Review Article

Nonlinear Random Stability via Fixed-Point Method

Yeol Je Cho, Shin Min Kang, and Reza Saadati

1 Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea
2 Department of Mathematics and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea
3 Department of Mathematics, Iran University of Science and Technology, Behshahr, Iran

Correspondence should be addressed to Shin Min Kang, smkang@gnu.ac.kr and Reza Saadati, rsaadati@eml.cc

Received 31 October 2011; Accepted 22 December 2011

Academic Editor: Yeong-Cheng Liou

Copyright © 2012 Yeol Je Cho et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

\[ f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \]

in various complete random normed spaces.

1. Introduction


The functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]  \hspace{1cm} (1.1)

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for
the quadratic functional equation was proved by Cholewa [6] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8–12]).

In [13], Jun and Kim consider the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (1.2)$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a cubic functional equation, and every solution of the cubic functional equation is said to be a cubic mapping.

Considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \quad (1.3)$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation, which is called a quartic functional equation, and every solution of the quartic functional equation is said to be a quartic mapping. One can easily show that an odd mapping $f : X \to Y$ satisfies the additive-quadratic-cubic-quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x)$$

$$+ f(2y) + f(-2y) - 4f(y) - 4f(-y) \quad (1.4)$$

if and only if it is an additive-cubic mapping, that is,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \quad (1.5)$$

It was shown in Lemma 2.2 of [14] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.4) if and only if it is a quadratic-quartic mapping, that is,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y). \quad (1.6)$$

Also $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and $f(x) = (1/12)g(x) - (1/12)h(x)$.

For a given mapping $f : X \to Y$, we define

$$Df(x, y) := f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x)$$

$$- f(2y) - f(-2y) + 4f(y) + 4f(-y) \quad (1.7)$$

for all $x, y \in X$. 

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the fixed-point alternative of Diaz and Margolis.

**Theorem 1.1** (see [15, 16]). Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$, then for each given element $x \in X$, either

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$,
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$,
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X | d(J^n x, y) < \infty\}$,
4. $d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [18–21]).

## 2. Preliminaries

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22–26]. Throughout this paper, $\Delta^*$ is the space of all probability distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$, such that $F$ is left continuous, nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $\{F(+\infty) = 1\}$. $D^*$ is a subset of $\Delta^*$ consisting of all functions $F \in \Delta^+$ for which $\underline{f} F(+\infty) = 1$, where $\underline{f} F(x)$ denotes the left limit of the function $f$ at the point $x$, that is, $\underline{f} F(x) = \lim_{t \to x^-} f(t)$. The space $\Delta^*$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^*$ in this order is the distribution function $\varepsilon_0$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (2.1)$$

A triangular norm (shortly t-norm) is a binary operation on the unit interval $[0, 1]$, that is, a function $T : [0, 1] \times [0, 1] \to [0, 1]$, such that for all $a, b, c \in [0, 1]$, the following four axioms satisfied:

1. $T(a, b) = T(b, a)$ (commutativity),
2. $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity),
(T3) \( T(a, 1) = a \) (boundary condition),
(T4) \( T(a, b) \leq T(a, c) \) whenever \( b \leq c \) (monotonicity).

Basic examples are the Łukasiewicz \( t \)-norm \( T_L, T_L(a, b) = \max(a + b - 1, 0) \) for all \( a, b \in [0, 1] \) and the \( t \)-norms \( T_P, T_M, T_D \), where \( T_P(a, b) := ab, T_M(a, b) := \min\{a, b\} \),

\[
T_D(a, b) := \begin{cases} 
    \min(a, b), & \text{if } \max(a, b) = 1, \\
    0, & \text{otherwise.}
\end{cases}
\]  

(2.2)

If \( T \) is a \( t \)-norm, then \( x^{(n)}_T \) is defined for every \( x \in [0, 1] \) and \( n \in \mathbb{N} \cup \{0\} \) by 1, if \( n = 0 \) and \( T(x^{(n-1)}_T, x) \) if \( n \geq 1 \). A \( t \)-norm \( T \) is said to be of Hadzić type (we denote by \( T \in \mathcal{H} \)) if the family \( \{x^{(n)}_T\}_{n \in \mathbb{N}} \) is equicontinuous at \( x = 1 \) (cf. [27]).

Other important triangular norms are the following (see [28]):

(1) The Sugeno-Weber family \( \{T^\text{SW}_{\lambda}\}_{\lambda \in [-1, \infty]} \) is defined by \( T^\text{SW}_{-1} = T_D, T^\text{SW}_{\infty} = T_P \) and

\[
T^\text{SW}_{\lambda}(x, y) = \max\left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right)
\]  

(2.3)

if \( \lambda \in (-1, \infty) \).

(2) The Domby family \( \{T^\text{D}_{\lambda}\}_{\lambda \in [0, \infty]} \) is defined by \( T_D \) if \( \lambda = 0 \), \( T_M \) if \( \lambda = \infty \), and

\[
T^\text{D}_{\lambda}(x, y) = \frac{1}{1 + \left(\frac{1}{x} + \frac{1}{y}\right)^{1/\lambda}}
\]  

(2.4)

if \( \lambda \in (0, \infty) \).

(3) The Aczel-Alsina family \( \{T^\text{AA}_{\lambda}\}_{\lambda \in [0, \infty]} \) is defined by \( T_D \) if \( \lambda = 0 \), \( T_M \) if \( \lambda = \infty \) and

\[
T^\text{AA}_{\lambda}(x, y) = e^{-\lambda \log|x|/|\log y|^{1/\lambda}}
\]  

(2.5)

if \( \lambda \in (0, \infty) \).

A \( t \)-norm \( T \) can be extended (by associativity) in a unique way to an \( n \)-array operation taking for \( (x_1, \ldots, x_n) \in [0, 1]^n \) the value \( T(x_1, \ldots, x_n) \) defined by

\[
T^0_{i=1} x_i = 1, \quad T^n_{i=1} x_i = T\left(T^{n-1}_{i=1} x_i, x_n\right) = T(x_1, \ldots, x_n).
\]  

(2.6)

\( T \) can also be extended to a countable operation taking for any sequence \( (x_n)_{n \in \mathbb{N}} \) in \([0, 1]\) the value

\[
T^\infty_{i=1} x_i = \lim_{n \to \infty} T^n_{i=1} x_i.
\]  

(2.7)

The limit on the right side of (2.7) exists since the sequence \( (T^n_{i=1} x_i)_{n \in \mathbb{N}} \) is nonincreasing and bounded from below.
Proposition 2.1 (see [28]). We have the following.

(1) For $T \geq T_L$, the following implication holds:

$$\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$  

(2) If $T$ is of Hadžic type, then

$$\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1$$

for every sequence $(x_n)_{n \in \mathbb{N}}$ in $[0,1]$ such that $\lim_{n \to \infty} x_n = 1$.

(3) If $T \in \{ T^{\Lambda \Lambda}_{1} \}_{\lambda \in (0,\infty)} \cup \{ T^{\Lambda}_{1} \}_{\lambda \in (0,\infty)}$, then

$$\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n)^a < \infty.$$  

(4) If $T \in \{ T^{SW}_{1} \}_{\lambda \in (-1,\infty)}$, then

$$\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$  

Definition 2.2 (see [26]). A Random normed space (briefly, RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm, and $\mu$ is a mapping from $X$ into $D^+$ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\mu_{ax}(t) = \mu_x(t/|a|)$ for all $x \in X$, and $a \neq 0$,

(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let $(X, \mu, T)$ be an RN-space.

(1) A sequence $\{ x_n \}$ in $X$ is said to be convergent to $x$ in $X$ if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $\mu_{x_n-x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{ x_n \}$ in $X$ is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $\mu_{x_n-x_m}(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN-space $(X, \mu, T)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent to a point in $X$. A complete RN-space is said to be random Banach space.

Theorem 2.4 (see [25]). If $(X, \mu, T)$ is an RN-space and $\{ x_n \}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.
The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces, and fuzzy normed spaces has been recently studied [20, 24, 29–39].

3. Non-Archimedean Random Normed Space

By a non-Archimedean field, we mean a field \( K \) equipped with a function (valuation) \( |·| \) from \( K \) into \([0, \infty)\) such that \(|r| = 0\) if and only if \( r = 0 \), \(|rs| = |r||s|\), and \(|r + s| \leq \max\{|r|, |s|\}\) for all \( r, s \in K \). Clearly, \(|1| = |−1| = 1\) and \(|n| \leq 1\) for all \( n \in \mathbb{N} \). By the trivial valuation, we mean the mapping \(|·|\) taking everything but 0 into 1 and \(|0| = 0\). Let \( X \) be a vector space over a field \( K \) with a non-Archimedean nontrivial valuation \(|·|\). A function \( \| · \| : X \to [0, \infty) \) is called a non-Archimedean norm if it satisfies the following conditions:

(NAN1) \( \|x\| = 0 \) if and only if \( x = 0 \),
(NAN2) for any \( r \in K \) and \( x \in X \), \( \|rx\| = |r|\|x\|\),
(NAN3) the strong triangle inequality (ultrametric), namely,

\[
\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X),
\]

then \((X, \| · \|)\) is called a non-Archimedean normed space. Due to the fact that

\[
\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m),
\]

a sequence \( \{x_n\} \) is a Cauchy sequence if and only if \( \{x_{n+1} - x_n\} \) converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [40] discovered the \( p \)-adic numbers of as a number theoretical analogues of power series in complex analysis. Fix a prime number \( p \). For any nonzero rational number \( x \), there exists a unique integer \( n_x \in \mathbb{Z} \) such that \( x = (a/b)p^{n_x} \), where \( a \) and \( b \) are integers not divisible by \( p \). Then \( |x|_p := p^{-n_x} \) defines a non-Archimedean norm on \( \mathbb{Q} \). The completion of \( \mathbb{Q} \) with respect to the metric \( d(x, y) = |x − y|_p \) is denoted by \( \mathbb{Q}_p \), which is called the \( p \)-adic number field.

Throughout the paper, we assume that \( X \) is a vector space and \( Y \) is a complete non-Archimedean normed space.

Definition 3.1. A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple \((X, \mu, T)\), where \( X \) is a linear space over a non-Archimedean field \( K \), \( T \) is a continuous \( t \)-norm, and \( \mu \) is a mapping from \( X \) into \( D^* \) such that the following conditions hold:

(NA-RN1) \( \mu_x(t) = \varepsilon_{0}(t) \) for all \( t > 0 \) if and only if \( x = 0 \),
(NA-RN2) \( \mu_{ax}(t) = \mu_x(t/|a|) \) for all \( x \in X, t > 0, \) and \( a \neq 0 \),
(NA-RN3) \( \mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s)) \) for all \( x, y, z \in X \) and \( t, s \geq 0 \).
It is easy to see that if (NA-RN3) holds, then so is
\[(RN3) \; \mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s)).\]

As a classical example, if \((X, \|\cdot\|)\) is a non-Archimedean normed linear space, then the triple \((X, \mu, T_M)\), where
\[
\mu_x(t) = \begin{cases} 
0, & t \leq \|x\|, \\
1, & t > \|x\|,
\end{cases}
\]  

(3.3)
is a non-Archimedean RN-space.

**Example 3.2.** Let \((X, \|\cdot\|)\) be a non-Archimedean normed linear space. Define
\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \; t > 0),
\]  

(3.4)
then \((X, \mu, T_M)\) is a non-Archimedean RN-space.

**Definition 3.3.** Let \((X, \mu, T)\) be a non-Archimedean RN-space. Let \(\{x_n\}\) be a sequence in \(X\), then \(\{x_n\}\) is said to be convergent if there exists \(x \in X\) such that
\[
\lim_{n \to \infty} \mu_{x_n - x}(t) = 1
\]  

(3.5)
for all \(t > 0\). In that case, \(x\) is called the limit of the sequence \(\{x_n\}\).

A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if for each \(\varepsilon > 0\) and each \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\) and all \(p > 0\), we have \(\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon\).

If each Cauchy sequence is convergent, then the random norm is said to be complete and the non-Archimedean RN-space is called a non-Archimedean random Banach space.

**Remark 3.4** (see [41]). Let \((X, \mu, T_M)\) be a non-Archimedean RN-space, then
\[
\mu_{x_{n+p} - x_n}(t) \geq \min \{ \mu_{x_{n+j} - x_{n+j-1}}(t) : j = 0, 1, 2, \ldots, p - 1 \}.
\]  

(3.6)
So, the sequence \(\{x_n\}\) is a Cauchy sequence if for each \(\varepsilon > 0\) and \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\),
\[
\mu_{x_{n+1} - x_n}(t) > 1 - \varepsilon.
\]  

(3.7)


Let \(K\) be a non-Archimedean field, let \(X\) be a vector space over \(K\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(K\).
Next, we define a random approximately AQCQ mapping. Let $\Psi$ be a distribution function on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing and

$$\Psi(cx, cx, t) \geq \Psi\left(x, x, \frac{t}{|c|}\right) \quad (x \in X, \ c \neq 0). \tag{4.1}$$

**Definition 4.1.** A mapping $f : X \to Y$ is said to be $\Psi$-approximately AQCQ if

$$\mu_{Df(x,y)}(t) \geq \Psi(x, y, t) \quad (x, y \in X, \ t > 0). \tag{4.2}$$

In this section, we assume that $2 \neq 0$ in $\mathcal{K}$ (i.e., characteristic of $\mathcal{K}$ is not 2). Our main result, in this section, is the following.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean random spaces, an odd case.

**Theorem 4.2.** Let $\mathcal{K}$ be a non-Archimedean field, let $X$ be a vector space over $\mathcal{K}$ and let $(Y, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an odd mapping and $\Psi$-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer $k$, $k > 3$ with $|2^k| < \alpha$,

$$\Psi\left(2^{-k}x, 2^k y, t\right) \geq \Psi(x, y, at) \quad (x \in X, \ t > 0), \tag{4.3}$$

$$\lim_{n \to \infty} \frac{\mu_{Df(x,y)}(t)}{T^n M(2x, \frac{\alpha^i t}{8^{|i|}})} = 1 \quad (x \in X, \ t > 0), \tag{4.4}$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{8^{|i|}}\right) \tag{4.5}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T_{k-1}\left[\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right), \ldots, \Psi\left(\frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{|4|}\right), \Psi\left(2^{k-1}x, 2^{k-1}x, t\right)\right] \quad (x \in X, \ t > 0). \tag{4.6}$$

**Proof.** Letting $x = y$ in (4.2), we get

$$\mu_{f(y)-4f(2y)+5f(y)}(t) \geq \Psi(y, y, t) \tag{4.7}$$

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (4.2), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \Psi(2y, y, t) \tag{4.8}$$

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (4.2), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \Psi(2y, y, t)$$
for all $y \in X$ and $t > 0$. By (4.7) and (4.8), we have

$$
\mu_{f(4y)−10f(2y)+16f(y)}(t) \geq T\left(\mu_{4f(3y)−4f(2y)+5f(y)}(t), \mu_{4f(3y)+6f(2y)−4f(y)}(t)\right)
$$

$$
= T\left(\mu_{f(3y)−4f(2y)+5f(y)}\left(\frac{t}{4}\right), \mu_{f(3y)+6f(2y)−4f(y)}(t)\right)
$$

$$
\geq T\left(\Psi\left(y, y, \frac{t}{4}\right), \Psi(2y, y, t)\right)
$$

(4.9)

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) − 2f(x)$ for all $x \in X$ in (4.9), we get

$$
\mu_{g(x)−8g(x/2)}(t) \geq T\left(\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{4}\right), \Psi(x, \frac{x}{2}, t)\right)
$$

(4.10)

for all $x \in X$ and $t > 0$. Now, we show by induction on $j$ that for all $x \in X$, $t > 0$ and $j \geq 1$,

$$
\mu_{g(2^{j+1}x)−8g(2^jx/2)}(t)
$$

$$
\geq M_j(x, t)
$$

$$
:= T^{2j+1}\left[\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{4}\right), \Psi(x, \frac{x}{2}, t), \ldots, \Psi\left(\frac{2^{j−1}x}{2}, \frac{2^{j−1}x}{2}, \frac{t}{4}\right), \Psi\left(\frac{2^{j−1}x}{2}, \frac{2^{j−1}x}{2}, t\right)\right].
$$

(4.11)

Putting $j = 1$ in (4.11), we obtain (4.10). Assume that (4.11) holds for some $j \geq 1$. Replacing $x$ by $2^j x$ in (4.10), we get

$$
\mu_{g(2^jx)−8g(2^{j−1}x)}(t) \geq T\left(\Psi\left(2^{j−1}x, 2^{j−1}x, \frac{t}{4}\right), \Psi(2^jx, 2^{j−1}x, t)\right).
$$

(4.12)

Since $|8| \leq 1$,

$$
\mu_{g(2^jx)−8g(2^{j−1}x)}(t) \geq T\left(\mu_{g(2^jx)−8g(2^{j−1}x)}(t), \mu_{8g(2^{j−1}x)−8g(2^jx)}(t)\right)
$$

$$
= T\left(\mu_{g(2^jx)−8g(2^{j−1}x)}(t), \mu_{g(2^{j−1}x)−8g(2^jx)}\left(\frac{t}{|8|}\right)\right)
$$

$$
\geq T^2\left(\Psi\left(2^{j−1}x, 2^{j−1}x, \frac{t}{|4|}\right), \Psi(2^jx, 2^{j−1}x, t), M_j(x, t)\right)
$$

$$
= M_{j+1}(x, t)
$$

(4.13)

for all $x \in X$ and $t > 0$. Thus, (4.11) holds for all $j \geq 2$. In particular,

$$
\mu_{g(2^{j+1}x)−8^jg(x/2)}(t) \geq M(x, t) \quad (x \in X, \ t > 0).
$$

(4.14)
Replacing $x$ by $2^{-(k_n+1)}x$ in (4.14) and using inequality (4.3), we obtain

$$
\mu_{g(x/2^{n+1})} - 8^k g(x/2^{(n+1)}) (t) \geq M \left( \frac{2x}{2^{k(n+1)}}, t \right) \quad (x \in X, \; t > 0, \; n = 0, 1, 2, \ldots). \tag{4.15}
$$

Then

$$
\mu_{g(x/2^{n+1})} - 8^k g(x/2^{(n+1)}) (t) \geq M \left( 2x, \frac{a^{n+1}}{|8^k|^{n+1}}, t \right) \quad (x \in X, \; t > 0, \; n = 0, 1, 2, \ldots). \tag{4.16}
$$

Hence

$$
\mu_{g(x/2^{n+1})} - 8^k g(x/2^{(n+1)}) (t) \geq T^n_{n+1} \left( \mu_{g(x/2^{k})} - 8^k g(x/2^{k}) (t) \right) \\
\geq T^n_{n+1} \left( 2x, \frac{a^{n+1}}{|8^k|^{n+1}}, t \right) \quad (x \in X, \; t > 0, \; n = 0, 1, 2, \ldots). \tag{4.17}
$$

Since

$$
\lim_{n \to \infty} \left. T^n_{n+1} \right|_{n \in \mathbb{N}} \left( 2x, \frac{a^{n+1}}{|8^k|^{n+1}}, t \right) = 1 \quad (x \in X, \; t > 0), \tag{4.18}
$$

then

$$
\left\{ 8^k g \left( \frac{x}{2^{kn}} \right) \right\}_{n \in \mathbb{N}} \tag{4.19}
$$

is a Cauchy sequence in the non-Archimedean random Banach space $(Y, \mu, T)$. Hence we can define a mapping $C : X \to Y$ such that

$$
\lim_{n \to \infty} \mu\left( 8^k \right) - 8^k g(x/2^{kn}) (t) = 1 \quad (x \in X, \; t > 0). \tag{4.20}
$$

Next for each $n \geq 1$, $x \in X$ and $t > 0$,

$$
\mu_{g(x)} - 8^k g(x/2^{kn}) (t) = \mu_{\sum_{i=0}^{n-1} (8^k)^i g(x/2^{kn}) - (8^k)^i g(x/2^{kn+1})} (t) \\
\geq T_n \left( \mu_{g(x)} - 8^k g(x/2^{kn}) - (8^k)^i g(x/2^{kn+1}) (t) \right) \\
\geq T_n \left( 2x, \frac{a^{n+1}}{|8^k|^{n+1}}, t \right). \tag{4.21}
$$
Therefore,

\[
\mu_{g(x)-C(x)}(t) \geq T \left( \mu_{g(x)-(8^i)g(x/2^i)}(t), \mu_{(8^i)g(x/2^i)-C(x)}(t) \right) \\
\geq T \left( T_{i=1}^{n-1} M \left( x, \frac{\alpha^{i+1}t}{|8^i|^{i+1}} \right), \mu_{(8^i)g(x/2^i)-C(x)}(t) \right). \tag{4.22}
\]

By letting \( n \to \infty \), we obtain

\[
\mu_{g(x)-C(x)}(t) \geq T_{i=1}^{\infty} M \left( x, \frac{\alpha^{i+1}t}{|8^i|^{i+1}} \right). \tag{4.23}
\]

So,

\[
\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M \left( x, \frac{\alpha^{i+1}t}{|8^i|^{i+1}} \right). \tag{4.24}
\]

This proves (4.5). From \( Dg(x, y) = Df(2x, 2y) - 2Df(x, y) \), by (4.2), we deduce that

\[
\mu_{Df(2x, 2y)}(t) \geq \Psi(2x, 2y, t), \\
\mu_{-2Df(x, y)}(t) = \mu_{Df(x, y)} \left( \frac{t}{|2|} \right) \geq \mu_{Df(x, y)}(t) \geq \Psi(x, y, t), \tag{4.25}
\]

and so, by (NA-RN3) and (4.2), we obtain

\[
\mu_{Dg(x, y)}(t) \geq T(\mu_{Df(2x, 2y)}(t), \mu_{-2Df(x, y)}(t)) \geq T(\Psi(2x, 2y, t), \Psi(x, y, t)) := N(x, y, t). \tag{4.26}
\]

It follows that

\[
\mu_{g(x)-Dg(x/2^i,y/2^i)}(t) = \mu_{Dg(x/2^i,y/2^i)} \left( \frac{t}{|8|^{kn}} \right) \\
\geq N \left( \frac{x}{2^{kn}}, \frac{y}{2^{kn}}, \frac{t}{|8|^{kn}} \right) \geq \cdots \geq N \left( x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}} \right) \tag{4.27}
\]

for all \( x, y \in X, t > 0, \) and \( n \in \mathbb{N} \). Since

\[
\lim_{n \to \infty} N \left( x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}} \right) = 1 \tag{4.28}
\]
for all \(x, y \in X \) and \(t > 0\), by Theorem 2.4, we deduce that
\[
\mu_{DC(x,y)}(t) = 1
\]
for all \(x, y \in X\) and \(t > 0\). Thus, the mapping \(C : X \to Y\) satisfies (1.4).

Now, we have
\[
C(2x) - 8C(x) = \lim_{n \to \infty} \left[ 8^n g \left( \frac{x}{2^{n-1}} \right) - 8^{n+1} g \left( \frac{x}{2^n} \right) \right]
= 8 \lim_{n \to \infty} \left[ 8^{n-1} g \left( \frac{x}{2^{n-1}} \right) - 8^n g \left( \frac{x}{2^n} \right) \right] = 0
\]
for all \(x \in X\). Since the mapping \(x \to C(2x) - 2C(x)\) is cubic (see Lemma 2.2 of [14]), from the equality \(C(2x) = 8C(x)\), we deduce that the mapping \(C : X \to Y\) is cubic.

\[\text{Corollary 4.3.} \quad \text{Let} \ K \text{ be a non-Archimedean field, let} \ X \text{ be a vector space over} \ K, \text{ and let} \ (Y, \mu, T) \text{ be a non-Archimedean random Banach space over} \ K \text{ under a} \ t\text{-norm} \ T \in \mathcal{K}. \text{Let} \ f : X \to Y \text{ be an odd and} \ \Psi\text{-approximately AQCQ mapping. If, for some} \ \alpha \in \mathbb{R}, \ \alpha > 0, \text{ and some integer} \ k, \ k > 3, \text{ with} |2^k| < \alpha, \]
\[\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0), \]
then there exists a unique cubic mapping \(C : X \to Y\) such that
\[
\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T \left[ \sum_{i=1}^{n} M \left( x, \frac{\alpha^{i+1} t}{|8|^{i}} \right) \right]
\]
for all \(x \in X\) and \(t > 0\).

\[\text{Proof.} \quad \text{Since}
\lim_{n \to \infty} M \left( x, \frac{\alpha^i t}{|8|^{i}} \right) = 1 \quad (x \in X, \ t > 0)
\]
and \(T\) is of Hadžić type, from Proposition 2.1, it follows that
\[
\lim_{n \to \infty} T \left[ \sum_{i=1}^{n} M \left( x, \frac{\alpha^{i+1} t}{|8|^{i}} \right) \right] = 1 \quad (x \in X, \ t > 0).
\]
Now, we can apply Theorem 4.2 to obtain the result.

\[\text{Example 4.4.} \quad \text{Let} \ (X, \mu, T_M) \text{ be non-Archimedean random normed space in which}
\mu_x(t) = \frac{t}{t + ||x||} \quad (x \in X, \ t > 0).
\]
And let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define

\[
\Psi(x, y, t) = \frac{t}{1 + t}.
\]  

(4.36)

It is easy to see that (4.3) holds for \(\alpha = 1\). Also, since

\[
M(x, t) = \frac{t}{1 + t},
\]  

(4.37)

we have

\[
\lim_{n \to \infty} T_{M, j=\infty}^{\infty} M\left(\frac{\alpha^j t}{8^{kj}}\right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T_{M, j=\infty}^{\infty} M\left(\frac{t}{8^{kj}}\right) \right)
\]

(4.38)

\[
= \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |8^{kj}|} \right)
\]

\[
= 1 \quad (x \in X, \ t > 0).
\]

Let \(f : X \to Y\) be an odd and \(\Psi\)-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique cubic mapping \(C : X \to Y\) such that

\[
\mu f(x) - 2f(x/2) - C(x/2) = \frac{t}{t + |8^k|}.
\]  

(4.39)

**Theorem 4.5.** Let \(\mathcal{K}\) be a non-Archimedean field, let \(X\) be a vector space over \(\mathcal{K}\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(\mathcal{K}\). Let \(f : X \to Y\) be an odd mapping and \(\Psi\)-approximately AQCQ mapping. If for some \(\alpha \in \mathbb{R}\), \(\alpha > 0\), and some integer \(k, k > 1\) with \(|2^k| < \alpha\),

\[
\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0),
\]  

(4.40)

\[
\lim_{n \to \infty} T_{j=\infty}^{\infty} M\left(\frac{2x, \alpha^j t}{2^{|kj|}}\right) = 1 \quad (x \in X, \ t > 0),
\]

then there exists a unique additive mapping \(A : X \to Y\) such that

\[
\mu f(x) - 8f(x/2) - A(x/2) = 1 \quad (x \in X, \ t > 0).
\]  

(4.41)
for all \( x \in X \) and \( t > 0 \), where

\[
M(x, t) := T^{k-1} \left[ \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \ldots, \Psi \left( \frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{4} \right), \Psi \left( \frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, t \right) \right] 
\]

(\( x \in X, \ t > 0 \))

\[(4.42)\]

Proof. Letting \( y := x/2 \) and \( g(x) := f(2x) - 8f(x) \) for all \( x \in X \) in (4.9), we get

\[
\mu g(x) - 8g(x/2)(t) \geq T \left( \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right) \right) 
\]

for all \( x \in X \) and \( t > 0 \).

The rest of the proof is similar to the proof of Theorem 4.2. \( \square \)

Corollary 4.6. Let \( \mathcal{K} \) be a non-Archimedean field, let \( X \) be a vector space over \( \mathcal{K} \), and let \( (Y, \mu, T) \) be a non-Archimedean random Banach space over \( \mathcal{K} \) under a t-norm \( T \in H. \) Let \( f : X \to Y \) be an odd and \( \Psi \)-approximately AQCQ mapping. If, for some \( \alpha \in \mathbb{R}, \ \alpha > 0 \), and some integer \( k, k > 1 \), with \( |2^k| < \alpha \),

\[
\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi (x, y, \alpha t) \quad (x \in X, \ t > 0),
\]

then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\mu f(x) - 8f(x/2) - A(x/2)(t) \geq T^{\infty}_{j=1} M \left( x, \frac{\alpha^{j+1}t}{|2|^{kj}} \right) 
\]

(\( x \in X, \ t > 0 \)).

Proof. Since

\[
\lim_{n \to \infty} M \left( x, \frac{\alpha^j t}{|2|^{k^j}} \right) = 1 \quad (x \in X, \ t > 0)
\]

(4.46)

and \( T \) is of Hadžić type, from Proposition 2.1, it follows that

\[
\lim_{n \to \infty} T^{\infty}_{j=1} M \left( x, \frac{\alpha^j t}{|2|^{k^j}} \right) = 1 \quad (x \in X, \ t > 0).
\]

(4.47)

Now, we can apply Theorem 4.5 to obtain the result. \( \square \)
Example 4.7. Let \((X, \mu, T_M)\) non-Archimedean random normed space in which
\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0),
\]
and let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define
\[
\Psi(x, y, t) = \frac{t}{1 + t}.
\]
It is easy to see that (4.3) holds for \(\alpha = 1\). Also, since
\[
M(x, t) = \frac{t}{1 + t},
\]
we have
\[
\lim_{n \to \infty} T_{M, j=n}^\infty M\left(x, \frac{\alpha^j t}{|2|^k}\right) = \lim_{m \to \infty} \lim_{n \to \infty} T_{M, j=n}^m M\left(x, \frac{t}{|2|^k}\right)
\]
\[
= \lim_{n \to \infty} \lim_{m \to \infty} \left(\frac{t}{t + |2|^k|^n}\right)
\]
\[
= 1 \quad (x \in X, \ t > 0).
\]
Let \(f : X \to Y\) be an odd and \(\Psi\)-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique additive mapping \(A : X \to Y\) such that
\[
\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq \frac{t}{t + |2|^k}.
\]

5. Generalized Hyers-Ulam Stability of the Functional Equation (1.4) in Non-Archimedean Random Normed Spaces: An Even Case

Now, we prove the generalized Hyers-Ulam stability of the functional equation \(Df(x, y) = 0\) in non-Archimedean Banach spaces, an even case.

Theorem 5.1. Let \(\mathcal{K}\) be a non-Archimedean field, let \(X\) be a vector space over \(\mathcal{K}\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(\mathcal{K}\). Let \(f : X \to Y\) be an even mapping, \(f(0) = 0\), and \(\Psi\)-approximately AQCQ mapping. If for some \(\alpha \in \mathbb{R}, \ \alpha > 0\), and some integer \(k, k > 4\) with \(|2|^k < \alpha\),
\[
\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0),
\]
\[
\lim_{n \to \infty} T_{M, j=n}^\infty M\left(2x, \frac{\alpha^j t}{|16|^k}\right) = 1 \quad (x \in X, \ t > 0),
\]

(5.1)
then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that

$$
\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M \left( x, \frac{\alpha^{i+1} t}{16^{|k|}} \right)
$$

(5.2)

for all $x \in X$ and $t > 0$, where

$$
M(x, t) := T^{k-1} \left[ \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right), \ldots, \Psi \left( \frac{2^{k-1} x}{2}, \frac{2^{k-1} x}{2}, \frac{t}{4} \right), \Psi \left( 2^{k-1} x, \frac{2^{k-1} x}{2}, t \right) \right]
$$

(5.3)

Proof. Letting $x = y$ in (4.2), we get

$$
\mu_{f(3y)+6f(2y)+15f(y)}(t) \geq \Psi(y, y, t)
$$

(5.4)

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (4.2), we get

$$
\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq \Psi(2y, y, t)
$$

(5.5)

for all $y \in X$ and $t > 0$. By (5.4) and (5.5), we have

$$
\mu_{f(4y)-20f(2y)+64f(y)}(t) \geq T(\mu_{f(3y)+6f(2y)+15f(y)}(t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t))
$$

$$
= T(\mu_{f(3y)-4f(2y)+5f(y)}(t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t))
$$

(5.6)

$$
\geq T(\Psi(y, y, \frac{t}{4}), \Psi(2y, y, t))
$$

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 4f(x)$ for all $x \in X$ in (5.6), we get

$$
\mu_{g(x)-16g(x/2)}(t) \geq T(\Psi\left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi\left( x, \frac{x}{2}, t \right))
$$

(5.7)

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.2. \qed

**Corollary 5.2.** Let $\mathcal{K}$ be a non-Archimedean field, let $X$ be a vector space over $\mathcal{K}$, and let $(Y, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$ under a $t$-norm $T \in \mathcal{K}$. Let $f : X \rightarrow Y$ be an even, $f(0) = 0$, and $\Psi$-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer $k$, $k > 4$, with $|2^k| < \alpha$,

$$
\Psi\left( 2^{-k} x, 2^{-k} y, t \right) \geq \Psi(x, y, at) \quad (x \in X, t > 0),
$$

(5.8)
then there exists a unique quartic mapping \( Q : X \to Y \) such that

\[
\mu f(x) - 4f(x/2) - Q(x/2) < t \geq T_\infty M\left(x, \frac{\alpha^j t}{16|k_j|}\right)
\]  

(5.9)

for all \( x \in X \) and \( t > 0 \).

Proof. Since

\[
\lim_{n \to \infty} M\left(x, \frac{\alpha^j t}{|16|^{k_j}}\right) = 1 \quad (x \in X, \ t > 0) \tag{5.10}
\]

and \( T \) is of Hadžić type, from Proposition 2.1, it follows that

\[
\lim_{n \to \infty} T_\infty M\left(x, \frac{\alpha^j t}{|16|^{k_j}}\right) = 1 \quad (x \in X, \ t > 0). \tag{5.11}
\]

Now, we can apply Theorem 5.1 to obtain the result. \( \square \)

Example 5.3. Let \((X, \mu, T_M)\) be non-Archimedean random normed space in which

\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0). \tag{5.12}
\]

And let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define

\[
\Psi(x, y, t) = \frac{t}{1 + t}. \tag{5.13}
\]

It is easy to see that (4.3) holds for \( \alpha = 1 \). Also, since

\[
M(x, t) = \frac{t}{1 + t}, \tag{5.14}
\]

we have

\[
\lim_{n \to \infty} T_M^{\infty, M, \infty} M\left(x, \frac{\alpha^j t}{16|k_j|}\right) = \lim_{m \to \infty} \lim_{n \to \infty} T_M^{m, \infty} M\left(x, \frac{t}{|16|^{k_j}}\right) = \lim_{m \to \infty} \lim_{n \to \infty} \left(\frac{t}{t + |16|^n}\right) = 1 \quad (x \in X, \ t > 0). \tag{5.15}
\]
Let \( f : X \to Y \) be an even, \( f(0) = 0 \), and \( \Psi \)-approximately AQCQ mapping. Thus all the conditions of Theorem 5.1 hold, and so there exists a unique quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq \frac{t}{t + |16k|}.
\] (5.16)

**Theorem 5.4.** Let \( \mathcal{K} \) be a non-Archimedean field, let \( X \) be a vector space over \( \mathcal{K} \) and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \( \mathcal{K} \). Let \( f : X \to Y \) be an even mapping, \( f(0) = 0 \) and \( \Psi \)-approximately AQCQ mapping. If for some \( \alpha \in \mathbb{R} \), \( \alpha > 0 \), and some integer \( k, k > 2 \) with \( |2^k| < \alpha \),

\[
\Psi(2^{-k}x, 2^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0),
\] (5.17)

\[
\lim_{n \to \infty} T_{j=n}^\infty M \left( 2x, \frac{\alpha^j t}{|4|^{k^j}} \right) = 1 \quad (x \in X, \ t > 0),
\] (5.18)

then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\mu_{f(x)-16f(x/2)-Q(x/2)}(t) \geq T_{i=1}^\infty M \left( x, \frac{\alpha^{i+1} t}{|4|^{k^i}} \right)
\] (5.19)

for all \( x \in X \) and \( t > 0 \), where

\[
M(x, t) := T^{-1} \left[ \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi \left( x, \frac{x}{2}, t \right), \ldots, \Psi \left( \frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{4} \right), \Psi \left( 2^{k-1}x, \frac{2^{k-1}x}{2}, t \right) \right]
\] (5.19)

\[
(x \in X, \ t > 0).
\]

**Proof.** Letting \( y := x/2 \) and \( g(x) := f(2x) - 16f(x) \) for all \( x \in X \) in (5.6), we get

\[
\mu_{f(x)-4g(x/2)}(t) \geq T \left( \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi \left( x, \frac{x}{2}, t \right) \right)
\] (5.20)

for all \( x \in X \) and \( t > 0 \).

The rest of the proof is similar to the proof of Theorem 5.1. \( \square \)

**Corollary 5.5.** Let \( \mathcal{K} \) be a non-Archimedean field, let \( X \) be a vector space over \( \mathcal{K} \), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \( \mathcal{K} \) under a \( t \)-norm \( T \in \mathcal{K} \). Let \( f : X \to Y \) be an even, \( f(0) = 0 \), and \( \Psi \)-approximately AQCQ mapping. If, for some \( \alpha \in \mathbb{R} \), \( \alpha > 0 \), and some integer \( k, k > 2 \), with \( |2^k| < \alpha \),

\[
\Psi(2^{-k}x, 2^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0),
\] (5.21)
then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$
\mu_{f(x) - 16f(x/2) - Q(x/2)}(t) \geq T^\infty_{i=1} M \left( x, \frac{\alpha t}{4^{ki}} \right) \tag{5.22}
$$

for all $x \in X$ and $t > 0$.

Proof. Since

$$
\lim_{n \to \infty} M \left( x, \frac{\alpha t}{4^{kj}} \right) = 1 \quad (x \in X, \ t > 0) \tag{5.23}
$$

and $T$ is of Hadžić type, from Proposition 2.1, it follows that

$$
\lim_{n \to \infty} T^\infty_{j=n} M \left( x, \frac{\alpha t}{4^{kj}} \right) = 1 \quad (x \in X, \ t > 0). \tag{5.24}
$$

Now, we can apply Theorem 5.4 to obtain the result. \hfill \Box

Example 5.6. Let $(X, \mu, T M)$ be a non-Archimedean random normed space in which

$$
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0). \tag{5.25}
$$

And let $(Y, \mu, T M)$ be a complete non-Archimedean random normed space (see Example 3.2). Define

$$
\Psi(x, y, t) = \frac{t}{1 + t}. \tag{5.26}
$$

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$
M(x, t) = \frac{t}{1 + t}, \tag{5.27}
$$

we have

$$
\lim_{n \to \infty} T^\infty_{M,j=n} M \left( x, \frac{\alpha t}{4^{kj}} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T^m_{M,j=n} M \left( x, \frac{t}{4^{kj}} \right) \right) \tag{5.28}
$$

$$
= \lim_{n \to \infty} \left( \lim_{m \to \infty} \frac{t}{t + 4^{kn}} \right) = 1 \quad (x \in X, \ t > 0).
$$
Let $f : X \rightarrow Y$ be an even, $f(0) = 0$, and $\Psi$-approximately AQCQ mapping. Thus, all the conditions of Theorem 5.4 hold, and so there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$
\mu_{f(x) - 16f(x/2) - Q(x/2)}(t) \geq \frac{t}{t + |4^k|}.
$$

(5.29)

6. Latticetic Random Normed Space

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum and infimum, and $0, 1_L = \inf L, 1_L = \sup L$. The space of latticetic random distribution functions, denoted by $\Delta^*_L$, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \rightarrow L$ such that $F$ is left continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0_L, F(+\infty) = 1_L$.

$D^*_L \subseteq \Delta^*_L$ is defined as $D^*_L = \{F : \Delta^*_L : t \in \mathcal{L}, \inf L \rightarrow t \},$ where $t \in \mathcal{L}$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta^*_L$ is partially ordered by the usual pointwise ordering of functions, that is, $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^*_L$ in this order is the distribution function given by

$$
\varepsilon_0(t) = \begin{cases} 0_L, & \text{if } t \leq 0, \\ 1_L, & \text{if } t > 0. \end{cases}
$$

(6.1)

In Section 2, we defined $t$-norms on $[0, 1]$, and now we extend $t$-norms on a complete lattice.

Definition 6.1 (see [42]). A triangular norm ($t$-norm) on $L$ is a mapping $\mathcal{T} : (L)^2 \rightarrow L$ satisfying the following conditions:

(a) (for all $x \in L)(\mathcal{T}(x, 1_L) = x)$ (boundary condition);
(b) (for all $(x, y) \in (L)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
(c) (for all $(x, y, z) \in (L)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
(d) (for all $(x, x', y, y') \in (L)^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (monotonicity).

Let $\{x_n\}$ be a sequence in $L$ converges to $x \in L$ (equipped order topology). The $t$-norm $\mathcal{T}$ is said to be a continuous $t$-norm if

$$
\lim_{n \rightarrow \infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y)
$$

(6.2)

for all $y \in L$.

A $t$-norm $\mathcal{T}$ can be extended (by associativity) in a unique way to an $n$-array operation taking for $(x_1, \ldots, x_n) \in L^n$ the value $\mathcal{T}(x_1, \ldots, x_n)$ defined by

$$
\mathcal{T}_{i=1}^0 x_i = 1, \quad \mathcal{T}_{i=1}^n x_i = \mathcal{T}(\mathcal{T}_{i=1}^{n-1} x_i, x_n) = \mathcal{T}(x_1, \ldots, x_n).
$$

(6.3)
\[ \mathcal{T}_{\infty} \mathcal{T} = \lim_{n \to \infty} \mathcal{T}^n \mathcal{T}. \]

The limit on the right side of (6.4) exists since the sequence \((\mathcal{T}^n \mathcal{T})\) is nonincreasing and bounded from below.

Note that we put \(\mathcal{T} = T\) whenever \(L = [0,1]\). If \(T\) is a \(t\)-norm, then \(x\) is defined for every \(x \in [0,1]\) and \(n \in N \cup \{0\}\) by 1 if \(n = 0\) and \(T(x, x^{n-1})\) if \(n \geq 1\). A \(t\)-norm \(T\) is said to be of Hadžić type, (we denote by \(T \in \mathcal{A}\)) if the family \((x^{n})\) is equicontinuous at \(x = 1\) (cf. [27]).

**Definition 6.2** (see [42]). A continuous \(t\)-norm \(\mathcal{T}\) on \(L = [0,1]^2\) is said to be **continuous \(t\)-representable** if there exist a continuous \(t\)-norm \(*\) and a continuous \(t\)-conorm \(\bigcirc\) on \([0,1]\) such that, for all \(x = (x_1, x_2), y = (y_1, y_2) \in L\),

\[ \mathcal{T}(x, y) = (x_1 * y_1, x_2 \bigcirc y_2). \]  

For example,

\[ \mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\}), \]
\[ M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \]

for all \(a = (a_1, a_2), b = (b_1, b_2) \in [0,1]^2\) are continuous \(t\)-representable. Define the mapping \(\mathcal{T}_\lambda\) from \(L^2\) to \(L\) by

\[ \mathcal{T}_\lambda(x, y) = \begin{cases} x, & \text{if } y \leq x, \\ y, & \text{if } x \leq y. \end{cases} \]

Recall (see [27, 28]) that if \(\{x_n\}\) is a given sequence in \(L\), \((\mathcal{T}_\lambda)_{i=1}^n x_i\) is defined recurrently by \((\mathcal{T}_\lambda)_{i=1}^1 x_i = x_1 \) and \((\mathcal{T}_\lambda)_{i=1}^n x_i = \mathcal{T}_\lambda((\mathcal{T}_\lambda)_{i=1}^{n-1} x_i, x_n)\) for all \(n \geq 2\).

A negation on \(\mathcal{L}\) is any decreasing mapping \(\mathcal{N} : L \to L\) satisfying \(\mathcal{N}(0, \mathcal{L}) = 1\) and \(\mathcal{N}(1, \mathcal{L}) = 0\). If \(\mathcal{N}(\mathcal{N}(x)) = x\), for all \(x \in L\), then \(\mathcal{N}\) is called an involutive negation. In the following, \(\mathcal{L}\) is endowed with a (fixed) negation \(\mathcal{N}\).

**Definition 6.3.** A **latticetic random normed space** (in short LRN-space) is a triple \((X, \mu, \mathcal{T}_\lambda)\), where \(X\) is a vector space and \(\mu\) is a mapping from \(X\) into \(D^*_\mathcal{T}\) such that the following conditions hold:

- (LRN1) \(\mu_x(t) = \epsilon_0(t)\) for all \(t > 0\) if and only if \(x = 0\),
- (LRN2) \(\mu_x(t/|a|) = \mu_x(t/|a|)\) for all \(x \in X, a \neq 0\) and \(t \geq 0\),
- (LRN3) \(\mu_x(t + s) \geq \mathcal{T}_\lambda(\mu_x(t), \mu_y(s))\) for all \(x, y \in X\) and \(t, s \geq 0\).

We note that from (LPN2) it follows that \(\mu_{-x}(t) = \mu_x(t)\) for all \(x \in X\) and \(t \geq 0\).
Example 6.4. Let \( L = [0, 1] \times [0, 1] \) and operation \( \leq_L \) be defined by

\[
L = \{(a_1, a_2) : (a_1, a_2) \in [0, 1] \times [0, 1], \ a_1 + a_2 \leq 1\},
\]

\[
(a_1, a_2) \leq_L (b_1, b_2) \iff a_1 \leq b_1, \ a_2 \geq b_2, \ \forall a = (a_1, a_2), \ b = (b_1, b_2) \in L.
\]

Then \( (L, \leq_L) \) is a complete lattice (see [42]). In this complete lattice, we denote its units by \( 0_L = (0, 1) \) and \( 1_L = (1, 0) \). Let \((X, \| \cdot \| ))\) be a normed space. Let \( T(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \) for all \( a = (a_1, a_2), \ b = (b_1, b_2) \in [0, 1] \times [0, 1] \) and \( \mu \) be a mapping defined by

\[
\mu_x(t) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right) \quad (t \in \mathbb{R}^+)
\]

then \((X, \mu, T)\) is a latticetic random normed spaces.

If \((X, \mu, T)\) is a latticetic random normed space, then

\[
\mathcal{U} = \{V(\varepsilon, \lambda) : \varepsilon > L, \ \lambda \in L \setminus \{0_L, 1_L\}\}, \quad V(\varepsilon, \lambda) = \{x \in X : F_\varepsilon(\varepsilon) > L A(\lambda)\},
\]

is a complete system of neighborhoods of null vector for a linear topology on \( X \) generated by the norm \( F \).

Definition 6.5. Let \((X, \mu, T, \lambda)\) be a latticetic random normed spaces.

1. A sequence \( \{x_n\} \) in \( X \) is said to be convergent to \( x \) in \( X \) if, for every \( t > 0 \) and \( \varepsilon \in L \setminus \{0_L\} \), there exists a positive integer \( N \) such that \( \mu_{x_n - x}(t) > L A(\varepsilon) \) whenever \( n \geq N \).
2. A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for every \( t > 0 \) and \( \varepsilon \in L \setminus \{0_L\} \), there exists a positive integer \( N \) such that \( \mu_{x_n - x_m}(t) > L A(\varepsilon) \) whenever \( n \geq m \geq N \).
3. A latticetic random normed spaces \((X, \mu, T, \lambda)\) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent to a point in \( X \).

Theorem 6.6. If \((X, \mu, T, \lambda)\) is a latticetic random normed space and \( \{x_n\} \) is a sequence such that \( x_n \to x \) in \( X \), then \( \lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t) \).

Proof. The proof is the same as classical random normed spaces, see [25]. \( \square \)


Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation \( Df(x, y) = 0 \) in random Banach spaces: an odd case.

Theorem 7.1. Let \( X \) be a linear space, let \((Y, \mu, T, \lambda)\) be a complete LRN-space, and \( \Phi \) let be a mapping from \( X^2 \) to \( D_L^+(\Phi(x, y)) \) is denoted by \( \Phi_{x,y} \) such that, for some \( 0 < a < 1/8 \),

\[
\Phi_{x,2y}(t) \leq_L \Phi_{x,y}(at) \quad (x, y \in X, \ t > 0).
\]
Let \( f : X \to Y \) be an odd mapping satisfying

\[
\mu_{Df(x,y)}(t) \geq_L \Phi_{x,y}(t)
\]

for all \( x, y \in X \) and \( t > 0 \). Then

\[
C(x) := \lim_{n \to \infty} 8^n \left( f\left( \frac{x}{2^n-1} \right) - 2f\left( \frac{x}{2^n} \right) \right)
\]

exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(2x) - 2f(x) - C(x)}(t) \geq_L \mathcal{T}_\alpha \left( \Phi_{x,x}(\frac{1-8\alpha}{5\alpha}t), \Phi_{2x,x}(\frac{1-8\alpha}{5\alpha}t) \right)
\]

for all \( x \in X \) and \( t > 0 \).

Proof. Letting \( x = y \) in (7.2), we get

\[
\mu_{f(3y) - 4f(2y) + 5f(y)}(t) \geq_L \Phi_{y,y}(t)
\]

for all \( y \in X \) and \( t > 0 \). Replacing \( x \) by \( 2y \) in (7.2), we get

\[
\mu_{f(4y) - 4f(3y) + 6f(2y) - 4f(y)}(t) \geq_L \Phi_{2y,y}(t)
\]

for all \( y \in X \) and \( t > 0 \). By (7.5) and (7.6),

\[
\mu_{f(4y) - 10f(2y) + 16f(y)}(5t) \geq_L \mathcal{T}_\alpha \left( \mu_{4f(3y) - 4f(2y) + 5f(y)}(4t), \mu_{f(4y) - 4f(3y) + 6f(2y) - 4f(y)}(t) \right)
\]

\[
= \mathcal{T}_\alpha \left( \mu_{f(3y) - 4f(2y) + 5f(y)}(t), \mu_{f(4y) - 4f(3y) + 6f(2y) - 4f(y)}(t) \right)
\]

\[
\geq_L \mathcal{T}_\alpha \left( \Phi_{y,y}(t), \Phi_{2y,y}(t) \right)
\]

for all \( y \in X \) and \( t > 0 \). Letting \( y := x/2 \) and \( g(x) := f(2x) - 2f(x) \) for all \( x \in X \), we get

\[
\mu_{g(x) - 8g(x/2)}(5t) \geq_L \mathcal{T}_\alpha \left( \Phi_{x/2,x/2}(t), \Phi_{x,x}(t) \right)
\]

for all \( x \in X \) and \( t > 0 \).

Consider the set

\[
S := \{ h : X \to Y, h(0) = 0 \}
\]

and introduce the generalized metric on \( S \):

\[
d(h, k) = \inf \{ u \in \mathbb{R}^+ : \mu_{h(x) - k(x)}(ut) \geq_L \mathcal{T}_\alpha (\Phi_{x,x}(t), \Phi_{2x,x}(t)), \forall x \in X, \forall t > 0 \}
\]
where, as usual, \( \inf \emptyset = +\infty \). It is easy to show that \((S, d)\) is complete (see the proof of Lemma 2.1 of [24]).

Now, we consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := 8h\left(\frac{x}{2}\right)
\]

for all \( x \in X \), and we prove that \( J \) is a strictly contractive mapping with the Lipschitz constant \( 8\alpha \).

Let \( h, k \in S \) be given such that \( d(h, k) < \varepsilon \). Then

\[
\mu_{h(x)-k(x)}(et) \geq L \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

for all \( x \in X \) and \( t > 0 \). Hence

\[
\mu_{Jh-Jk}(x) (8at) = \mu_{8h(x/2)-8k(x/2)}(8at) \\
= \mu_{h(x/2)-k(x/2)}(at) \\
\geq L \mathcal{T}_\alpha(\Phi_{x,x/2}(at), \Phi_{2x,x/2}(at)) \\
\geq L \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

for all \( x \in X \) and \( t > 0 \). So, \( d(h, k) < \varepsilon \) implies that

\[
d(Jh, Jk) \leq \frac{\alpha}{8}\varepsilon.
\]

This means that

\[
d(Jh, Jk) \leq \frac{\alpha}{8}d(h, k)
\]

for all \( h, k \in S \). It follows from (7.8) that

\[
\mu_{g(x)-8g(x/2)}(5at) \geq L \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5\alpha \leq 5/8 \).

By Theorem 1.1, there exists a mapping \( C : X \to Y \) satisfying the following:

1. \( C \) is a fixed point of \( J \), that is,

\[
C\left(\frac{x}{2}\right) = \frac{1}{8}C(x)
\]

for all \( x \in X \). Since \( g : X \to Y \) is odd, \( C : X \to Y \) is an odd mapping. The mapping \( C \) is a unique fixed point of \( J \) in the set

\[
M = \{ h \in S : d(h, g) < \infty \}.
\]
This implies that $C$ is a unique mapping satisfying (7.17) such that there exists a $u \in (0, \infty)$ satisfying
\[
\mu_{g(x)-C(x)}(ut) \geq_1 \mathcal{T}_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\] (7.19)
for all $x \in X$ and $t > 0$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality
\[
\lim_{n \to \infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)
\] (7.20)
for all $x \in X$.

(3) $d(h, C) \leq (1/(1-8\alpha))d(h, Jh)$ with $h \in M$, which implies the inequality
\[
d(g, C) \leq \frac{5\alpha}{1-8\alpha},
\] (7.21)
from which it follows that
\[
\mu_{g(x)-C(x)}\left(\frac{5\alpha}{1-8\alpha} t\right) \geq_1 \mathcal{T}_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t)).
\] (7.22)

This implies that the inequality (7.4) holds. From $Dg(x, y) = Df(2x, 2y) - 2Df(x, y)$, by (7.2), we deduce that
\[
\mu_{Df(2x, 2y)}(t) \geq_1 \Phi_{2x,2y}(t),
\]
\[
\mu_{-2Df(x, y)}(t) = \mu_{Df(x, y)}\left(\frac{1}{2}\right) \geq_1 \Phi_{x,y}\left(\frac{1}{2}\right)
\] (7.23)
and so, by (LRN3) and (7.1), we obtain
\[
\mu_{Dg(x,y)}(3t) \geq_1 \mathcal{T}_\lambda(\mu_{Df(2x, 2y)}(t), \mu_{-2Df(x, y)}(2t))
\]
\[
\geq_1 \mathcal{T}_\lambda(\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \geq_1 \Phi_{2x,2y}(t).
\] (7.24)

It follows that
\[
\mu_{g(x/2^n, y/2^n)}(3t) = \mu_{Dg(x/2^n, y/2^n)}\left(\frac{3t}{8^n}\right)
\]
\[
\geq \Phi_{x/2^n, y/2^n}\left(\frac{t}{8^n}\right) \geq_1 \Phi_{x,y}\left(\frac{1}{8(8\alpha)^{-1}}\right)
\] (7.25)
for all $x, y \in X$, $t > 0$ and $n \in \mathbb{N}$. 
Since \( \lim_{n \to \infty} \Phi_{x,y}(3/8)(t/(8\alpha)^{n-1}) = 1 \) for all \( x, y \in X \) and \( t > 0 \), by Theorem 2.4, we deduce that

\[
\mu_{DC(x,y)}(3t) = 1_x
\]  

for all \( x, y \in X \) and \( t > 0 \). Thus the mapping \( C : X \to Y \) satisfies (1.4).

Now, we have

\[
C(2x) - 8C(x) = \lim_{n \to \infty} \left[ 8^n g \left( \frac{x}{2n^{1/2}} \right) - 8^{n+1} g \left( \frac{x}{2n^{1/2}} \right) \right] = 0
\]  

for all \( x \in X \). Since the mapping \( x \to C(2x) - 2C(x) \) is cubic (see Lemma 2.2 of [14]), from the equality \( C(2x) = 8C(x) \), we deduce that the mapping \( C : X \to Y \) is cubic. \( \square \)

**Corollary 7.2.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 3 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]  

for all \( x, y \in X \) and \( t > 0 \). Note that \( (X, \mu, T_M) \) is a complete LRN-space, in which \( L = [0,1] \), then

\[
C(x) := \lim_{n \to \infty} 8^n \left( f \left( \frac{x}{2n^{1/2}} \right) - 2f \left( \frac{x}{2n^{1/2}} \right) \right)
\]  

exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(2x) - 2f(x) - C(x)}(t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(1 + 2^p)\theta \|x\|^p}
\]  

for all \( x \in X \) and \( t > 0 \).

**Proof.** The proof follows from Theorem 7.1 by taking

\[
\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]
for all \(x, y \in X\) and \(t > 0\). Then we can choose \(\alpha = 2^{-p}\), and we get

\[
\mu_{f(2x) - 2f(x) - C(x)}(t) \geq \min \left( \frac{(1 - 2^{3-p})t}{(1 - 2^{3-p})t + 5 \cdot 2^{-p} \theta(2\|x\|^p)}, \frac{(1 - 2^{3-p})t}{(1 - 2^{3-p})t + 5 \cdot 2^{-p} \theta(\|2x\|^p + \|x\|^p)} \right)
\]

\[
\geq \frac{(1 - 2^{3-p})t}{(1 - 2^{3-p})t + 5 \cdot 2^{-p} \theta(\|2x\|^p + \|x\|^p)}
\]

\[
= \frac{(2^p - 8)t}{(2^p - 8)t + 5 \cdot (2^p + 1)\theta \|x\|^p},
\]

(7.32)

which is the desired result. \(\square\)

**Theorem 7.3.** Let \(X\) be a linear space, let \((Y, \mu, \mathcal{T}_\cdot)\) be a complete LRN-space, and let \(\Phi\) be a mapping from \(X^2\) to \(D^r_L\) (\(\Phi(x, y)\) is denoted by \(\Phi_{x,y}\)) such that, for some \(0 < \alpha < 8\),

\[
\Phi_{x/2,y/2}(t) \leq_L \Phi_{x,y}(at) \quad (x, y \in X, \ t > 0).
\]

(7.33)

Let \(f : X \to Y\) be an odd mapping satisfying (7.2), then

\[
C(x) := \lim_{n \to \infty} \frac{1}{8^n} \left( f \left( 2^{n+1}x \right) - 2f(2^n x) \right)
\]

(7.34)

exists for each \(x \in X\) and defines a cubic mapping \(C : X \to Y\) such that

\[
\mu_{f(2x) - 2f(x) - C(x)}(t) \geq_L \mathcal{T}_\lambda \left( \Phi_{x,x} \left( \frac{8 - \alpha}{5} \frac{t}{\|x\|^p} \right), \Phi_{2x,x} \left( \frac{8 - \alpha}{5} \frac{t}{\|2x\|^p} \right) \right)
\]

(7.35)

for all \(x \in X\) and \(t > 0\).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1.

Consider the linear mapping \(J : S \to S\) such that

\[
Jh(x) := \frac{1}{8} h(2x)
\]

(7.36)

for all \(x \in X\), and we prove that \(J\) is a strictly contractive mapping with the Lipschitz constant \(\alpha/8\).

Let \(h, k \in S\) be given such that \(d(h, k) < \varepsilon\), then

\[
\mu_{h(x) - k(x)}(\varepsilon t) \geq_L \mathcal{T}_\lambda \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\]

(7.37)
for all \( x \in X \) and \( t > 0 \). Hence

\[
\mu_{J}(x) - J(k(x)) \left( \frac{\alpha}{8} \varepsilon t \right) = \mu_{J}(1/8)h(2x) - (1/8)k(2x) \left( \frac{\alpha}{8} \varepsilon t \right)
\]

\[
= \mu_{h}(2x) - k(2x) (\alpha \varepsilon t)
\]

\[
\geq L \mathcal{T}_{\lambda}(\Phi_{2x,2x}(at), \Phi_{4x,2x}(at))
\]

\[
\geq \mathcal{T}_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

for all \( x \in X \) and \( t > 0 \). So, \( d(h, k) < \varepsilon \) implies that

\[
d(Jh, Jk) \leq \frac{\alpha}{8} \varepsilon.
\]

(7.39)

This means that

\[
d(Jh, Jk) \leq \frac{\alpha}{8} d(h, k)
\]

(7.40)

for all \( g, h \in S \). Letting \( g(x) := f(2x) - 2f(x) \) for all \( x \in X \), from (7.8), we get that

\[
\mu_{g}(x) - (1/8)g(2x) \left( \frac{5}{8} t \right) \geq \mathcal{T}_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.41)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5/8 \).

By Theorem 1.1, there exists a mapping \( C : X \to Y \) satisfying the following:

1. \( C \) is a fixed point of \( J \), that is,

\[
C(2x) = 8C(x)
\]

(7.42)

for all \( x \in X \). Since \( g : X \to Y \) is odd, \( C : X \to Y \) is an odd mapping. The mapping \( C \) is a unique fixed point of \( J \) in the set

\[
M = \{ h \in S : d(h, g) < \infty \}.
\]

(7.43)

This implies that \( C \) is a unique mapping satisfying (7.42) such that there exists a \( u \in (0, \infty) \) satisfying

\[
\mu_{g}(x) - C(x)(ut) \geq \mathcal{T}_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.44)

for all \( x \in X \) and \( t > 0 \).

2. \( d(J^{n}g, C) \to 0 \) as \( n \to \infty \). This implies the equalit

\[
\lim_{n \to \infty} \frac{1}{8^{n}} s(2^{n}x) = C(x)
\]

(7.45)

for all \( x \in X \).
(3) \(d(h, C) \leq (1/(1-\alpha/8))d(h, Jh)\) for every \(h \in M\), which implies the inequality

\[
d(g, C) \leq \frac{5}{8-\alpha},
\]

(7.46)

from which it follows that

\[
\mu_{g(x)-C(x)}(\frac{5}{8-\alpha}t) \geq L, \mathcal{C}_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.47)

for all \(x \in X\) and \(t > 0\). This implies that the inequality (7.35) holds.

From

\[
\mu_{Dg(x,y)}(3t) \geq L, \mathcal{C}_\lambda(\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \geq L, \mathcal{C}_\lambda(\Phi_{2x,2y}(t), \Phi_{x,y}(\frac{t}{\alpha}))
\]

(7.48)

by (7.33), we deduce that

\[
\mu_{8^nDg(2^n x, 2^n y)}(3t) = \mu_{Dg(2^n x, 2^n y)}(3 \cdot 8^n t) \geq L, \mathcal{C}_\lambda(\Phi_{2x,2y}(t), \Phi_{x,y}(8^n t)) \geq L, \mathcal{C}_\lambda(\Phi_{2x,2y}(t), \Phi_{x,y}(\frac{8^n t}{\alpha}))
\]

(7.49)

for all \(x, y \in X\), \(t > 0\), and \(n \in \mathbb{N}\). As \(n \to \infty\), we deduce that

\[
\mu_{DC(x,y)}(3t) = 1_x
\]

(7.50)

for all \(x, y \in X\) and \(t > 0\). Thus the mapping \(C : X \to Y\) satisfies (1.4).

Now, we have

\[
C(2x) - 8C(x) = \lim_{n \to \infty} \left[ \frac{1}{8^n} g(2^{n+1} x) - \frac{1}{8^{n-1}} g(2^n x) \right]
\]

(7.51)

\[
= \lim_{n \to \infty} \left[ \frac{1}{8^n} g(2^{n+1} x) - \frac{1}{8^n} g(2^n x) \right] = 0
\]

for all \(x \in X\). Since the mapping \(x \to C(2x) - 2C(x)\) is cubic (see Lemma 2.2 of [14]), from the equality \(C(2x) = 8C(x)\), we deduce that the mapping \(C : X \to Y\) is cubic.

\(\square\)

**Corollary 7.4.** Let \(\theta \geq 0\) and let \(p\) be a real number with \(0 < p < 3\). Let \(X\) be a normed vector space with norm \(\|\cdot\|\). Let \(f : X \to Y\) be an odd mapping satisfying (7.28), then

\[
C(x) := \lim_{n \to \infty} \frac{1}{8^n} \left( f \left(2^{n+1} x\right) - 2 f \left(2^n x\right) \right)
\]

(7.52)
exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8-2^p)t}{(8-2^p)t + 5(1+2^p)\theta\|x\|^p}$$  \hspace{1cm} (7.53)

for all $x \in X$ and $t > 0$. Note that $(X, \mu, T_M)$ is a complete LRN-space, in which $L = [0, 1]$.

Proof. The proof follows from Theorem 7.3 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$  \hspace{1cm} (7.54)

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^p$, and we get the desired result. \hfill \Box

**Theorem 7.5.** Let $X$ be a linear space, let $(Y, \mu, T_n)$ be a complete LRN-space, and let $\Phi$ be a mapping from $X^2$ to $D^*_L(\Phi(x, y)$ is denoted by $\Phi_{x,y})$ such that, for some $0 < \alpha < 1/2$,

$$\Phi_{2x,2y}(t) \leq L \Phi_{x,y}(at) \hspace{1cm} (x, y \in X, \ t > 0).$$  \hspace{1cm} (7.55)

Let $f : X \to Y$ be an odd mapping satisfying (7.2), then

$$A(x) := \lim_{n \to \infty} 2^n \left( f\left( \frac{x}{2^{n-1}} \right) - 8f\left( \frac{x}{2^n} \right) \right)$$  \hspace{1cm} (7.56)

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq L \mathcal{T}_L \left( \Phi_{x,x}\left( \frac{1-2\alpha}{5\alpha}t \right), \Phi_{2x,2x}\left( \frac{1-2\alpha}{5\alpha}t \right) \right)$$  \hspace{1cm} (7.57)

for all $x \in X$ and $t > 0$.

Proof. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 7.1. Letting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$ in (7.7), we get

$$\mu_{g(x)-2g(x/2)}(5t) \geq L \mathcal{T}_L \left( \Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t) \right)$$  \hspace{1cm} (7.58)

for all $x \in X$ and $t > 0$.

Now, we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left( \frac{x}{2} \right)$$  \hspace{1cm} (7.59)

for all $x \in X$. It is easy to see that $J$ is a strictly contractive self-mapping on $S$ with the Lipschitz constant $2\alpha$. 

It follows from (7.58) and (7.55) that

$$
\mu_{g(x) - 2g(x/2)}(5at) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t))
$$

(7.60)

for all $x \in X$ and $t > 0$. So, $d(g, fg) \leq 5\alpha < \infty$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

1. $A$ is a fixed point of $J$, that is,

$$
A\left(\frac{x}{2}\right) = \frac{1}{2}A(x)
$$

(7.61)

for all $x \in X$. Since $g : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$
M = \{ h \in S : d(h, g) < \infty \}.
$$

(7.62)

This implies that $A$ is a unique mapping satisfying (7.61) such that there exists a $u \in (0, \infty)$ satisfying

$$
\mu_{g(x) - A(x)}(ut) \geq L \cdot \Phi_{x,x}(t), \Phi_{2x,x}(t)
$$

(7.63)

for all $x \in X$ and $t > 0$.

2. $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$
\lim_{n \to \infty} 2^n g\left(\frac{x}{2^n}\right) = A(x)
$$

(7.64)

for all $x \in X$.

3. $d(h, A) \leq (1/ (1 - 2\alpha))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$
d(g, A) \leq \frac{5\alpha}{1 - 2\alpha}.
$$

(7.65)

This implies that the inequality (7.57) holds. Since $\mu_{Dg(x,y)}(3t) \geq L \Phi_{2x,2y}(t)$, it follows that

$$
\mu_{2^n Dg(x/2^n,y/2^n)}(3t) = \mu_{Dg(x/2^n,y/2^n)}\left(\frac{3}{2^n} t\right)
$$

$$
\geq \Phi_{x/2^n-1,y/2^n-1} \left(\frac{t}{2^n}\right) \geq L \cdot \cdots \geq L \Phi_{x,y}\left(\frac{1}{2} \left(\frac{t}{(2\alpha)^{n-1}}\right)\right)
$$

(7.66)
for all \( x, y \in X, t > 0, \) and \( n \in \mathbb{N} \). As \( n \to \infty \), we deduce that

\[
\mu_{DA(x,y)}(3t) = 1.
\]

(7.67)

for all \( x, y \in X \) and \( t > 0 \). Thus, the mapping \( A : X \to Y \) satisfies (1.4).

Now, we have

\[
A(2x) - 2A(x) = \lim_{n \to \infty} \left[ 2^n g \left( \frac{x}{2^{n-1}} \right) - 2^{n+1} g \left( \frac{x}{2^n} \right) \right]
\]

(7.68)

\[
= 2 \lim_{n \to \infty} \left[ 2^{n-1} g \left( \frac{x}{2^{n-1}} \right) - 2^n g \left( \frac{x}{2^n} \right) \right] = 0
\]

for all \( x \in X \). Since the mapping \( x \to A(2x) - 8A(x) \) is additive (see Lemma 2.2 of [14]), from the equality \( A(2x) = 2A(x) \), we deduce that the mapping \( A : X \to Y \) is additive. \( \square \)

**Corollary 7.6.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying (7.28), then

\[
A(x) := \lim_{n \to \infty} 2^n \left( f \left( \frac{x}{2^{n-1}} \right) - 8f \left( \frac{x}{2^n} \right) \right)
\]

(7.69)

exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 5(1 + 2^p)\theta \|x\|^p}
\]

(7.70)

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0, 1] \).

**Proof.** The proof follows from Theorem 7.5 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]

(7.71)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^{-p} \), and we get the desired result. \( \square \)

**Theorem 7.7.** Let \( X \) be a linear space, let \( (Y, \mu, T_n) \) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D^*_L(\Phi(x,y)) \) is denoted by \( \Phi_{x,y}(t) \) such that, for some \( 0 < \alpha < 2 \),

\[
\Phi_{x,y}(at) \geq L. \Phi_{x/2,y/2}(t) \quad (x, y \in X, \ t > 0).
\]

(7.72)

Let \( f : X \to Y \) be an odd mapping satisfying (7.2), then

\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} \left( f \left( 2^{n+1}x \right) - 8f(2^n x) \right)
\]

(7.73)
exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
\mu_{f(2x) - 8f(x) - A(x)}(t) \geq L \tau_{\lambda}(\Phi_{x,x}(\frac{2 - \alpha}{5\alpha} t), \Phi_{2x,x}(\frac{2 - \alpha}{5\alpha} t))
\]  

(7.74)

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1. Consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{2} h(2x)
\]  

(7.75)

for all \( x \in X \). It is easy to see that \( J \) is a strictly contractive self-mapping on \( S \) with the Lipschitz constant \( \alpha/2 \). Let \( g(x) = f(2x) - 8f(x) \), from (7.58), it follows that

\[
\mu_{g(x) - 1/2g(2x)} \left( \frac{5}{2} t \right) \geq L \tau_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]  

(7.76)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5/2 \). By Theorem 1.1, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( J \), that is,

\[
A(2x) = 2A(x)
\]  

(7.77)

for all \( x \in X \). Since \( h : X \to Y \) is odd, \( A : X \to Y \) is an odd mapping. The mapping \( A \) is a unique fixed point of \( J \) in the set

\[
M = \{ h \in S : d(h, g) < \infty \}.
\]  

(7.78)

This implies that \( A \) is a unique mapping satisfying (7.77) such that there exists a \( u \in (0, \infty) \) satisfying

\[
\mu_{g(x) - A(x)}(ut) \geq L \tau_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]  

(7.79)

for all \( x \in X \) and \( t > 0 \).

2. \( d(J^n g, A) \to 0 \) as \( n \to \infty \). This implies the equality

\[
\lim_{n \to \infty} \frac{1}{2^n} g(2^n x) = A(x)
\]  

(7.80)

for all \( x \in X \).
(3) \( d(h, A) \leq (1/(1 - \alpha/2))d(h, Jh) \), which implies the inequality

\[
d(g, A) \leq \frac{5}{2 - \alpha}.
\] (7.81)

This implies that the inequality (7.74) holds.

Proceeding as in the proof of Theorem 7.5, we obtain that the mapping \( A : X \rightarrow Y \) satisfies (1.4). Now, we have

\[
A(2x) - 2A(x) = \lim_{n \rightarrow \infty} \left( \frac{1}{2^n} g(2^{n+1}x) - \frac{1}{2^{n-1}} g(2^n x) \right)
\]

\[
= 2 \lim_{n \rightarrow \infty} \left( \frac{1}{2^n} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x) \right) = 0
\] (7.82)

for all \( x \in X \). Since the mapping \( x \rightarrow A(2x) - 8A(x) \) is additive (see Lemma 2.2 of [14]), from the equality \( A(2x) = 2A(x) \), we deduce that the mapping \( A : X \rightarrow Y \) is additive. \( \square \)

**Corollary 7.8.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \rightarrow Y \) be an odd mapping satisfying (7.28), then

\[
A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} \left( f(2^{n+1}x) - 8f(2^n x) \right)
\] (7.83)

exists for each \( x \in X \) and defines an additive mapping \( A : X \rightarrow Y \) such that

\[
\mu_{f(2x) - 8f(x) - A(x)}(t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 5(1 + 2^p)\theta\|x\|^p}
\] (7.84)

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, TM) \) is a complete LRN-space in which \( L = [0,1] \).

**Proof.** The proof follows from Theorem 7.7 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\] (7.85)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

**8. Generalized Hyers-Ulam Stability of the Functional Equation (1.4): An Even Case via Fixed-Point Method**

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation \( Df(x, y) = 0 \) in random Banach spaces, an even case.
Theorem 8.1. Let $X$ be a linear space, let $(Y, \mu, \mathcal{T}_0)$ be a complete LRN-space, and let $\Phi$ be a mapping from $X^2$ to $D_L^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/16$,

$$\Phi_{x,y}(at) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X, \ t > 0). \quad (8.1)$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$Q(x) := \lim_{n \to \infty} 16^n \left( f \left( \frac{x}{2^n} \right) - 4f \left( \frac{x}{2^n} \right) \right) \quad (8.2)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq_L \mathcal{T}_0 \left( \Phi_{x,x} \left( \frac{1-16\alpha}{5\alpha} t \right), \Phi_{2x,x} \left( \frac{1-16\alpha}{5\alpha} t \right) \right) \quad (8.3)$$

for all $x \in X$ and $t > 0$.

Proof. Letting $x = y$ in (7.2), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq_L \Phi_{y,y}(t) \quad (8.4)$$

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (7.2), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq_L \Phi_{2y,y}(t) \quad (8.5)$$

for all $y \in X$ and $t > 0$. By (8.4) and (8.5),

$$\mu_{f(4x)-20f(2x)+64f(x)}(5t) \geq_L \mathcal{T}_0 \left( \mu_{4f(3x)-6f(2x)+15f(x)}(4t), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \right) \quad (8.6)$$

for all $x \in X$ and $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-16g(x/2)}(5t) \geq_L \mathcal{T}_0 \left( \Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t) \right) \quad (8.7)$$

for all $x \in X$ and $t > 0$. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 7.1.

Now we consider the linear mapping $J : S \to S$ such that $Jh(x) := 16h(x/2)$ for all $x \in X$. It is easy to see that $J$ is a strictly contractive self-mapping on $S$ with the Lipschitz constant $16\alpha$. It follows from (8.7) that

$$\mu_{g(x)-16g(x/2)}(5at) \geq_L \mathcal{T}_0 \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right) \quad (8.8)$$

for all $x \in X$ and $t > 0$. So,

$$d(g, J) \leq 5\alpha \leq \frac{5}{16} < \infty. \quad (8.9)$$
By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

1. $Q$ is a fixed point of $J$, that is,
   
   $$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \quad (8.10)$$

   for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping $Q$ is a unique fixed point of $J$ in the set
   
   $$M = \{ h \in S : d(h, g) < \infty \}. \quad (8.11)$$

   This implies that $Q$ is a unique mapping satisfying (8.10) such that there exists a $u \in (0, \infty)$ satisfying
   
   $$\mu_{g(x) - Q(x)}(ut) \geq \mathcal{T}_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \quad (8.12)$$

   for all $x \in X$ and $t > 0$.

2. $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality
   
   $$\lim_{n \to \infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x) \quad (8.13)$$

   for all $x \in X$.

3. $d(h, Q) \leq (1/(1 - 16\alpha))d(h, Jh)$ for every $h \in M$, which implies the inequality
   
   $$d(g, Q) \leq \frac{5\alpha}{1 - 16\alpha}. \quad (8.14)$$

   This implies that the inequality (8.3) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$Q(2x) - 16Q(x) = \lim_{n \to \infty} \left[16^n g\left(\frac{x}{2^{n-1}}\right) - 16^{n+1} g\left(\frac{x}{2^n}\right)\right]$$

$$= 16 \lim_{n \to \infty} \left[16^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 16^n g\left(\frac{x}{2^n}\right)\right] = 0 \quad (8.15)$$

for all $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \to Y$ is quartic.

**Corollary 8.2.** Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then

$$Q(x) := \lim_{n \to \infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right) \quad (8.16)$$
Theorem 8.3. Let \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x) - 4f(x) - Q(x)}(t) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 5(1 + 2^p)\theta\|x\|^p} \tag{8.17}
\]

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0,1] \).

Proof. The proof follows from Theorem 8.1 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{8.18}
\]

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^{-p} \), and we get the desired result. \(\square\)

**Theorem 8.3.** Let \( X \) be a linear space, let \( (Y, \mu, \mathcal{T}_\lambda) \) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D_L^+ (\Phi(x,y)) \) is denoted by \( \Phi_{x,y} \) such that, for some \( 0 < \alpha < 16 \),

\[
\Phi_{x,y}(at) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, t > 0). \tag{8.19}
\]

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (7.2), then

\[
Q(x) := \lim_{n \to \infty} \frac{1}{16^n} \left( f \left( 2^{n+1}x \right) - 4f(2^nx) \right) \tag{8.20}
\]

exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x) - 4f(x) - Q(x)}(t) \geq \lambda \left( \Phi_{x,x} \left( \frac{16 - \alpha}{5}t \right), \Phi_{2x,x} \left( \frac{16 - \alpha}{5}t \right) \right) \tag{8.21}
\]

for all \( x \in X \) and \( t > 0 \).

Proof. In the generalized metric space \( (S,d) \) defined in the proof of Theorem 7.1, we consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{16}h(2x) \tag{8.22}
\]

for all \( x \in X \). It is easy to see that \( J \) is a strictly contractive self-mapping on \( S \) with the Lipschitz constant \( \alpha/16 \).

Letting \( g(x) := f(2x) - 4f(x) \) for all \( x \in X \), by (8.7), we get

\[
\mu_{g(x) - (1/16)g(2x)} \left( \frac{5}{16}t \right) \geq \lambda \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right) \tag{8.23}
\]

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5/16 \).
By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, that is,

$$Q(2x) = 16Q(x)$$

(8.24)

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{ h \in S : d(h, g) < \infty \}.$$  

(8.25)

This implies that $Q$ is a unique mapping satisfying (8.24) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq T^\lambda(\varphi_{x,x}(t), \varphi_{2x,x}(t))$$

(8.26)

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{1}{16^n} g(2^n x) = Q(x)$$

(8.27)

for all $x \in X$.

(3) $d(g, Q) \leq (16/(16 - \alpha))d(g, Jg)$ for each $h \in M$, which implies the inequality

$$d(g, Q) \leq 5/(16 - \alpha).$$

(8.28)

This implies that the inequality (8.21) holds.

Proceeding as in the proof of Theorem 7.3, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$Q(2x) - 16Q(x) = \lim_{n \to \infty} \left[ \frac{1}{16^n} g^n(2^{n+1} x) - \frac{1}{16^{n+1}} g^n(2^n x) \right]$$

$$= 16 \lim_{n \to \infty} \left[ \frac{1}{16^{n+1}} g^n(2^{n+1} x) - \frac{1}{16^n} g^n(2^n x) \right] = 0$$

(8.29)

for all $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \to Y$ is quartic.

\[ \square \]

**Corollary 8.4.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then

$$Q(x) := \lim_{n \to \infty} \frac{1}{16^n} \left( f \left( 2^{n+1} x \right) - 4f(2^n x) \right)$$

(8.30)
exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x) - 4f(x) - Q(x)}(t) \geq \frac{(16 - 2^p)t}{(16 - 2^p)t + 5(1 + 2^p)\theta \|x\|^p}
\]  \hspace{1cm} (8.31)

for all \( x \in X \) and \( t > 0 \), where \((X, \mu, T_M)\) is a complete LRN-space in which \( L = [0, 1] \).

**Proof.** The proof follows from Theorem 8.3 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta (\|x\|^p + \|y\|^p)}
\]  \hspace{1cm} (8.32)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

**Theorem 8.5.** Let \( X \) be a linear space, let \((Y, \mu, T_\alpha)\) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D^4_\alpha (\Phi(x,y)) \) is by denoted \( \Phi_{x,y} \) such that, for some \( 0 < \alpha < 1/4 \),

\[
\Phi_{x,y}(at) \geq \Phi_{2x,2y}(t) \quad (x, y \in X, \ t > 0).
\]  \hspace{1cm} (8.33)

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (7.2), then

\[
T(x) := \lim_{n \to \infty} 4^n \left( f\left( \frac{x}{2^{n-1}} \right) - 16f\left( \frac{x}{2^n} \right) \right)
\]  \hspace{1cm} (8.34)

exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
\mu_{f(2x) - 16f(x) - T(x)}(t) \geq \lambda_\alpha \left( \Phi_{x,x}\left( \frac{1 - 4\alpha}{5\alpha} - t \right), \Phi_{2x,2x}\left( \frac{1 - 4\alpha}{5\alpha} - t \right) \right)
\]  \hspace{1cm} (8.35)

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1. Letting \( g(x) := f(2x) - 16f(x) \) for all \( x \in X \) in (8.6), we get

\[
\mu_{g(x) - 4g(x/2)}(5t) \geq \lambda_\alpha (\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t))
\]  \hspace{1cm} (8.36)

for all \( x \in X \) and \( t > 0 \). It is easy to see that the linear mapping \( f : S \to S \) such that

\[
f h(x) := 4h\left( \frac{x}{2} \right)
\]  \hspace{1cm} (8.37)

for all \( x \in X \), is a strictly contractive self-mapping with the Lipschitz constant \( 4\alpha \).

It follows from (8.36) that

\[
\mu_{g(x) - 4g(x/2)}(5\alpha t) \geq \lambda_\alpha (\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]  \hspace{1cm} (8.38)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Ig) \leq 5\alpha < \infty \).
By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following:

1. $T$ is a fixed point of $J$, that is,

$$T\left(\frac{x}{2}\right) = \frac{1}{4} T(x) \quad (8.39)$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping $T$ is a unique fixed point of $J$ in the set $M = \{ h \in S : d(h, \mathcal{G}) < \infty \}$. This implies that $T$ is a unique mapping satisfying (8.39) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-T(x)}(ut) \geq \mathcal{T}_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \quad (8.40)$$

for all $x \in X$ and $t > 0$.

2. $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 4^n g\left(\frac{x}{2^n}\right) = T(x) \quad (8.41)$$

for all $x \in X$.

3. $d(h, T) \leq (1/(1 - 4\alpha))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, T) \leq \frac{5\alpha}{1 - 4\alpha} \quad (8.42)$$

This implies that the inequality (8.35) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping $T : X \to Y$ satisfies (1.4). Now, we have

$$T(2x) - 4T(x) = \lim_{n \to \infty} \left[ 4^n g\left(\frac{x}{2^n}\right) - 4^{n+1} g\left(\frac{x}{2^{n+1}}\right) \right]$$

$$= 4 \lim_{n \to \infty} \left[ 4^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 4^n g\left(\frac{x}{2^n}\right) \right] = 0 \quad (8.43)$$

for all $x \in X$. Since the mapping $x \to T(2x) - 16T(x)$ is quadratic, we get that the mapping $T : X \to Y$ is quadratic. \qed

**Corollary 8.6.** Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then

$$T(x) := \lim_{n \to \infty} 4^n \left( f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right) \right) \quad (8.44)$$
exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(x) - 16f(x) - T(x)}(t) \geq \frac{(2^n - 4)t}{(2^n - 4)t + 5(1 + 2^n)\theta \|x\|^p}$$

(8.45)

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 8.5 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

(8.46)

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$, and we get the desired result. \(\Box\)

**Theorem 8.7.** Let $X$ be a linear space, let $(Y, \mu, T_M)$ be a complete RN-space, and let $\Phi$ be a mapping from $X^2$ to $D^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 4$,

$$\Phi_{x,y}(\alpha t) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, \ t > 0).$$

(8.47)

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$T(x) := \lim_{n \to \infty} \frac{1}{4^n} \left( f \left( 2^{n+1}x \right) - 16f(2^n x) \right)$$

(8.48)

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(x) - 16f(x) - T(x)}(t) \geq T_M \left( \Phi_{x,x} \left( \frac{4 - \alpha}{5} t \right), \Phi_{2x,x} \left( \frac{4 - \alpha}{5} t \right) \right)$$

(8.49)

for all $x \in X$ and $t > 0$.

Proof. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 7.1.

It is easy to see that the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{4} h(2x)$$

(8.50)

for all $x \in X$ is a strictly contractive self-mapping with the Lipschitz constant $\alpha/4$.

Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$, from (8.36), we get

$$\mu_{g(x) - 1/4g(2x)} \left( \frac{5}{4} t \right) \geq T_M \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)$$

(8.51)

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/4$. 

Journal of Applied Mathematics 41
By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following:

(1) $T$ is a fixed point of $f$, that is,

$$ T(2x) = 4T(x) \quad (8.52) $$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping $T$ is a unique fixed point of $f$ in the set

$$ M = \{ h \in S : d(h, g) < \infty \}. \quad (8.53) $$

This implies that $T$ is a unique mapping satisfying (8.52) such that there exists a $u \in (0, \infty)$ satisfying

$$ \mu_{g(x) - T(x)}(ut) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \quad (8.54) $$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$ \lim_{n \to \infty} \frac{1}{4^n} g(2^n x) = T(x) \quad (8.55) $$

for all $x \in X$.

(3) $d(h, T) \leq (1/(1 - a/4))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$ d(g, T) \leq 5/(4 - a). \quad (8.56) $$

This implies that the inequality (8.49) holds.

Proceeding as in the proof of Theorem 2.3, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$ T(2x) - 4T(x) = \lim_{n \to \infty} \left[ \frac{1}{4^n} g(2^{n+1} x) - \frac{1}{4^{n-1}} g(2^n x) \right] $$

$$ = 4 \lim_{n \to \infty} \left[ \frac{1}{4^{n+1}} g(2^{n+1} x) - \frac{1}{4^n} g(2^n x) \right] = 0 \quad (8.57) $$

for all $x \in X$. Since the mapping $x \to T(2x) - 16T(x)$ is quadratic, we get that the mapping $T : X \to Y$ is quadratic.

**Corollary 8.8.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 2$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28). Then

$$ T(x) := \lim_{n \to \infty} \frac{1}{4^n} \left( f(2^{n+1} x) - 16 f(2^n x) \right) \quad (8.58) $$
exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 5(1 + 2^p)\theta \|x\|^p}
\]  

(8.59)

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0,1] \).

Proof. The proof follows from Theorem 8.5 by taking

\[
\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]  

(8.60)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

Acknowledgments

The authors are grateful to the area Editor Professor Yeong-Cheng Liou and the reviewer for their valuable comments and suggestions. Y. J. Cho was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant no. 2011-0021821).

References


Submit your manuscripts at
http://www.hindawi.com