Research Article

General Common Fixed Point Theorems and Applications

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The main result is a common fixed point theorem for a pair of multivalued maps on a complete metric space extending a recent result of Đorić and Lazović (2011) for a multivalued map on a metric space satisfying Ćirić-Suzuki-type-generalized contraction. Further, as a special case, we obtain a generalization of an important common fixed point theorem of Ćirić (1974). Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

1. Introduction

Consistent with Nadler [1, page 620], (X, d) will denote a metric space and $CL(X)$, the collection of all nonempty closed subsets of X . For $A, B \in CL(X)$ and $\varepsilon > 0$,

$$\begin{aligned} N(\varepsilon, A) &= \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}, \\ E_{A,B} &= \{\varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\}, \\ H(A, B) &= \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset \\ +\infty, & \text{if } E_{A,B} = \emptyset. \end{cases} \end{aligned} \tag{1.1}$$

The hyperspace $(CL(X), H)$ is called the generalized Hausdorff metric space induced by the metric d on X .

For nonempty subsets A, B of X , $d(A, B)$ denotes the gap between the subsets A and B , while

$$\begin{aligned}\rho(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}, \\ BN(X) &= \{A : \emptyset \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}.\end{aligned}\tag{1.2}$$

As usual, we write $d(x, B)$ (resp. $\rho(x, B)$) for $d(A, B)$ (resp. $\rho(A, B)$) when $A = \{x\}$.

Let $S, T : X \rightarrow CL(X)$. Then $u \in X$ is a fixed point of S if and only if $u \in Su$ and a common fixed point of S and T if and only if $u \in Su \cap Tu$.

Let S and T be maps to be defined specifically in a particular context, while x and y are the elements of a metric space (X, d) :

$$M(Sx, Ty) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}.\tag{1.3}$$

Recently Suzuki [2] and Kikkawa and Suzuki [3] obtained interesting generalizations of the Banach's classical fixed point theorem and other fixed point results by Nadler [4], Jungck [5], and Meir and Keeler [6]. These results have important outcomes (see, e.g., [7–14]). The following result, due to Ćirić and Lazović [9], extends and generalizes fixed point theorems from Ćirić [15], Kikkawa and Suzuki [3], Nadler [4], Reich [16], Rus [17], and others.

Theorem 1.1. *Define a nonincreasing function φ from $[0, 1)$ onto $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases}\tag{1.4}$$

Let X be a complete metric space and $T : X \rightarrow CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq rM(Tx, Ty).\tag{1.5}$$

Then there exists $z \in X$ such that $z \in Tz$.

We remark that, for every $x, y \in X$, the generalized contraction $H(Tx, Ty) \leq rM(Tx, Ty)$, $0 \leq r < 1$, was first studied by Ćirić [15]. The following important common fixed point theorem is due to Ćirić [18].

Theorem 1.2. *Let X be a complete metric space and $S, T : X \rightarrow X$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(Sx, Ty) \leq rM(Sx, Ty).\tag{1.6}$$

Then S and T have a unique common fixed point.

For an excellent discussion on several special cases and variants of Theorem 1.2, one may refer to Rus [17]. However, the generality of Theorem 1.2 may be appreciated from the fact that (1.6) in Theorem 1.2 cannot be replaced by

$$d(Sx, Ty) \leq r \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}. \quad (1.7)$$

Indeed, Sastry and Naidu [19, Example 5] have shown that maps S and T satisfying (1.7) need not have a common fixed point on a complete metric space. Notice that the condition (1.7) with $S = T$ is the quasicontraction due to Ćirić [20].

The main result of this paper (cf. Theorem 2.2) generalizes Theorems 1.1 and 1.2. Further, a corollary of Theorem 2.2 is used to obtain a unique common fixed point theorem for multivalued maps on a metric space with values in $BN(X)$. As another application, we deduce the existence of a common solution for a general class of functional equations under much weaker conditions than those in [12, 14, 21–24].

2. Main Results

We shall need the following result essentially due to Nadler [4] (see also [15, 25], [26, page 4], [27], [17, page 76]).

Lemma 2.1. *If $A, B \in CL(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.*

Theorem 2.2. *Let X be a complete metric space and $S, T : X \rightarrow CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \text{ implies } H(Sx, Ty) \leq rM(Sx, Ty). \quad (2.1)$$

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$.

Proof. Obviously $M(Sx, Ty) = 0$ iff $x = y$ is a common fixed point of S and T . So, we may take without any loss of generality that $M(Sx, Ty) > 0$ for distinct $x, y \in X$. Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$. Then by Lemma 2.1, there exists $u_2 \in Su_1$ such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + \varepsilon M(Su_1, Tu_0). \quad (2.2)$$

Similarly, there exists $u_3 \in Tu_2$ such that

$$d(u_3, u_2) \leq H(Tu_2, Su_1) + \varepsilon M(Tu_2, Su_1). \quad (2.3)$$

Continuing in this manner, we find a sequence $\{u_n\}$ in X such that

$$\begin{aligned} u_{2n+1} &\in Tu_{2n}, \quad u_{2n+2} \in Su_{2n+1} \text{ such that} \\ d(u_{2n+1}, u_{2n}) &\leq H(Tu_{2n}, Su_{2n-1}) + \varepsilon M(Tu_{2n}, Su_{2n-1}), \\ d(u_{2n+2}, u_{2n+1}) &\leq H(Su_{2n+1}, Tu_{2n}) + \varepsilon M(Su_{2n+1}, Tu_{2n}). \end{aligned} \quad (2.4)$$

Now, we consider two cases and show that for any $n \in N$,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \quad (2.5)$$

Case 1. If $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$, then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.6)$$

Therefore by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \quad (2.7)$$

Case 2. If $d(u_{2n}, Tu_{2n}) \geq d(u_{2n-1}, Su_{2n-1})$, then

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.8)$$

So by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \quad (2.9)$$

Hence in either case we obtain by (2.7) and (2.9),

$$\begin{aligned} d(u_{2n}, u_{2n+1}) &\leq H(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) \\ &\leq rM(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) = \beta M(Su_{2n-1}, Tu_{2n}) \\ &= \beta \max\left\{d(u_{2n-1}, u_{2n}), d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n}), \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2}\right\} \\ &\leq \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}. \end{aligned} \quad (2.10)$$

This yields (2.5). Analogously, we obtain $d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n})$, and conclude that for any $n \in N$,

$$d(u_{n+1}, u_n) \leq \beta d(u_n, u_{n-1}). \quad (2.11)$$

Therefore $\{u_n\}$ is a Cauchy sequence and has a limit in X . Call it u .

Now we show that for any $y \in X - \{u\}$,

$$d(u, Ty) \leq r \max\{d(u, y), d(y, Ty)\}, \quad (2.12)$$

$$d(u, Sy) \leq r \max\{d(u, y), d(y, Sy)\}. \quad (2.13)$$

Since $u_n \rightarrow u$, there exists $n_0 \in \mathbb{N}$ (natural numbers) such that

$$d(u, u_n) \leq \frac{1}{3}d(u, y) \quad \text{for } y \neq u \text{ and all } n \geq n_0. \quad (2.14)$$

Then as in [2, page 1862],

$$\begin{aligned} \varphi(r)d(u_{2n-1}, Su_{2n-1}) &\leq d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, u_{2n}) \leq d(u_{2n-1}, u) + d(u, u_{2n}) \\ &\leq \frac{2}{3}d(y, u) = d(y, u) - \frac{1}{3}d(y, u) \leq d(y, u) - d(u_{2n-1}, u) \\ &\leq d(u_{2n-1}, y). \end{aligned} \quad (2.15)$$

Therefore

$$\varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, y). \quad (2.16)$$

Now either $d(u_{2n-1}, Su_{2n-1}) \leq d(y, Ty)$ or $d(y, Ty) \leq d(u_{2n-1}, Su_{2n-1})$.

So in either case by (2.16),

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(y, Ty)\} \leq d(u_{2n-1}, y). \quad (2.17)$$

Hence by the assumption (2.1),

$$\begin{aligned} d(u_{2n}, Ty) &\leq H(Su_{2n-1}, Ty) \leq rM(Su_{2n-1}, Ty) \\ &\leq r \max\left\{d(u_{2n-1}, y), d(u_{2n-1}, Su_{2n-1}), d(y, Ty), \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2}\right\}. \end{aligned} \quad (2.18)$$

Making $n \rightarrow \infty$,

$$\begin{aligned} d(u, Ty) &\leq r \max\left\{d(u, y), d(u, u), d(y, Ty), \frac{d(u, Ty) + d(y, u)}{2}\right\} \\ &\leq r \max\{d(u, y), d(y, Ty), d(u, Ty)\}. \end{aligned} \quad (2.19)$$

This yields (2.12). Similarly, we can show (2.13).

Now, we show that $u \in Su \cap Tu$.

For $0 \leq r < 1/2$, the following cases arise.

Case 1. Suppose $u \notin Su$ and $u \notin Tu$. Then as in [8, page 6], let $a \in Tu$ be such that

$$2rd(a, u) < d(u, Tu), \quad (2.20)$$

and $a \in Su$ be such that $2rd(a, u) < d(u, Su)$.

Since $a \in Tu$ implies $a \neq u$, we have from (2.12) and (2.13),

$$d(u, Ta) \leq r \max\{d(u, a), d(a, Ta)\}, \quad (2.21)$$

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \quad (2.22)$$

On the other hand, since $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \quad (2.23)$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa, a) &\leq H(Sa, Tu) \leq r \max\left\{d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2}\right\} \\ &= r \max\left\{d(a, u), d(a, Sa), \frac{1}{2}d(u, Sa)\right\}. \end{aligned} \quad (2.24)$$

This gives $d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u)$.

So by (2.22), $d(Sa, u) \leq rd(a, u)$. Thus

$$\begin{aligned} d(u, Tu) &\leq d(u, Sa) + H(Sa, Tu) \\ &\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad (\text{by the assumption of Case 1}). \end{aligned} \quad (2.25)$$

This contradicts $u \notin Tu$. Consequently $u \in Tu$. Similarly $u \in Su$.

Case 2. Let $u \in Su$ and $u \notin Tu$. Then as in the previous case, let $a \in Tu$ be such that

$$2rd(a, u) < d(u, Tu). \quad (2.26)$$

Since $a \neq u$, we have from (2.13),

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \quad (2.27)$$

On the other hand, Since $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \quad (2.28)$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa, a) &\leq H(Sa, Tu) \leq r \max \left\{ d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2} \right\} \\ &= r \max \left\{ d(a, u), d(a, Sa), \frac{1}{2}d(u, Sa) \right\}. \end{aligned} \quad (2.29)$$

This gives $d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u)$.

So by (2.22), $d(Sa, u) \leq rd(a, u)$. Thus

$$\begin{aligned} d(u, Tu) &\leq d(u, Sa) + H(Sa, Tu) \\ &\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad (\text{by the assumption of Case 2}). \end{aligned} \quad (2.30)$$

This contradicts $u \notin Tu$. Consequently $u \in Tu$.

Case 3. $u \in Tu$ and $u \notin Su$. As in the previous case, it follows that $u \in Su$.

Now we consider the case $1/2 \leq r < 1$.

First we show that

$$H(Sx, Tu) \leq r \max \left\{ d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2} \right\}. \quad (2.31)$$

Assume that $x \neq u$. Then for every $n \in \mathbb{N}$, there exists $z_n \in Sx$ such that

$$d(u, z_n) \leq d(u, Sx) + \frac{1}{n}d(x, u). \quad (2.32)$$

Therefore

$$\begin{aligned} d(x, Sx) &\leq d(x, z_n) \leq d(x, u) + d(u, z_n) \\ &\leq d(x, u) + d(u, Sx) + \frac{1}{n}d(x, u). \end{aligned} \quad (2.33)$$

Using (2.13) with $y = x$, (2.33) implies

$$d(x, Sx) \leq d(x, u) + r \max \{ d(x, u), d(x, Sx) \} + \frac{1}{n}d(u, x). \quad (2.34)$$

If $d(x, u) \geq d(x, Sx)$, then (2.34) gives

$$\begin{aligned} d(x, Sx) &\leq d(x, u) + rd(x, u) + \frac{1}{n}d(u, x) \\ &= \left(1 + r + \frac{1}{n} \right) d(x, u). \end{aligned} \quad (2.35)$$

Making $n \rightarrow \infty$,

$$d(x, Sx) \leq (1 + r)d(x, u). \quad (2.36)$$

Thus $\varphi(r)d(x, Sx) = (1 - r)d(x, Sx) \leq (1/(1 + r))d(x, Sx) \leq d(x, u)$.

Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption (2.1),

$$H(Sx, Tu) \leq r \max\left\{d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2}\right\}. \quad (2.37)$$

If $d(x, u) < d(x, Sx)$, then (2.34) gives

$$d(x, Sx) \leq d(x, u) + rd(x, Sx) + \frac{1}{n}d(u, x), \quad (2.38)$$

that is, $(1 - r)d(x, Sx) \leq (1 + (1/n))d(x, u)$. □

Making $n \rightarrow \infty$,

$$\varphi(r)d(x, Sx) \leq d(x, u). \quad (2.39)$$

Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption, we get (2.37).

Taking $x = u_{2n+1}$ in (2.37) and passing to the limit, we obtain

$$d(u, Tu) \leq rd(u, Tu). \quad (2.40)$$

This gives $u \in Tu$. Analogously, $u \in Su$.

The following result generalizes Theorem 1.2.

Corollary 2.3. *Let X be a complete metric space and S, T maps from X into X . Suppose there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \text{ implies } d(Sx, Ty) \leq rM(Sx, Ty). \quad (2.41)$$

Then S and T have a unique common fixed point.

Proof. For single-valued maps S and T , it comes from Theorem 2.2 that they have a common fixed point. The uniqueness of the common fixed point follows easily. □

Remark 2.4. Theorem 1.1 is obtained as a particular case of Theorem 2.2 when $S = T$.

Now we derive the following result due to Đorić and Lazović [9, Corollary 2.3].

Corollary 2.5. Let X be a complete metric space and T a map from X into X . Suppose there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rM(Tx, Ty). \quad (2.42)$$

Then T has a unique fixed point.

Proof. It comes from Corollary 2.3 when $S = T$. $\square$

The following example shows the generality of our results.

Example 2.6. Let $X = \{(0, 0), (0, 4), (4, 0), (0, 5), (5, 0), (4, 5), (5, 4)\}$ be endowed with the metric d defined by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|. \quad (2.43)$$

Let S and T be such that

$$S(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, 0) & \text{if } x_1 > x_2, \end{cases} \quad T(x_1, x_2) = \begin{cases} (x_2, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases} \quad (2.44)$$

Then S and T do not satisfy the condition (1.6) of Theorem 1.2 at $x = (4, 5)$, $y = (5, 4)$. However, this is readily verified that all the hypotheses of Corollary 2.3 are satisfied for the maps S and T .

Theorem 2.7. Let X be a complete metric space and $P, Q : X \rightarrow BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y) \quad (2.45)$$

implies

$$\rho(Px, Qy) \leq r \max\left\{d(x, y), \rho(x, Px), \rho(y, Qy), \frac{d(x, Qy) + d(y, Px)}{2}\right\}. \quad (2.46)$$

Then there exists a unique point $z \in X$ such that $z \in Pz \cap Qz$.

Proof. Choose $\lambda \in (0, 1)$. Define single-valued maps $S, T : X \rightarrow X$ as follows. For each $x \in X$, let Sx be a point of Px which satisfies

$$d(x, Sx) \geq r^\lambda \rho(x, Px). \quad (2.47)$$

Similarly, for each $y \in X$, let Ty be a point of Qy such that

$$d(y, Ty) \geq r^\lambda \rho(y, Qy). \quad (2.48)$$

Since $Sx \in Px$ and $Ty \in Qy$,

$$d(x, Sx) \leq \rho(x, Px), \quad d(y, Ty) \leq \rho(y, Qy). \quad (2.49)$$

So, (2.45) gives

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y), \quad (2.50)$$

and this implies (2.46). Therefore

$$\begin{aligned} d(Sx, Ty) &\leq \rho(Px, Qy) \\ &\leq r \cdot r^{-\lambda} \max\left\{r^\lambda d(x, y), r^\lambda \rho(x, Px), r^\lambda \rho(y, Qy), \frac{r^\lambda d(x, Qy) + r^\lambda d(y, Px)}{2}\right\} \\ &\leq r^{1-\lambda} \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}. \end{aligned} \quad (2.51)$$

So (2.50), namely, $\varphi(r') \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$ implies

$$d(Sx, Ty) \leq r' \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}, \quad (2.52)$$

where $r' = r^{1-\lambda} < 1$.

Hence by Theorem 2.2, S and T have a unique point $z \in X$ such that $Sz = Tz = z$. This implies $z \in Pz \cap Qz$. $\square$

Corollary 2.8. *Let X be a complete metric space and $P : X \rightarrow BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\begin{aligned} \rho(x, Px) &\leq (1+r)d(x, y) \text{ implies} \\ \rho(Px, Py) &\leq r \max\left\{d(x, y), \rho(x, Px), \rho(y, Py), \frac{d(x, Py) + d(y, Px)}{2}\right\}. \end{aligned} \quad (2.53)$$

Then there exists a unique point $z \in X$ such that $z \in Pz$.

Proof. It comes from Theorem 2.7 when $Q = P$. $\square$

3. Applications

Throughout this section, we assume that Y and Z are Banach spaces, $W \subseteq Y$ and $D \subseteq Z$. Let R denotes the field of reals, $g_1, g_2 : W \times D \rightarrow R$ and $G_1, G_2 : W \times D \times R \rightarrow R$. Taking W and D

as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

$$p_i = \sup_{y \in D} \{g_i(x, y) + H_i(x, y, p_i(x, y))\}, \quad x \in W, \quad i = 1, 2. \quad (3.1)$$

In the multistage process, some functional equations arise in a natural way (cf. [22, 23]; see also [21, 24, 28, 29]). In this section, we study the existence of common solution of the functional equations (3.1) arising in dynamic programming.

Let $B(W)$ denotes the set of all bounded real-valued functions on W . For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) H_1, H_2, g_1 , and g_2 are bounded.

(DP-2) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D, h, k \in B(W)$ and $t \in W$,

$$\varphi(r) \min\{|h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|\} \leq |h(t) - k(t)| \quad (3.2)$$

implies

$$\begin{aligned} & |H_1(x, y, h(t)) - H_2(x, y, k(t))| \\ & \leq r \max \left\{ |h(t) - k(t)|, |h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|, \frac{|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|}{2} \right\}, \end{aligned} \quad (3.3)$$

where A_1, A_2 are defined as follows:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(x, y)), \quad x \in W, \quad h \in B(W), \quad i = 1, 2. \quad (3.4)$$

Theorem 3.1. Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.1), $i = 1, 2$, have a unique common solution in $B(W)$.

Proof. For any $h, k \in B(W)$, let $d(h, k) = \sup\{|h(x) - k(x)| : x \in W\}$. Then $(B(W), d)$ is a complete metric space.

Let λ be any arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$A_i h_i < H_i(x, y_i, h_i(x_i)) + \lambda, \quad (3.5)$$

where $x_i = (x, y_i)$, $i = 1, 2$.

Further,

$$A_1 h_1 \geq H_1(x, y_2, h_1(x_2)), \quad (3.6)$$

$$A_2 h_2 \geq H_2(x, y_1, h_2(x_1)). \quad (3.7)$$

Therefore, the first inequality in (DP-2) becomes

$$\varphi(r) \min\{|h_1(x) - A_1 h_1(x)|, |h_2(x) - A_2 h_2(x)|\} \leq |h_1(x) - h_2(x)|, \quad (3.8)$$

and this together with (3.5) and (3.7) implies

$$\begin{aligned} A_1 h_1 - A_2 h_2 &< H_1(x, y_1, h_1(x_1)) - H_2(x, y, h_2(x_1)) + \lambda \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \lambda \\ &\leq rM(H_1 h_1, H_2 h_2) + \lambda. \end{aligned} \quad (3.9)$$

Similarly, (3.5), (3.6), and (3.8) imply

$$A_2 h_2(x) - A_1 h_1(x) \leq rM(A_1 h_1, A_2 h_2) + \lambda. \quad (3.10)$$

So, from (3.10) and (3.11), we obtain

$$|A_1 h_1(x) - A_2 h_2(x)| \leq rM(A_1 h_1, A_2 h_2) + \lambda. \quad (3.11)$$

Since this inequality is true for any $x \in W$, and $\lambda > 0$ is arbitrary, on taking supremum, we find from (3.8) and (3.11) that

$$\varphi(r) \min\{d(h_1, A_1 h_1), d(h_2, A_2 h_2)\} \leq d(h_1, h_2) \quad (3.12)$$

implies

$$d(A_1 h_1, A_2 h_2) \leq rM(A_1 h_1, A_2 h_2). \quad (3.13)$$

Therefore, Corollary 2.3 applies, wherein A_1 and A_2 correspond, respectively, to the maps S and T . So A_1 and A_2 have a unique common fixed point h^* , that is, $h^*(x)$ is the unique bounded common solution of the functional equations (3.1), $i = 1, 2$. $\square$

The following result generalizes a recent result of Singh and Mishra [12, Corollary 4.2] which in turn extends certain results from [21, 23, 24].

Corollary 3.2. *Suppose that the following conditions hold.*

- (i) G and g are bounded.
- (ii) There exists $r \in [0, 1)$ such that for every $x, y \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\begin{aligned} \varphi(r)|h(t) - Kh(t)| &\leq |h(t) - k(t)| \text{ implies} \\ |G(x, y, h(t)) - G(x, y, k(t))| &\leq r \max M(K, h(t), k(t)), \end{aligned} \quad (3.14)$$

where K is defined as

$$Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(t, y))\}, \quad t \in W, \quad h \in B(W). \quad (3.15)$$

Then the functional equation (3.1) with $H_1 = H_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in W .

Proof. It comes from Theorem 3.1 when $g_1 = g_2 = g$ and $H_1 = H_2 = G$. $\square$

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References

- [1] S. B. Nadler Jr., *Hyperspaces of Sets: A Text with Research Questions*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49, Marcel Dekker, New York, NY, USA, 1978.
- [2] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.
- [3] M. Kikkawa and T. Suzuki, "Three fixed point theorems for generalized contractions with constants in complete metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2942–2949, 2008.
- [4] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
- [5] G. Jungck, "Commuting mappings and fixed points," *The American Mathematical Monthly*, vol. 83, no. 4, pp. 261–263, 1976.
- [6] A. Meir and E. Keeler, "A theorem on contraction mappings," *Journal of Mathematical Analysis and Applications*, vol. 28, pp. 326–329, 1969.
- [7] A. Abkar and M. Eslamian, "Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 457935, 10 pages, 2010.
- [8] S. Dhompongsa and H. Yingtaeewesittikul, "Fixed points for multivalued mappings and the metric completeness," *Fixed Point Theory and Applications*, vol. 2009, Article ID 972395, 15 pages, 2009.
- [9] D. Đorić and R. Lazović, "Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications," *Fixed Point Theory and Applications*, vol. 2011, article 40, 2011.
- [10] G. Moş and A. Petruşel, "Fixed point theory for a new type of contractive multivalued operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3371–3377, 2009.
- [11] O. Popescu, "Two fixed point theorems for generalized contractions with constants in complete metric space," *Central European Journal of Mathematics*, vol. 7, no. 3, pp. 529–538, 2009.
- [12] S. L. Singh and S. N. Mishra, "Coincidence theorems for certain classes of hybrid contractions," *Fixed Point Theory and Applications*, vol. 2010, Article ID 898109, 14 pages, 2010.
- [13] S. L. Singh and S. N. Mishra, "Remarks on recent fixed point theorems," *Fixed Point Theory and Applications*, vol. 2010, Article ID 452905, 18 pages, 2010.
- [14] S. L. Singh, H. K. Pathak, and S. N. Mishra, "On a Suzuki type general fixed point theorem with applications," *Fixed Point Theory and Applications*, vol. 2010, Article ID 234717, 15 pages, 2010.
- [15] L. B. Ćirić, "Fixed points for generalized multi-valued contractions," *Matematički Vesnik*, vol. 9(24), pp. 265–272, 1972.
- [16] S. Reich, "Fixed points of multi-valued functions," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 51, pp. 32–35, 1971.
- [17] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.
- [18] L. B. Ćirić, "On a family of contractive maps and fixed points," *Institut Mathématique. Publications. Nouvelle Série*, vol. 17(31), pp. 45–51, 1974.
- [19] K. P. R. Sastry and S. V. R. Naidu, "Fixed point theorems for generalised contraction mappings," *Yokohama Mathematical Journal*, vol. 28, no. 1-2, pp. 15–29, 1980.

- [20] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.
- [21] R. Baskaran and P. V. Subrahmanyam, "A note on the solution of a class of functional equations," *Applicable Analysis*, vol. 22, no. 3-4, pp. 235–241, 1986.
- [22] R. Bellman, *Methods of Nonlinear Analysis. Vol. II*, Mathematics in Science and Engineering, Vol. 61-II, Academic Press, New York, NY, USA, 1973.
- [23] R. Bellman and E. S. Lee, "Functional equations in dynamic programming," *Aequationes Mathematicae*, vol. 17, no. 1, pp. 1–18, 1978.
- [24] P. C. Bhakta and S. Mitra, "Some existence theorems for functional equations arising in dynamic programming," *Journal of Mathematical Analysis and Applications*, vol. 98, no. 2, pp. 348–362, 1984.
- [25] N. A. Assad and W. A. Kirk, "Fixed point theorems for set-valued mappings of contractive type," *Pacific Journal of Mathematics*, vol. 43, pp. 553–562, 1972.
- [26] B. Djafari Rouhani and S. Moradi, "Common fixed point of multivalued generalized ϕ -weak contractive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 708984, 13 pages, 2010.
- [27] I. A. Rus, "Fixed point theorems for multi-valued mappings in complete metric spaces," *Mathematica Japonica*, vol. 20, pp. 21–24, 1975.
- [28] H. K. Pathak, Y. J. Cho, S. M. Kang, and B. S. Lee, "Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming," *Le Matematiche*, vol. 50, no. 1, pp. 15–33, 1995.
- [29] S. L. Singh and S. N. Mishra, "On a Ljubomir Ćirić fixed point theorem for nonexpansive type maps with applications," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 4, pp. 531–542, 2002.