Bigg\} ds \Bigg\| \end{aligned}$$

$$\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left[\left\| f(s, x_n(s)) - f(s, x(s)) \right\| + \left\| \int_0^s k(s,\tau)(h(s,\tau,x_n(\tau)) - h(s,\tau,x(\tau)))d\tau \right\| \right] ds. \tag{2.34}$$

By (2.33) and our assumptions, we see that $\mathcal{F}$ is continuous.

Since χ is the Hausdorff measure of noncompactness in X, we know that χ is monotone, nonsingular, invariant with respect to union with compact sets, algebraically semiadditive, and regular. This means that

(i) for any $\Omega_0, \Omega_1 \in 2^E$ with $\Omega_0 \subset \Omega_1$,

$$\chi(\Omega_0) \leq \chi(\Omega_1); \tag{2.35}$$

(ii) for every $a_0 \in E$, $\Omega \in 2^E$,

$$\chi(\{a_0\} \cup \Omega) = \chi(\Omega); \tag{2.36}$$

(iii) for every relatively compact set $D \subset E$, $\Omega \in 2^E$,

$$\chi(\{D\} \cup \Omega) = \chi(\Omega); \tag{2.37}$$

(iv) for each $\Omega_0, \Omega_1 \in 2^E$,

$$\chi(\Omega_0 + \Omega_1) \leq \chi(\Omega_0) + \chi(\Omega_1); \tag{2.38}$$

(v) $\chi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

Noting that for any $\psi \in L^1([0,T], X)$, we have

$$\lim_{L \to +\infty} \sup_{t \in [0,T]} \int_0^t e^{-L(t-s)} \psi(s) ds = 0. \tag{2.39}$$

So, there exists a positive constant L such that

$$\begin{aligned} \frac{qM}{\Gamma(1+q)} \sup_{t \in [0,T]} \int_0^t (t-s)^{q-1} \eta(s) e^{-L(t-s)} ds &= L_1 < \frac{1}{3}, \\ \frac{qMk^*\zeta^*}{\Gamma(1+q)} \sup_{t \in [0,T]} \int_0^t (t-s)^{q-1} e^{-L(t-s)} ds &= L_2 < \frac{1}{3}, \\ \frac{qM}{\Gamma(1+q)} \sup_{t \in [0,T]} \int_0^t (t-s)^{q-1} (\mu(s) + m^* k^*) e^{-L(t-s)} ds &= L_3 < \frac{1}{3}. \end{aligned} \tag{2.40}$$

For every bounded subset $\Omega \subset C([0,T],X)$, we define

$$\begin{aligned}
\operatorname{mod}_c(\Omega) &:= \lim_{\delta\to 0}\sup_{v\in\Omega}\max_{|t_1-t_2|\le\delta}\|v(t_1)-v(t_2)\|,\\
\Psi(\Omega) &:= \sup_{t\in[0,T]}\left(e^{-Lt}\chi(\Omega(t))\right),\\
\alpha(\Omega) &:= (\Psi(\Omega),\ \operatorname{mod}_c(\Omega)).
\end{aligned} \tag{2.41}$$

Then $\operatorname{mod}_c(\Omega)$ is the module of equicontinuity of Ω, and α is a measure of noncompactness in the space $C([0,T],X)$ with values in the cone $\mathbf{R}_+^2$.

Let $\Omega \subset C([0,T],X)$ be a nonempty, bounded set such that

$$\alpha(\mathcal{F}(\Omega)) \ge \alpha(\Omega). \tag{2.42}$$

By the assumptions and the continuity of $T(t)$ in the uniform operator topology for $t>0$, we get

$$\operatorname{mod}_c(\mathcal{F}_1\Omega) = 0. \tag{2.43}$$

Clearly,

$$\|f(s,x(s))\| + \|a(x)(s)\| \le (\mu(s)+m^*k^*)\|x\|_{[0,T]}. \tag{2.44}$$

Let $\delta>0$, $t_1,t_2\in(0,T]$ such that $0<t_1-t_2\le\delta$ and $x\in\Omega$. Then

$$\begin{aligned}
&\left\|\int_0^{t_1} Z(t_1-s)\left[f(s,x(s))+a(x)(s)\right]ds - \int_0^{t_2} Z(t_2-s)\left[f(s,x(s))+a(x)(s)\right]ds\right\|\\
&\quad\le \|x\|_{[0,T]}\left(\int_0^{t_2}\|Z(t_1-s)-Z(t_2-s)\|(\mu(s)+m^*k^*)ds + \int_{t_2}^{t_1}\|Z(t_1-s)\|(\mu(s)+m^*k^*)ds\right)\\
&\quad\le q\|x\|_{[0,T]}\left[\int_0^{t_2}\int_0^{\infty}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|\sigma\xi_q(\sigma)\left\|T\left((t_1-s)^q\sigma\right)\right\|(\mu(s)+m^*k^*)d\sigma\,ds\right.\\
&\qquad\quad + \int_0^{t_2}\int_0^{\infty}(t_2-s)^{q-1}\sigma\xi_q(\sigma)\left\|T\left((t_1-s)^q\sigma\right)-T\left((t_2-s)^q\sigma\right)\right\|\\
&\qquad\quad \left.\times(\mu(s)+m^*k^*)d\sigma\,ds\right] + \frac{qM\|x\|_{[0,T]}}{\Gamma(1+q)}\int_{t_2}^{t_1}(t_1-s)^{q-1}(\mu(s)+m^*k^*)ds
\end{aligned}$$

$$\leq \frac{qM\|x\|_{[0,T]}}{\Gamma(1+q)}\left[\int_0^{t_2}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|\left(\mu(s)+m^*k^*\right)d\sigma\, ds\right.$$
$$\left.+\int_{t_2}^{t_1}(t_1-s)^{q-1}\left(\mu(s)+m^*k^*\right)ds\right]$$
$$+\int_0^{t_2}\int_0^{\infty}(t_2-s)^{q-1}\sigma\xi_q(\sigma)\left\|T\left((t_1-s)^q\sigma\right)-T\left((t_2-s)^q\sigma\right)\right\|\left(\mu(s)+m^*k^*\right)d\sigma\, ds. \tag{2.45}$$

It is not hard to see that the right-hand side of (2.45) tend to 0 as $t_2 \to t_1$. Thus, the set $\{(\mathcal{F}_2x)(\cdot) : x \in \Omega\}$ is equicontinuous, then $\mathrm{mod}_c(\mathcal{F}_2\Omega) = 0$. Combining with (2.43), we have $\mathrm{mod}_c(\mathcal{F}\Omega) = 0$, which implies $\mathrm{mod}_c(\Omega) = 0$ from (2.42). Next, we show that $\Psi(\Omega) = 0$.

It is easy to see that

$$\Psi(\mathcal{F}_1\Omega) = 0. \tag{2.46}$$

For any $t \in [0,T]$, we define

$$\widehat{\mathcal{F}}_2(\Omega)(t) := \left\{\int_0^t Z(t-s)f(s,x(s))ds : x \in \Omega\right\}. \tag{2.47}$$

We consider the multifunction $s \in [0,t] \multimap G(s)$:

$$G(s) = \left\{Z(t-s)f(s,x(s)) : x \in \Omega\right\}. \tag{2.48}$$

Obviously, G is integrable, that is, G admits a Bochner integrable selection $\mathfrak{g} : [0,h] \to E$, and

$$\mathfrak{g}(t) \in G(t), \quad \text{for a.e. } t \in [0,h]. \tag{2.49}$$

From (2.27) and our assumptions, it follows that G is integrably bounded, that is, there exists a function $\varrho \in L^1([0,h],E)$ such that

$$\|G(t)\| := \sup\{\|\mathfrak{g}\| : \mathfrak{g} \in G(t)\} \leq \varrho(t), \quad \text{a.e. } t \in [0,h]. \tag{2.50}$$

Moreover, we have the following estimate for a.e. $s \in [0,t]$:

$$\begin{aligned}\chi(G(s)) &\leq \frac{qM}{\Gamma(1+q)}(t-s)^{q-1}\chi\big(f(s,\Omega(s))\big)\\ &\leq \frac{qM}{\Gamma(1+q)}(t-s)^{q-1}\eta(s)\chi(\Omega(s))\\ &= \frac{qM}{\Gamma(1+q)}(t-s)^{q-1}\eta(s)e^{Ls}e^{-Ls}\chi(\Omega(s))\\ &\leq \frac{qM}{\Gamma(1+q)}(t-s)^{q-1}\eta(s)e^{Ls}\Psi(\Omega).\end{aligned} \tag{2.51}$$

Therefore, since X is a separable Banach space, we know by [20, Theorem 4.2.3] that

$$\chi\Big(\widehat{\mathcal{F}}_2(\Omega)(t)\Big) = \chi\left(\int_0^t G(s)ds\right) \leq \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\eta(s)e^{Ls}ds\cdot\Psi(\Omega). \tag{2.52}$$

So

$$\begin{aligned}\sup_{t\in[0,T]}\Big(e^{-Lt}\chi\Big(\widehat{\mathcal{F}}_2(\Omega)(t)\Big)\Big) &\leq \frac{qM}{\Gamma(1+q)}\sup_{t\in[0,T]}\int_0^t (\mathrm{t}-s)^{q-1}\eta(s)e^{-L(t-s)}ds\cdot\Psi(\Omega)\\ &= L_1\Psi(\Omega).\end{aligned} \tag{2.53}$$

Similarly, if we set

$$\widetilde{\mathcal{F}}_2(\Omega)(t) = \left\{\int_0^t Z(t-s)a(x)(s)ds:\ x\in\Omega\right\}, \tag{2.54}$$

then we see that the multifunction $s \in [0,t] \multimap \widetilde{G}(s)$,

$$\widetilde{G}(s) = \{Z(t-s)a(x)(s):\ x\in\Omega\} \tag{2.55}$$

is integrable and integrably bounded. Thus, we obtain the following estimate for a.e. $s \in [0,t]$:

$$\begin{aligned}\chi\Big(\widetilde{G}(s)\Big) &\leq \frac{qMk^*\zeta^*}{\Gamma(1+q)}(t-s)^{q-1}e^{Ls}\Psi(\Omega),\\ \sup_{t\in[0,T]}\Big(e^{-Lt}\chi\Big(\widetilde{\mathcal{F}}_2(\Omega)(t)\Big)\Big) &\leq \frac{qMk^*\zeta^*}{\Gamma(1+q)}\sup_{t\in[0,T]}\int_0^t (t-s)^{q-1}e^{-L(t-s)}ds\cdot\Psi(\Omega)\\ &= L_2\Psi(\Omega).\end{aligned} \tag{2.56}$$

Now, from (2.53) and (2.56), it follows that

$$\Psi(\mathcal{F}(\Omega)) \leq \Psi(\mathcal{F}_1(\Omega)) + \Psi(\mathcal{F}_2(\Omega)) \leq (L_1 + L_2)\Psi(\Omega) = \widetilde{L}\Psi(\Omega), \tag{2.57}$$

where $0 < \widetilde{L} < 1$. Then by (2.42), we get $\Psi(\Omega) = 0$. Hence $\alpha(\Omega) = (0,0)$. Thus, Ω is relatively compact due to the regularity property of α. This means that $\mathcal{F}$ is α-condensing.

Let us introduce in the space $C([0,T],X)$ the equivalent norm defined as

$$\|x\|_* = \sup_{t\in[0,T]} \left(e^{-Lt}\|x(t)\|\right). \tag{2.58}$$

Consider the set

$$B_r = \{x \in C([0,T],X) : \ \|x\|_* \leq r\}. \tag{2.59}$$

Next, we show that there exists some $r > 0$ such that $\mathcal{F}B_r \subset B_r$. Suppose on the contrary that for each $r > 0$ there exist $x_r(\cdot) \in B_r$, and some $t \in [0,T]$ such that $\|(\mathcal{F}x_r)(t)\|_* > r$.

From the assumptions, we have

$$\|(\mathcal{F}_1 x_r)(t)\|_* \leq M(b + \|x_0\|). \tag{2.60}$$

Moreover,

$$\begin{aligned}
\|(\mathcal{F}_2 x_r)(t)\| &\leq \int_0^t \left\| Z(\mathrm{t}-s)\left[f(s,x_r(s)) + a(x_r)(s)\right]\right\| ds \\
&\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\left(\mu(s)\|x_r(s)\| + m^* k^* e^{Ls}\|x_r\|_*\right) ds \\
&= \frac{qM}{\Gamma(1+q)} \left[\int_0^t (t-s)^{q-1}\mu(s)e^{Ls}e^{-Ls}\|x_r(s)\| ds \right. \\
&\qquad \left. + m^* k^* \int_0^t (t-s)^{q-1} e^{Ls} ds \cdot \|x_r\|_*\right] \\
&\leq \frac{qMr}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\left(\mu(s) + m^* k^*\right) e^{Ls} ds.
\end{aligned} \tag{2.61}$$

Therefore,

$$\begin{aligned}
r &< \sup_{t\in[0,T]} \left(e^{-Lt}\|(\mathcal{F}x_r)(t)\|\right) \\
&\leq M(b + \|x_0\|) + \frac{qMr}{\Gamma(1+q)} \sup_{t\in[0,T]} \int_0^t (t-s)^{q-1}\left(\mu(s) + m^* k^*\right) e^{-L(t-s)} ds.
\end{aligned} \tag{2.62}$$

Dividing both sides of (2.62) by r, and taking $r \to \infty$, we have

$$\frac{qM}{\Gamma(1+q)} \sup_{t\in[0,T]} \int_0^t (t-s)^{q-1}(\mu(s)+m^*k^*)e^{-L(t-s)}ds \geq 1. \tag{2.63}$$

This is a contradiction. Hence for some positive number r, $\mathcal{F}B_r \subset B_r$. According to the following known fact.

Let $\mathfrak{M}$ be a bounded convex closed subset of E and $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ a α-condensing map. Then $\operatorname{Fix}\mathfrak{F} = \{x : x = \mathfrak{F}(x)\}$ *is nonempty.*

we see that problem (1.2) has at least one mild solution.

Next, for $c \in (0,1]$, we consider the following one-parameter family of maps:

$$\begin{gathered} \mathcal{H} : [0,1] \times C([0,T],X) \longrightarrow C([0,T],X) \\ (c,x) \longrightarrow \mathcal{H}(c,x) = c\mathcal{F}(x). \end{gathered} \tag{2.64}$$

We will demonstrate that the fixed point set of the family $\mathcal{H}$,

$$\operatorname{Fix}\mathcal{H} = \{x \in \mathcal{H}(c,x) \text{ for some } c \in (0,1]\} \tag{2.65}$$

is a priori bounded. Indeed, let $x \in \operatorname{Fix}\mathcal{H}$, for $t \in [0,T]$, we have

$$\begin{aligned} \|x(t)\| &\leq M\|g(x)+x_0\| + \int_0^t \|Z(t-s)\|\|f(s,x(s))+a(x)(s)\|ds \\ &\leq M(b+\|x_0\|) + \frac{qM}{\Gamma(1+q)}\left[\int_0^t (t-s)^{q-1}\mu(s)\|x(s)\|ds \right. \\ &\qquad \left. + m^*k^* \int_0^t (t-s)^{q-1} \sup_{\tau\in[0,s]} \|x(\tau)\|ds\right]. \end{aligned} \tag{2.66}$$

Noting that the Hölder inequality, we have

$$\begin{aligned} \int_0^t (t-s)^{q-1}\mu(s)ds &\leq \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \cdot t^{(pq-1)/p} \cdot \|\mu\|_{L^p} \\ &\leq \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} \cdot T^{q-1/p} \cdot \|\mu\|_{L^p}. \end{aligned} \tag{2.67}$$

Therefore, from (2.66), we obtain

$$\begin{aligned} \|x(t)\| \leq M(b+\|x_0\|) &+ \frac{qM}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p} T^{q-1/p}\|\mu\|_{L^p} \sup_{s\in[0,t]}\|x(s)\| \\ &+ \frac{qMm^*k^*}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \sup_{\tau\in[0,s]}\|x(\tau)\|ds. \end{aligned} \tag{2.68}$$

We denote

$$y(t) := \sup_{s\in[0,t]} \|x(s)\|. \tag{2.69}$$

Let $\tilde{t} \in [0,t]$ such that $y(t) = \|x(\tilde{t})\|$. Then, by (2.68), we can see

$$\begin{aligned} y(t) \le M(b+\|x_0\|) &+ \frac{qM}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p} T^{q-1/p}\|\mu\|_{L^p} y(t) \\ &+ \frac{qMm^*k^*}{\Gamma(1+q)}\int_0^t (t-s)^{q-1} y(s)ds. \end{aligned} \tag{2.70}$$

By a generalization of Gronwall's lemma for singular kernels ([22, Lemma 7.1.1]), we deduce that there exists a constant $\kappa = \kappa(q)$ such that

$$\begin{aligned} y(t) &\le \frac{M(b+\|x_0\|)}{1-(qM/\Gamma(1+q))((p-1)/(pq-1))^{(p-1)/p}T^{q-1/p}\|\mu\|_{L^p}} \\ &\quad + \frac{\kappa M(b+\|x_0\|)(qMm^*k^*/\Gamma(1+q))}{\left(1-(qM/\Gamma(1+q))((p-1)/(pq-1))^{(p-1)/p}T^{q-1/p}\|\mu\|_{L^p}\right)^2}\int_0^t (t-s)^{q-1}ds \\ &\le \frac{M(b+\|x_0\|)}{1-(qM/\Gamma(1+q))((p-1)/(pq-1))^{(p-1)/p}T^{q-1/p}\|\mu\|_{L^p}} \\ &\quad + \frac{\kappa M(b+\|x_0\|)(qMm^*k^*/\Gamma(1+q))T^q}{q\left(1-(qM/\Gamma(1+q))((p-1)/(pq-1))^{(p-1)/p}T^{q-1/p}\|\mu\|_{L^p}\right)^2} \\ &:= w. \end{aligned} \tag{2.71}$$

Hence, $\sup_{t\in[0,T]}\|x(t)\| \le w$.

Now we consider a closed ball:

$$B_R = \left\{x \in C([0,T],X) : \ \|x\|_{[0,T]} \le R\right\} \subset C([0,T],X). \tag{2.72}$$

We take the radius $R > 0$ large enough to contain the set $\operatorname{Fix}\mathcal{H}$ inside itself. Moreover, from the proof above, $\mathcal{F} : B_R \to C([0,T],X)$ is α-condensing. Consequently, the following known fact implies our conclusion: Let $V \subset E$ *be a bounded open neighborhood of zero and* $\mathfrak{F} : \overline{V} \to E$ *a* α*-condensing map satisfying the boundary condition:*

$$x \ne \overline{\lambda}\mathfrak{F}(x), \tag{2.73}$$

for all $x \in \partial V$ and $0 < \overline{\lambda} \le 1$. *Then,* $\operatorname{Fix}\mathfrak{F}$ *is nonempty compact.* □

3. Example

In this section, let $X = L^2([0,\pi])$, we consider the following nonlocal Cauchy problem for an integrodifferential problem:

$$\begin{gathered}\partial_t^q u(t,\xi) = \frac{\partial^2}{\partial\xi^2}u(t,\xi) + \frac{1}{k\sqrt[k]{t}}\cdot\frac{u(t,\xi)}{1+u(t,\xi)} + \int_0^t (t-s)\sin\left(\frac{\sqrt{s}\cdot u(s,\xi)}{t}\right)ds, \qquad t\in(0,1] \\ u(t,0) = u(t,\pi) = 0, \qquad t\in[0,1] \\ u(0,\xi) = \sum_{i=0}^{j}\int_0^\pi c_i(\xi,y)\frac{u(t_i,y)}{1+u(t_i,y)}dy + u_0(\xi),\end{gathered} \tag{3.1}$$

where ∂_t^q is the Caputo fractional partial derivative of order $0 < q < 1$; $\xi \in [0,\pi]$; $k > 0$ is a constant to be specified later;

$$u_0(\xi) \in X; \qquad j\in\mathbf{N}^+; \qquad 0 < t_0 < t_1 < \cdots < t_j < 1; \tag{3.2}$$

$c_i(\cdot,\cdot)$ $(i = 0,1,\ldots,j)$ are continuous functions and there exists a positive constant b such that

$$\sum_{i=0}^{j}\int_0^\pi \|c_i(\xi,y)\|dy \le b. \tag{3.3}$$

For $t\in(0,1]$, $\xi\in[0,\pi]$, we set

$$\begin{aligned} x(t)(\xi) &= u(t,\xi), \\ g(x)(\xi) &= \sum_{i=0}^{j}\int_0^\pi c_i(\xi,y)\frac{x(t_i)(y)}{1+x(t_i)(y)}dy, \\ k(t,s) &= t-s, \\ h(t,s,x(s))(\xi) &= \sin\left(\frac{\sqrt{s}\cdot x(s)(\xi)}{t}\right), \\ f(t,x(t))(\xi) &= \frac{1}{k\sqrt[k]{t}}\cdot\frac{x(t)(\xi)}{1+x(t)(\xi)}. \end{aligned} \tag{3.4}$$

On the other hand, it is known that the operator A ($Au = u''$ with $D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$) generates an analytic semigroup and uniformly bounded semigroup $\{T(t)\}_{t\ge0}$ on X with $\|T(t)\|_{L(X)} \le 1$. Therefore, (3.1) is a special case of (1.2).

Moreover, we have

(1) for all $t\in(0,1]$,

$$\|f(t,x)\| \le \frac{1}{k\sqrt[k]{t}}\|x\| := \mu(t)\|x\|; \tag{3.5}$$

(2) for any $w, \tilde{w} \in X$,

$$\left\| f(t,w) - f(t,\tilde{w}) \right\| \leq \frac{1}{k\sqrt[k]{t}} \|w - \tilde{w}\|, \tag{3.6}$$

that is, for any bounded set $D \subset X$,

$$\chi\big(f(t,D)\big) \leq \frac{1}{k\sqrt[k]{t}} \chi(D), \tag{3.7}$$

for a.e. $t \in [0,1]$;

(3) for almost all $(t,s) \in \Delta$,

$$\|h(t,s,x)\| = \left\| \sin\left(\frac{\sqrt{s} \cdot x(s)(\xi)}{t} \right) \right\| \leq m(t,s)\|x\|, \tag{3.8}$$

where $m(t,s) := \sqrt{s}/t$, and

$$m^* = \sup_{t\in[0,1]} \int_0^t m(t,s)ds = \sup_{t\in[0,1]} \int_0^t \frac{\sqrt{s}}{t} ds = \frac{2}{3}; \tag{3.9}$$

(4)

$$\|h(t,s,w) - h(t,s,\tilde{w})\| \leq \frac{\sqrt{s}}{t} \|w - \tilde{w}\|, \tag{3.10}$$

that is, for any bounded set $D \subset X$,

$$\chi(h(t,s,D)) \leq \zeta(t,s)\chi(D), \tag{3.11}$$

where $\zeta(t,s) := \sqrt{s}/t$, and

$$\sup_{t\in[0,1]} \int_0^t \zeta(t,s)ds = \frac{2}{3}. \tag{3.12}$$

Therefore, Theorem 2.4 implies that the problem (3.1) has at least a mild solution when

$$\frac{q \cdot \big((p-1)/(pq-1)\big)^{(p-1)/p}}{\Gamma(1+q)} \|\mu\|_{L^p} < 1. \tag{3.13}$$

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