Research Article

General Common Fixed Point Theorems and Applications

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The main result is a common fixed point theorem for a pair of multivalued maps on a complete metric space extending a recent result of Đorić and Lazović (2011) for a multivalued map on a metric space satisfying Cirić-Suzuki-type-generalized contraction. Further, as a special case, we obtain a generalization of an important common fixed point theorem of Cirić (1974). Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

1. Introduction

Consistent with Nadler [1, page 620], (X, d) will denote a metric space and CL(X), the collection of all nonempty closed subsets of X. For A, B ∈ CL(X) and ε > 0,

\[ N(\varepsilon, A) = \{ x \in X : d(x, a) < \varepsilon \text{ for some } a \in A \}, \]

\[ E_{A,B} = \{ \varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A) \}, \]

\[ H(A, B) = \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset \\ +\infty, & \text{if } E_{A,B} = \emptyset. \end{cases} \tag{1.1} \]

The hyperspace (CL(X), H) is called the generalized Hausdorff metric space induced by the metric d on X.
For nonempty subsets \( A, B \) of \( X \), \( d(A, B) \) denotes the gap between the subsets \( A \) and \( B \), while

\[
\rho(A, B) = \sup \{d(a, b) : a \in A, b \in B\},
\]

\[
BN(X) = \{A : \emptyset \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}.
\]

As usual, we write \( d(x, B) \) (resp. \( \rho(x, B) \)) for \( d(A, B) \) (resp. \( \rho(A, B) \)) when \( A = \{x\} \).

Let \( S, T : X \to \text{CL}(X) \). Then \( u \in X \) is a fixed point of \( S \) if and only if \( u \in Su \) and a common fixed point of \( S \) and \( T \) if and only if \( u \in Su \cap Tu \).

Let \( S \) and \( T \) be maps to be defined specifically in a particular context, while \( x \) and \( y \) are the elements of a metric space \((X, d)\):

\[
M(Sx, Ty) = \max \left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}.
\]

Recently Suzuki [2] and Kikkawa and Suzuki [3] obtained interesting generalizations of the Banach’s classical fixed point theorem and other fixed point results by Nadler [4], Jungck [5], and Meir and Keeler [6]. These results have important outcomes (see, e.g., [7–14]). The following result, due to Đorić and Lazović [9], extends and generalizes fixed point theorems from Ćirić [15], Kikkawa and Suzuki [3], Nadler [4], Reich [16], Rus [17], and others.

**Theorem 1.1.** Define a nonincreasing function \( \varphi \) from \([0, 1)\) onto \((0, 1]\) by

\[
\varphi(r) = \begin{cases} 
1 & \text{if } 0 \leq r < \frac{1}{2} \\
1 - r & \text{if } \frac{1}{2} \leq r < 1.
\end{cases}
\]

Let \( X \) be a complete metric space and \( T : X \to \text{CL}(X) \). Assume there exists \( r \in [0, 1) \) such that for every \( x, y \in X \),

\[
\varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq rM(Tx, Ty).
\]

Then there exists \( z \in X \) such that \( z \in Tz \).

We remark that, for every \( x, y \in X \), the generalized contraction \( H(Tx, Ty) \leq rM(Tx, Ty), 0 \leq r < 1 \), was first studied by Ćirić [15]. The following important common fixed point theorem is due to Ćirić [18].

**Theorem 1.2.** Let \( X \) be a complete metric space and \( S, T : X \to X \). Assume there exists \( r \in [0, 1) \) such that for every \( x, y \in X \),

\[
d(Sx, Ty) \leq rM(Sx, Ty).
\]

Then \( S \) and \( T \) have a unique common fixed point.
For an excellent discussion on several special cases and variants of Theorem 1.2, one may refer to Rus [17]. However, the generality of Theorem 1.2 may be appreciated from the fact that (1.6) in Theorem 1.2 cannot be replaced by
\[ d(Sx,Ty) \leq r \max\{d(x,y),d(x,Sx),d(y,Ty),d(x,Ty),d(y,Sx)\}. \] (1.7)

Indeed, Sastry and Naidu [19, Example 5] have shown that maps \( S \) and \( T \) satisfying (1.7) need not have a common fixed point on a complete metric space. Notice that the condition (1.7) with \( S = T \) is the quasicontraction due to Ćirić [20].

The main result of this paper (cf. Theorem 2.2) generalizes Theorems 1.1 and 1.2. Further, a corollary of Theorem 2.2 is used to obtain a unique common fixed point theorem for multivalued maps on a metric space with values in \( BN(X) \). As another application, we deduce the existence of a common solution for a general class of functional equations under much weaker conditions than those in [12, 14, 21–24].

2. Main Results

We shall need the following result essentially due to Nadler [4] (see also [15, 25], [26, page 4], [27], [17, page 76]).

**Lemma 2.1.** If \( A, B \in \text{CL}(X) \) and \( a \in A \), then for each \( \varepsilon > 0 \), there exists \( b \in B \) such that \( d(a,b) \leq H(A,B) + \varepsilon \).

**Theorem 2.2.** Let \( X \) be a complete metric space and \( S, T : X \to \text{CL}(X) \). Assume there exists \( r \in [0,1) \) such that for every \( x, y \in X \),
\[ \varphi(r) \min\{d(x,Sx),d(y,Ty)\} \leq d(x,y) \text{ implies } H(Sx,Ty) \leq r M(Sx,Ty). \] (2.1)

Then there exists an element \( u \in X \) such that \( u \in Su \cap Tu \).

**Proof.** Obviously \( M(Sx,Ty) = 0 \) iff \( x = y \) is a common fixed point of \( S \) and \( T \). So, we may take without any loss of generality that \( M(Sx,Ty) > 0 \) for distinct \( x, y \in X \). Let \( \varepsilon > 0 \) be such that \( \beta = r + \varepsilon < 1 \). Let \( u_0 \in X \) and \( u_1 \in Tu_0 \). Then by Lemma 2.1, their exists \( u_2 \in Su_1 \) such that
\[ d(u_2,u_1) \leq H(Su_1,Tu_0) + \varepsilon M(Su_1,Tu_0). \] (2.2)

Similarly, their exists \( u_3 \in Tu_2 \) such that
\[ d(u_3,u_2) \leq H(Tu_2,Su_1) + \varepsilon M(Tu_2,Su_1). \] (2.3)

Continuing in this manner, we find a sequence \( \{u_n\} \) in \( X \) such that
\[ u_{2n+1} \in Tu_{2n}, \quad u_{2n+2} \in Su_{2n+1} \text{ such that} \]
\[ d(u_{2n+1},u_{2n}) \leq H(Tu_{2n},Su_{2n-1}) + \varepsilon M(Tu_{2n},Su_{2n-1}), \] (2.4)
\[ d(u_{2n+2},u_{2n+1}) \leq H(Su_{2n+1},Tu_{2n}) + \varepsilon M(Su_{2n+1},Tu_{2n}). \]
Now, we consider two cases and show that for any $n \in N$,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \quad (2.5)$$

**Case 1.** If $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$, then

$$\varphi(r) \min \{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.6)$$

Therefore by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq r M(Su_{2n-1}, Tu_{2n}). \quad (2.7)$$

**Case 2.** If $d(u_{2n}, Tu_{2n}) \geq d(u_{2n-1}, Su_{2n-1})$, then

$$\varphi(r) \min \{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \quad (2.8)$$

So by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq r M(Su_{2n-1}, Tu_{2n}). \quad (2.9)$$

Hence in either case we obtain by (2.7) and (2.9),

$$d(u_{2n}, u_{2n+1})$$

$$\leq H(Su_{2n-1}, Tu_{2n}) + \epsilon M(Su_{2n-1}, Tu_{2n})$$

$$\leq r M(Su_{2n-1}, Tu_{2n}) + \epsilon M(Su_{2n-1}, Tu_{2n}) = \beta M(Su_{2n-1}, Tu_{2n})$$

$$= \beta \max \left\{ d(u_{2n-1}, u_{2n}), d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n}), \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2} \right\}$$

$$\leq \beta \max \{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}. \quad (2.10)$$

This yields (2.5). Analogously, we obtain $d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n})$, and conclude that for any $n \in N$,

$$d(u_{n+1}, u_{n}) \leq \beta d(u_{n}, u_{n-1}). \quad (2.11)$$

Therefore $\{u_n\}$ is a Cauchy sequence and has a limit in $X$. Call it $u$. Now we show that for any $y \in X - \{u\}$,

$$d(u, Ty) \leq r \max \{d(u, y), d(y, Ty)\}, \quad (2.12)$$

$$d(u, Sy) \leq r \max \{d(u, y), d(y, Sy)\}. \quad (2.13)$$
Since \( u_n \to u \), there exists \( n_0 \in N \) (natural numbers) such that

\[ d(u, u_n) \leq \frac{1}{3} d(u, y) \quad \text{for } y \neq u \text{ and all } n \geq n_0. \]  

(2.14)

Then as in [2, page 1862],

\[ \varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, Su_{2n}) \leq d(u_{2n-1}, u_{2n}) \leq d(u, u_{2n}) + d(u, u_{2n}) \]

\[ \leq \frac{2}{3} d(y, u) = d(y, u) - \frac{1}{3}d(y, u) \leq d(y, u) - d(u_{2n-1}, u) \]  

(2.15)

\[ \leq d(u_{2n-1}, y). \]

Therefore

\[ \varphi(r)d(u_{2n-1}, Su_{2n-1}) \leq d(u_{2n-1}, y). \]  

(2.16)

Now either \( d(u_{2n-1}, Su_{2n-1}) \leq d(y, Ty) \) or \( d(y, Ty) \leq d(u_{2n-1}, Su_{2n-1}) \).

So in either case by (2.16),

\[ \varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(y, Ty)\} \leq d(u_{2n-1}, y). \]  

(2.17)

Hence by the assumption (2.1),

\[ d(u_{2n}, Ty) \leq H(Su_{2n-1}, Ty) \leq r M(Su_{2n-1}, Ty) \]

\[ \leq r \max\left\{ d(u_{2n-1}, y), d(u_{2n-1}, Su_{2n}), d(y, Ty), \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2} \right\}. \]  

(2.18)

Making \( n \to \infty \),

\[ d(u, Ty) \leq r \max\left\{ d(u, y), d(u, u), d(y, Ty), \frac{d(u, Ty) + d(y, u)}{2} \right\} \]  

\[ \leq r \max\{d(u, y), d(y, Ty), d(u, Ty)\}. \]  

(2.19)

This yields (2.12). Similarly, we can show (2.13).

Now, we show that \( u \in Su \cap Tu \).

For \( 0 \leq r < 1/2 \), the following cases arise.

Case 1. Suppose \( u \notin Su \) and \( u \notin Tu \). Then as in [8, page 6], let \( a \in Tu \) be such that

\[ 2rd(a, u) < d(u, Tu), \]

(2.20)

and \( a \in Su \) be such that \( 2rd(a, u) < d(u, Su) \).
Case 2. Let \( u \neq a \). This contradicts a/.

Since \( a \in Tu \) implies \( a \neq u \), we have from (2.12) and (2.13),

\[
d(u, Ta) \leq r \max\{d(u, a), d(a, Ta)\},
\]
\[
d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}.
\]

On the other hand, since \( \varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u), \)

\[
\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u).
\]

Therefore by the assumption (2.1),

\[
d(Sa, a) \leq H(Sa, Tu) \leq r \max\left\{d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2}\right\}
\]
\[
= r \max\left\{d(a, u), d(a, Sa), \frac{1}{2}d(u, Sa)\right\}.
\]

This gives \( d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u). \)

So by (2.22), \( d(Sa, u) \leq rd(a, u). \) Thus

\[
d(u, Tu) \leq d(u, Sa) + H(Sa, Tu)
\]
\[
\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad \text{(by the assumption of Case 1).}
\]

This contradicts \( u \notin Tu \). Consequently \( u \in Tu \). Similarly \( u \in Su \).

Case 2. Let \( u \in Su \) and \( u \notin Tu \). Then as in the previous case, let \( a \in Tu \) be such that

\[
2rd(a, u) < d(u, Tu).
\]

Since \( a \neq u \), we have from (2.13),

\[
d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}.
\]

On the other hand, since \( \varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u), \)

\[
\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u).
\]
Therefore by the assumption (2.1),
\[
d(Sa, a) \leq H(Sa,Tu) \leq r \max \left\{ d(a,u), d(u,Tu), d(a,Sa), \frac{d(u,Sa) + d(a,Tu)}{2} \right\}
= r \max \left\{ d(a,u), d(a,Sa), \frac{1}{2} d(u,Sa) \right\}.
\] (2.29)

This gives \(d(a,Sa) \leq H(Sa,Tu) \leq rd(a,u) < d(a,u)\).
So by (2.22), \(d(Sa,u) \leq rd(a,u)\). Thus
\[
d(u,Tu) \leq d(u,Sa) + H(Sa,Tu)
\leq rd(a,u) + rd(a,u) = 2rd(a,u) < d(u,Tu) \quad \text{(by the assumption of Case 2)}.
\] (2.30)

This contradicts \(u \not\in Tu\). Consequently \(u \in Tu\).

Case 3. \(u \in Tu\) and \(u \not\in Su\). As in the previous case, it follows that \(u \in Su\).
Now we consider the case \(1/2 \leq r < 1\).
First we show that
\[
H(Sx,Tu) \leq r \max \left\{ d(x,u), d(x,Sx), d(u,Tu), \frac{d(x,Tu) + d(u,Sx)}{2} \right\}.
\] (2.31)

Assume that \(x \neq u\). Then for every \(n \in N\), there exists \(z_n \in Sx\) such that
\[
d(u,z_n) \leq d(u,Sx) + \frac{1}{n} d(x,u).
\] (2.32)

Therefore
\[
d(x,Sx) \leq d(x,z_n) \leq d(x,u) + d(u,z_n)
\leq d(x,u) + d(u,Sx) + \frac{1}{n} d(x,u).
\] (2.33)

Using (2.13) with \(y = x\), (2.33) implies
\[
d(x,Sx) \leq d(x,u) + r \max \{d(x,u),d(x,Sx)\} + \frac{1}{n} d(u,x).
\] (2.34)

If \(d(x,u) \geq d(x,Sx)\), then (2.34) gives
\[
d(x,Sx) \leq d(x,u) + rd(x,u) + \frac{1}{n} d(u,x)
\leq \left(1 + r + \frac{1}{n}\right) d(x,u).
\] (2.35)
Making $n \to \infty$,\[ d(x, Sx) \leq (1 + r)d(x, u). \] (2.36)

Thus $\varphi(r)d(x, Sx) = (1 - r)d(x, Sx) \leq (1/(1 + r))d(x, Sx) \leq d(x, u)$. Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption (2.1),
\[ H(Sx, Tu) \leq r \max \left\{ d(x, u), d(x, Sx), d(u, Tu), \frac{d(x, Tu) + d(u, Sx)}{2} \right\}. \] (2.37)

If $d(x, u) < d(x, Sx)$, then (2.34) gives
\[ d(x, Sx) \leq d(x, u) + rd(x, Sx) + \frac{1}{n}d(u, x), \] (2.38)
that is, $(1 - r)d(x, Sx) \leq (1 + (1/n))d(x, u)$.

Making $n \to \infty$,
\[ \varphi(r)d(x, Sx) \leq d(x, u). \] (2.39)

Then $\varphi(r) \min\{d(x, Sx), d(u, Tu)\} \leq d(x, u)$, and by the assumption, we get (2.37).
Taking $x = u_{2n+1}$ in (2.37) and passing to the limit, we obtain
\[ d(u, Tu) \leq rd(u, Tu). \] (2.40)

This gives $u \in Tu$. Analogously, $u \in Su$.

The following result generalizes Theorem 1.2.

**Corollary 2.3.** Let $X$ be a complete metric space and $S, T$ maps from $X$ into $X$. Suppose there exists $r \in [0, 1)$ such that for every $x, y \in X$,
\[ \varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \text{ implies } d(Sx, Ty) \leq rM(Sx, Ty). \] (2.41)

Then $S$ and $T$ have a unique common fixed point.

**Proof.** For single-valued maps $S$ and $T$, it comes from Theorem 2.2 that they have a common fixed point. The uniqueness of the common fixed point follows easily. \[ \square \]

**Remark 2.4.** Theorem 1.1 is obtained as a particular case of Theorem 2.2 when $S = T$.

Now we derive the following result due to Đorić and Lazović [9, Corollary 2.3].
Corollary 2.5. Let $X$ be a complete metric space and $T$ a map from $X$ into $X$. Suppose there exists $r \in [0, 1)$ such that for every $x, y \in X$,
\[ \varphi(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rM(Tx, Ty). \] (2.42)

Then $T$ has a unique fixed point.

Proof. It comes from Corollary 2.3 when $S = T$. \qed

The following example shows the generality of our results.

Example 2.6. Let $X = \{(0, 0), (0, 4), (4, 0), (0, 5), (5, 0), (4, 5), (5, 4)\}$ be endowed with the metric $d$ defined by
\[ d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|. \] (2.43)

Let $S$ and $T$ be such that
\[ S(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \leq x_2 \\ (0, 0) & \text{if } x_1 > x_2, \end{cases} \quad T(x_1, x_2) = \begin{cases} (x_2, 0) & \text{if } x_1 \leq x_2 \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases} \] (2.44)

Then $S$ and $T$ do not satisfy the condition (1.6) of Theorem 1.2 at $x = (4, 5), y = (5, 4)$. However, this is readily verified that all the hypotheses of Corollary 2.3 are satisfied for the maps $S$ and $T$.

Theorem 2.7. Let $X$ be a complete metric space and $P, Q : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,
\[ \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y) \] (2.45)

implies
\[ \rho(Px, Qy) \leq r \max\left\{d(x, y), \rho(x, Px), \rho(y, Qy), \frac{d(x, Qy) + d(y, Px)}{2}\right\}. \] (2.46)

Then there exists a unique point $z \in X$ such that $z \in Px \cap Qz$.

Proof. Choose $\lambda \in (0, 1)$. Define single-valued maps $S, T : X \to X$ as follows. For each $x \in X$, let $Sx$ be a point of $Px$ which satisfies
\[ d(x, Sx) \geq r^4 \rho(x, Px). \] (2.47)

Similarly, for each $y \in X$, let $Ty$ be a point of $Qy$ such that
\[ d(y, Ty) \geq r^4 \rho(y, Qy). \] (2.48)
Since $Sx \in Px$ and $Ty \in Qy$,
\[ d(x, Sx) \leq \rho(x, Px), \quad d(y, Ty) \leq \rho(y, Qy). \] (2.49)

So, (2.45) gives
\[ \varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y), \] (2.50)
and this implies (2.46). Therefore
\[
d(Sx, Ty) \leq \rho(Px, Qy) \\
\leq r \cdot r^{-1} \max\left\{ r^1 d(x, y), r^1 \rho(x, Px), r^1 \rho(y, Qy), \frac{r^1 d(x, Qy) + r^1 d(y, Px)}{2} \right\} \\
\leq r^{-1} \max\left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}. \] (2.51)

So (2.50), namely, $\varphi(r') \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$ implies
\[
d(Sx, Ty) \leq r' \max\left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}, \] (2.52)
where $r' = r^{-1} < 1$.

Hence by Theorem 2.2, $S$ and $T$ have a unique point $z \in X$ such that $Sz = Tz = z$. This implies $z \in Pz \cap Qz$. \hfill \qed

**Corollary 2.8.** Let $X$ be a complete metric space and $P : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,
\[
\rho(x, Px) \leq (1 + r)d(x, y) \] implies
\[
\rho(Px, Py) \leq r \max\left\{ d(x, y), \rho(x, Px), \rho(y, Py), \frac{d(x, Py) + d(y, Px)}{2} \right\}. \] (2.53)

Then there exists a unique point $z \in X$ such that $z \in Pz$.

**Proof.** It comes from Theorem 2.7 when $Q = P$. \hfill \qed

### 3. Applications

Throughout this section, we assume that $Y$ and $Z$ are Banach spaces, $W \subseteq Y$ and $D \subseteq Z$. Let $R$ denotes the field of reals, $g_1, g_2 : W \times D \to R$ and $G_1, G_2 : W \times D \times R \to R$. Taking $W$ and $D$
as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

\[ p_i = \sup_{y \in D} \{ g_i(x, y) + H_i(x, y, p_i(x, y)) \}, \quad x \in W, \ i = 1, 2. \quad (3.1) \]

In the multistage process, some functional equations arise in a natural way (cf. [22, 23]; see also [21, 24, 28, 29]). In this section, we study the existence of common solution of the functional equations (3.1) arising in dynamic programming.

Let \( B(W) \) denotes the set of all bounded real-valued functions on \( W \). For an arbitrary \( h \in B(W) \), define \( \| h \| = \sup_{x \in W} |h(x)| \). Then \( (B(W), \| \cdot \|) \) is a Banach space. Suppose that the following conditions hold:

(DP-1) \( H_1, \ H_2, \ g_1, \) and \( g_2 \) are bounded.

(DP-2) There exists \( r \in (0, 1) \) such that for every \( (x, y) \in W \times D, h, k \in B(W) \) and \( t \in W, \)

\[ \varphi(r) \min\{|h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|\} \leq |h(t) - k(t)| \quad (3.2) \]

implies

\[ |H_1(x, y, h(t)) - H_2(x, y, k(t))| \]

\[ \leq r \max \left\{ |h(t) - k(t)|, |h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|, \frac{|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|}{2} \right\}, \quad (3.3) \]

where \( A_1, \ A_2 \) are defined as follows:

\[ A_i h(x) = \sup_{y \in D} H_i(x, y, h(x, y)), \quad x \in W, \ h \in B(W), \ i = 1, 2. \quad (3.4) \]

**Theorem 3.1.** Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.1), \( i = 1, 2, \) have a unique common solution in \( B(W) \).

**Proof.** For any \( h, k \in B(W) \), let \( d(h, k) = \sup\{|h(x) - k(x)| : x \in W\} \). Then \( (B(W), d) \) is a complete metric space.

Let \( \lambda \) be any arbitrary positive number and \( h_1, h_2 \in B(W) \). Pick \( x \in W \) and choose \( y_1, y_2 \in D \) such that

\[ A_i h_i < H_i(x, y_i, h_i(x_i)) + \lambda, \quad (3.5) \]

where \( x_i = (x, y_i), \ i = 1, 2. \)

Further,

\[ A_1 h_1 \geq H_1(x, y_2, h_1(x_2)), \quad (3.6) \]

\[ A_2 h_2 \geq H_2(x, y_1, h_2(x_1)). \quad (3.7) \]
Therefore, the first inequality in (DP-2) becomes

\[ \varphi(r) \min \{|h_1(x) - A_1h_1(x)|, |h_2(x) - A_2h_2(x)|\} \leq |h_1(x) - h_2(x)|, \]  

(3.8)

and this together with (3.5) and (3.7) implies

\[ A_1h_1 - A_2h_2 < H_1(x, y_1, h_1(x_1)) - H_2(x, y, h_2(x_1)) + \lambda \]
\[ \leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \lambda \]
\[ \leq rM(H_1h_1, H_2h_2) + \lambda. \]  

(3.9)

Similarly, (3.5), (3.6), and (3.8) imply

\[ A_2h_2(x) - A_1h_1(x) \leq rM(A_1h_1, A_2h_2) + \lambda. \]  

(3.10)

So, from (3.10) and (3.11), we obtain

\[ |A_1h_1(x) - A_2h_2(x)| \leq r M(A_1h_1, A_2h_2) + \lambda. \]  

(3.11)

Since this inequality is true for any \( x \in W \), and \( \lambda > 0 \) is arbitrary, on taking supremum, we find from (3.8) and (3.11) that

\[ \varphi(r) \min \{d(h_1, A_1h_1), d(h_2, A_2h_2)\} \leq d(h_1, h_2) \]  

(3.12)

implies

\[ d(A_1h_1, A_2h_2) \leq rM(A_1h_1, A_2h_2). \]  

(3.13)

Therefore, Corollary 2.3 applies, wherein \( A_1 \) and \( A_2 \) correspond, respectively, to the maps \( S \) and \( T \). So \( A_1 \) and \( A_2 \) have a unique common fixed point \( h^* \), that is, \( h^*(x) \) is the unique bounded common solution of the functional equations (3.1), \( i = 1, 2 \).

\( \square \)

The following result generalizes a recent result of Singh and Mishra [12, Corollary 4.2] which in turn extends certain results from [21, 23, 24].

**Corollary 3.2.** Suppose that the following conditions hold.

(i) \( G \) and \( g \) are bounded.

(ii) There exists \( r \in [0, 1) \) such that for every \( x, y \in W \times D, \ h, k \in B(W) \) and \( t \in W \),

\[ \varphi(r)|h(t) - Kh(t)| \leq |h(t) - k(t)| \]  

implies

\[ |G(x, y, h(t)) - G(x, y, k(t))| \leq r \max M(K, h(t), k(t)), \]  

(3.14)

where \( K \) is defined as

\[ Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(y))\}, \quad t \in W, \ h \in B(W). \]  

(3.15)
Then the functional equation (3.1) with $H_1 = H_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in $W$.

Proof. It comes from Theorem 3.1 when $g_1 = g_2 = g$ and $H_1 = H_2 = G$. □

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