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Research Article

On Convergence Results for Lipschitz Pseudocontractive Mappings

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We establish the strong convergence for the Ishikawa iteration scheme associated with Lipschitz pseudocontractive mappings in real Banach spaces. Moreover, our technique of proofs is of independent interest.

1. Introduction and Preliminaries

Let E be a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2,\ \|f^*\| = \|x\| \right\}, \quad \forall x \in E, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality mapping by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1.1. T is said to be *Lipschitz* if there exists a constant $L > 1$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in D(T). \tag{1.2}$$

Definition 1.2. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \tag{1.3}$$

Definition 1.3. T is said to be *pseudocontractive* if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x - y\|^2. \tag{1.4}$$

Remark 1.4. It is well known that every nonexpansive mapping is pseudocontractive. Indeed, if T is nonexpansive mapping, then for all $x,y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|Tx - Ty\|\|x - y\| \leq \|x - y\|^2. \tag{1.5}$$

Rhoades [1] showed that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings.

The class of pseudocontractions is, perhaps, the most important generalization of the class of nonexpansive mappings because of its strong relationship with the class of accretive mappings. A mapping $A : E \to E$ is accretive if and only if $I - A$ is pseudocontractive.

For a nonempty convex subset K of a normed space E and a mapping $T : K \to K$.

The Mann iteration scheme [2]: the sequence $\{x_n\}$ is defined by

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad n \geq 1, \end{gathered} \tag{1.6}$$

where $\{a_n\}$ is a sequence in $[0,1]$.

The Ishikawa iteration scheme [3]: the sequence $\{x_n\}$ is defined by

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \geq 1, \end{gathered} \tag{1.7}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$.

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive mappings using the Ishikawa iteration scheme (e.g., [3]). Results which had been known only in Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (e.g., [4–6] and the references cited therein).

In 1974, Ishikawa [3] introduced an iteration scheme which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T. He proved the following result.

Theorem 1.5. *Let K be a compact convex subset of a Hilbert space H and let $T : K \to K$ be a Lipschitz pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences satisfying conditions

(i) $0 \leq \alpha_n \leq \beta_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

In [4], Chidume extended the results of Schu [7] from Hilbert spaces to the much more general class of real Banach spaces and approximated the fixed points of strongly pseudocontractive mappings.

In this paper, we establish the strong convergence for the Ishikawa iteration scheme associated with Lipschitz pseudocontractive mappings in real Banach spaces. Moreover, our technique of proofs is of independent interest.

2. Main Results

We will need the following results.

Lemma 2.1 (see [8]). *Let $J : E \to 2^E$ be the normalized duality mapping. Then, for any $x, y \in E$, one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{2.1}$$

Lemma 2.2 (see [9]). *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$ (the set of all positive integers)*

$$\rho_{n+1} \leq \left(1-\delta_n^2\right)\rho_n + b_n, \tag{2.2}$$

where $\delta_n \in [0,1)$, $\sum_{n=1}^{\infty} \delta_n^2 = \infty$ and $b_n = o(\delta_n)$, then

$$\lim_{n\to\infty} \rho_n = 0. \tag{2.3}$$

We now prove our main results.

Theorem 2.3. *Let K be a nonempty closed convex subset of a real Banach space E and $T : K \to K$ be a Lipschitz pseudocontractive mapping such that $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying the conditions:*

(iv) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$;

(v) $\lim_{n\to\infty} \alpha_n = 0$;

(vi) $\lim_{n\to\infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be defined iteratively by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1. \end{aligned} \tag{2.4}$$

Then the following conditions are equivalent:

(a) *$\{x_n\}_{n=1}^{\infty}$ converges strongly to the fixed point p of T.*

(b) *$\{Tx_n\}_{n=1}^{\infty}$ and $\{Ty_n\}_{n=1}^{\infty}$ are bounded.*

Proof. Because p is a fixed point of T, then the set $F(T)$ of fixed points of T is nonempty.

Suppose that $\lim_{n\to\infty} x_n = p$, then since T is Lipschitz, so

$$\begin{aligned} \lim_{n\to\infty} T x_n &= p, \\ \lim_{n\to\infty} y_n &= \lim_{n\to\infty} \left[(1-\beta_n)x_n + \beta_n T x_n\right] = p, \end{aligned} \tag{2.5}$$

which implies that $\lim_{n\to\infty} T y_n = p$. Therefore $\{Tx_n\}_{n=1}^{\infty}$ and $\{Ty_n\}_{n=1}^{\infty}$ are bounded.

Set

$$M_1 = \|x_0 - p\| + \sup_{n\geq 1}\|Tx_n - p\| + \sup_{n\geq 1}\|Ty_n - p\|. \tag{2.6}$$

Obviously $M_1 < \infty$.

It is clear that $\|x_0 - p\| \leq M_1$. Let $\|x_n - p\| \leq M_1$. Next we will prove that $\|x_{n+1} - p\| \leq M_1$. Consider

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n T y_n - p\| \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n (T y_n - p)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|T y_n - p\| \\ &\leq M_1. \end{aligned} \tag{2.7}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n=1}^{\infty}$ is bounded. Let $M_2 = \sup_{n\geq 1}\|x_n - p\|$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

Now from Lemma 2.1 we obtain for all $n \geq 1$

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T y_n - p\|^2 \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n (T y_n - p)\|^2 \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T y_n - p, j(x_{n+1} - p)\rangle \\
&= (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n \langle T y_n - T x_{n+1}, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n \|T y_n - T x_{n+1}\| \|x_{n+1} - p\| \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|x_{n+1} - p\|^2 + 2\alpha_n \lambda_n,
\end{aligned} \tag{2.8}$$

where

$$\lambda_n = M\|T y_n - T x_{n+1}\|. \tag{2.9}$$

Using (2.4) we have

$$\begin{aligned}
\|y_n - x_{n+1}\| &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| \\
&= \beta_n \|x_n - T x_n\| + \alpha_n \|x_n - T y_n\| \\
&\leq 2M(\alpha_n + \beta_n).
\end{aligned} \tag{2.10}$$

From the conditions $\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n$ and (2.10), we obtain

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0, \tag{2.11}$$

and since T is Lipschitz,

$$\lim_{n\to\infty} \|T y_n - T x_{n+1}\| = 0, \tag{2.12}$$

thus, we have

$$\lim_{n\to\infty} \lambda_n = 0. \tag{2.13}$$

The real function $f : [0,\infty) \to [0,\infty)$ defined by $f(t) = t^2$ is increasing and convex. For all $\lambda \in [0,1]$ and t_1, $t_2 > 0$ we have

$$((1-\lambda)t_1 + \lambda t_2)^2 \leq (1-\lambda)t_1^2 + \lambda t_2^2. \tag{2.14}$$

Consider

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)x_n+\alpha_n Ty_n-p\|^2\\ &= \|(1-\alpha_n)(x_n-p)+\alpha_n(Ty_n-p)\|^2\\ &\le \left[(1-\alpha_n)\|x_n-p\|+\alpha_n\|Ty_n-p\|\right]^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|Ty_n-p\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+M^2\alpha_n.\end{aligned} \tag{2.15}$$

Substituting (2.15) in (2.8), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \left[(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)\right]\|x_n-p\|^2\\ &\quad+2\alpha_n\left(M^2\alpha_n+\lambda_n\right)\\ &= \left(1-\alpha_n^2\right)\|x_n-p\|^2+\varepsilon_n\alpha_n,\end{aligned} \tag{2.16}$$

where $\varepsilon_n = 2(M^2\alpha_n+\lambda_n)$. Now, with the help of $\sum_{n=1}^{\infty}\alpha_n^2=\infty$, $\lim_{n\to\infty}\alpha_n=0$, (2.13), and Lemma 2.2, we obtain from (2.16) that

$$\lim_{n\to\infty}\|x_n-p\|=0. \tag{2.17}$$

This completes the proof. □

Remark 2.4. Our technique of proofs is of independent interest.

Corollary 2.5. *Let K be a nonempty closed convex subset of a real Hilbert space E and let $T:K\to K$ be a Lipschitz pseudocontractive mapping such that $p\in F(T)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying the conditions (iv), (v), and (vi).*

For arbitrary $x_1\in K$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by (2.4). Then the following conditions are equivalent:

(a) *$\{x_n\}_{n=1}^{\infty}$ converges strongly to the fixed point p of T.*

(b) *$\{Tx_n\}_{n=1}^{\infty}$ and $\{Ty_n\}_{n=1}^{\infty}$ are bounded.*

The proof of the following result runs on the lines of proof of the Theorem 2.3, so is omitted.

Theorem 2.6. *Let K be a nonempty closed convex subset of a real Banach space E and let $T,S:K\to K$ be two Lipschitz pseudocontractive mappings such that $p\in F(T)\cap F(S):=\{x\in K: Tx=x=$*

$Sx\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying the conditions (iv), (v), and (vi). For arbitrary $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n S x_n, \quad n \geq 1. \end{aligned} \tag{2.18}$$

Then the following conditions are equivalent:

(a) *$\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point p of T and S.*

(b) *$\{Tx_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ are bounded.*

Corollary 2.7. *Let K be a nonempty closed convex subset of a real Hilbert space E and let $T, S : K \to K$ be two Lipschitz pseudocontractive mappings such that $p \in F(T) \cap F(S)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying conditions (iv), (v), and (vi).*

For arbitrary $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by (2.18). Then the following conditions are equivalent:

(a) *$\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point p of T and S.*

(b) *$\{Tx_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ are bounded.*

Remark 2.8. It is worth to mentioning that we have the following.

(1) The results of Chidume [4] and Zhou and Jia [10] depend on the geometry of the Banach space, whereas in our case we do not need such geometry.

(2) We remove the boundedness assumption on K introduced in [4, 10].

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