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Research Article

An Optimal Iteration Method for Strongly Nonlinear Oscillators

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We introduce a new method, namely, the Optimal Iteration Perturbation Method (OIPM), to solve nonlinear differential equations of oscillators with cubic and harmonic restoring force. We illustrate that OIPM is very effective and convenient and does not require linearization or small perturbation. Contrary to conventional methods, in OIPM, only one iteration leads to high accuracy of the solutions. The main advantage of this approach consists in that it provides a convenient way to control the convergence of approximate solutions in a very rigorous way and allows adjustment of convergence regions where necessary. A very good agreement was found between approximate and numerical solutions, which prove that OIPM is very efficient and accurate.

1. Introduction

Mathematical modelling of many physical systems leads to nonlinear ordinary or partial differential equations in various fields of physics, mathematics, or engineering. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. In many cases, it is possible to replace a nonlinear differential equation by a corresponding linear differential equation that approximates closely the original one to give useful results. In general, the study of nonlinear differential equations is restricted to a variety of special classes of equations and the method of solution usually involves one or more techniques to achieve analytical approximations to the solutions. Solving the governing equations of nonlinear oscillators has been one of the most time-consuming and difficult affairs among researchers. Therefore, many researchers and scientists of both vibrations and

mathematics have recently paid much attention to find and develop approximate solutions. Perturbation methods are well established tools to study diverse aspects of nonlinear problems [1–3]. However, the use of perturbation theory in many important practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings. Such a new technique should work over a larger range of parameters and yield accurate analytical approximate solutions beyond the coverage and ability of the classical perturbation methods.

It is noted that several methods have been used to obtain approximate solutions for strongly nonlinear oscillators. An interesting approach which combines the harmonic balance method and linearization of nonlinear oscillation equation was proposed in [4]. There also exists a wide range of literature dealing with approximate periodic solutions for nonlinear problems with large parameters by using a mixture of methodologies: the variational iteration method [5–8], some linearization methods [9, 10], the optimal homotopy asymptotic method [11], the optimal parametric iteration method [12], some modified Lindstedt-Poincare methods [13, 14], or a simple approach [15].

In this paper, coupling the iteration perturbation method [16] with the least square technology, a new approach, namely, the Optimal Iteration Perturbation Method (OIPM), is proposed to find explicit analytical periodic solutions to nonlinear oscillators with cubic and harmonic restoring force. Recently, in the same way, the variational iteration method [5] and the homotopy perturbation method [17] have been coupled with the least square technology resulting in two new powerful methods, namely, the optimal variational iteration method (OVIM) [7] and the optimal homotopy perturbation method (OHPM) [18].

The efficiency of the present procedure is proved while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. The proposed method does not require a small parameter into the equation and provides a convenient and rigorous way to optimally control the convergence of the solutions by means of a finite number of unknown parameters.

2. Formulation and Solution Approach

In this work, we consider a nonlinear oscillator in the form

$$u'' + f\left(u, u', u''\right) = 0, \tag{2.1}$$

with initial conditions

$$u(0) = A, \qquad u'(0) = 0, \tag{2.2}$$

where prime denotes derivative with respect to variable τ.

For (2.1) and (2.2) we propose the following iteration scheme:

$$u''_{n+1} + f\left(u_n, u'_n, u''_n\right) = 0, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

where the initial approximation $u_0(\tau)$ can be chosen in the general form

$$u_0(\tau) = \sum_{i=1}^{m} C_i f_i(\tau), \tag{2.4}$$

where C_i are unknown constants, m is a positive integer number, and the functions f_i are trigonometric functions sine or/and cosine in case of nonlinear oscillators.

Integrating (2.3) twice with respect to τ, we have, respectively,

$$\begin{aligned} &\text{(i) } u'_{n+1}(\tau) + F_n(\tau, C_1, C_2, \ldots, C_m) + C' = 0, \\ &\text{(ii) } u_{n+1}(\tau) + G_n(\tau, C_1, C_2, \ldots, C_m) + C'\tau + C'' = 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} F_n(\tau, C_1, C_2, \ldots, C_m) &= \int f\left(u_n(\tau), u'_n(\tau), u''_n(\tau)\right) d\tau, \\ G_n(\tau, C_1, C_2, \ldots, C_m) &= \int F_n(\tau, C_1, C_2, \ldots, C_m)\ d\tau. \end{aligned} \tag{2.6}$$

From the initial conditions (2.2), we consider

$$\begin{aligned} &\text{(i) } F_n(0, C_1, C_2, \ldots, C_m) = 0, \\ &\text{(ii) } G_n(0, C_1, C_2, \ldots, C_m) = -A \end{aligned} \tag{2.7}$$

such that the integration constants C' and C'' into (2.7)(i) and (2.5)(ii) become $C' = C'' = 0$.

In this way, the approximate solution of n+1 order can be written in the form

$$u_{n+1}(\tau) = -G_n(\tau, C_1, C_2, \ldots, C_m), \tag{2.8}$$

where the constants $C_1, C_2, \ldots, C_m$ which are considered in the initial approximation (2.4) can be identified via various methods, such as, for example, the least square method, the Galerkin method, the Ritz method, and the collocation method. For example, imposing that the residual functional given by

$$J(C_1, C_2, \ldots, C_m) = \int_0^T \left[u''_n + f\left(u_n, u'_n, u''_n\right)\right]^2 d\tau \tag{2.9}$$

is minimum, one can obtain the optimal values of the unknown constants. Taking into consideration (2.7), the constants C_i, $i = 1, 2, \ldots, m$ can be determined in this case from the equations (conditioned minimum)

$$\frac{\partial J}{\partial C_j} + \lambda_1 \frac{\partial F_n(0, C_1, C_2, \ldots, C_m)}{\partial C_j} + \lambda_2 \frac{\partial G_n(0, C_1, C_2, \ldots, C_m)}{\partial C_j} = 0, \quad j = 3, 4, \ldots, m, \tag{2.10}$$

where

$$\lambda_1 = \frac{(\partial J/\partial C_1)(\partial G_n/\partial C_2) - (\partial J/\partial C_2)(\partial G_n/\partial C_1)}{(\partial G_n/\partial C_1)(\partial F_n/\partial C_2) - (\partial G_n/\partial C_2)(\partial F_n/\partial C_1)},$$
$$\lambda_2 = \frac{(\partial J/\partial C_1)(\partial F_n/\partial C_2) - (\partial J/\partial C_2)(\partial F_n/\partial C_1)}{(\partial F_n/\partial C_1)(\partial G_n/\partial C_2) - (\partial F_n/\partial C_2)(\partial G_n/\partial C_1)} \tag{2.11}$$

and if (2.7)(i) is not identity. Now, if (2.7)(i) becomes identity, the constants $C_i,\ i = 1, 2, \ldots, m$ then can be determined from (2.7)(ii) and from the following equations:

$$\frac{\partial J}{\partial C_j} - \frac{\partial J/\partial C_1}{\partial G_n/\partial C_1}\frac{\partial G_n}{\partial C_j} = 0, \quad j = 2, 3, \ldots, m. \tag{2.12}$$

Therefore, the solution (2.8) with the known constants $C_1, C_2, \ldots, C_m$ is well determined.

In the present paper we consider a nonlinear oscillator with cubic and harmonic restoring force

$$\ddot{u} + u + au^3 + b\ \sin u = 0, \tag{2.13}$$

where a and b are known constants and dot denotes derivative with respect to time t. The initial conditions are given by

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{2.14}$$

If Ω is the frequency of the system described by (2.13) and introducing a new independent variable

$$\tau = \Omega t \tag{2.15}$$

then (2.13) becomes

$$u'' + f(u) = 0, \tag{2.16}$$

where $' = d/d\tau$ and

$$f(u) = \frac{1}{\Omega^2}\left(u + au^3 + b\sin u\right). \tag{2.17}$$

The initial conditions (2.14) become

$$u(0) = A, \qquad u'(0) = 0. \tag{2.18}$$

We consider the initial approximation in the form

$$u_0(\tau) = C_1 \cos\tau + 2C_2 \cos 3\tau + 2C_3 \cos 5\tau + 2C_4 \cos 7\tau, \tag{2.19}$$

where C_1, C_2, C_3, and C_4 are unknown constants at this moment.

For $n = 0$ into (2.3) we obtain the first iteration given by

$$u_1'' + f(u_0) = 0 \tag{2.20}$$

but it is difficult to calculate $f(u_0)$ with u_0 given by (2.19). Now, the function f can be expanded in a series using the well-known formula

$$f(t_0 + h) = f(t_0) + \frac{h}{1!} f_u(t_0) + \cdots, \tag{2.21}$$

where $f_u = df/du$. In the following, we consider

$$t_0 = C_1 \cos\tau, \qquad h = 2C_2 \cos 3\tau + 2C_3 \cos 5\tau + 2C_4 \cos 7\tau \tag{2.22}$$

such that, from (2.19), (2.21), and (2.22), we obtain

$$f(u_0) = f(C_1 \cos\tau) + (2C_2 \cos 3\tau + 2C_3 \cos 5\tau + 2C_4 \cos 7\tau) f_u(C_1 \cos\tau). \tag{2.23}$$

The first term in the right-hand side of (2.23) becomes

$$f(C_1 \cos\tau) = -\frac{1}{\Omega^2}\left[C_1 \cos\tau + \frac{aC_1^3}{4}(\cos 3\tau + 3\cos\tau) + b\sin(C_1 \cos\tau)\right]. \tag{2.24}$$

The last term in (2.24) can be expanded in the power series

$$\sin(C_1 \cos\tau) = C_1 \cos\tau - \frac{1}{3!}C_1^3 \cos^3\tau + \frac{1}{5!}C_1^5 \cos^5\tau - \frac{1}{7!}C_1^7 \cos^7\tau + \frac{1}{9!}C_1^9 \cos^9\tau + \cdots. \tag{2.25}$$

Substituting (2.25) into (2.24), after some simple manipulations we obtain

$$f(C_1 \cos\tau) = \alpha_1 \cos\tau + \alpha_3 \cos 3\tau + \alpha_5 \cos 5\tau + \alpha_7 \cos 7\tau + \alpha_9 \cos 9\tau + \cdots, \tag{2.26}$$

where

$$\begin{aligned}
\alpha_1 &= -\frac{C_1}{\Omega^2}\left[1+\frac{3}{4}aC_1^2+b\left(1-\frac{C_1^2}{8}+\frac{C_1^4}{192}-\frac{C_1^6}{9216}+\frac{C_1^8}{737280}+\cdots\right)\right];\\
\alpha_3 &= -\frac{C_1^3}{\Omega^2}\left[\frac{1}{4}a-b\left(\frac{1}{24}-\frac{C_1}{384}+\frac{C_1^4}{15360}-\frac{C_1^6}{1105920}+\cdots\right)\right];\\
\alpha_5 &= -\frac{bC_1^5}{\Omega^2}\left(\frac{1}{1920}-\frac{C_1^2}{46080}+\frac{C_1^4}{2580480}+\cdots\right);\\
\alpha_7 &= \frac{bC_1^7}{\Omega^2}\left(\frac{1}{322560}-\frac{C_1^2}{10321920}+\cdots\right); \qquad \alpha_9 = -\frac{bC_1^9}{\Omega^2}\left(\frac{1}{92897280}+\cdots\right).
\end{aligned} \tag{2.27}$$

The last term in the right-side of (2.23) is

$$f_u(C_1\cos\tau) = -\frac{1}{\Omega^2}\left[1+3aC_1^2\cos^2\tau+b\cos(C_1\cos\tau)\right]. \tag{2.28}$$

In (2.28), the last term can be written as

$$\cos(C_1\cos\tau) = 1-\frac{C_1^2\cos^2\tau}{2!}+\frac{C_1^4\cos^4\tau}{4!}-\frac{C_1^6\cos^6\tau}{6!}+\frac{C_1^8\cos^8\tau}{8!}+\cdots. \tag{2.29}$$

Substituting (2.29) into (2.28), we obtain

$$f_u(C_1\cos\tau) = \beta_0+\beta_2\cos 2\tau+\beta_4\cos 4\tau+\beta_6\cos 6\tau+\beta_8\cos 8\tau+\cdots, \tag{2.30}$$

where

$$\begin{aligned}
\beta_0 &= -\frac{1}{\Omega^2}\left[1+\frac{3}{2}aC_1^2+b\left(1-\frac{C_1^2}{4}+\frac{C_1^4}{64}-\frac{C_1^6}{2304}+\frac{C_1^8}{147456}+\cdots\right)\right];\\
\beta_2 &= \frac{1}{\Omega^2}\left[\frac{3}{2}aC_1^2-\frac{1}{4}C_1^2\left(1-\frac{C_1^2}{12}+\frac{C_1^4}{384}-\frac{C_1^6}{23040}+\cdots\right)\right];\\
\beta_4 &= \frac{bC_1^4}{192\Omega^2}\left(1-\frac{C_1^2}{20}+\frac{C_1^4}{960}+\cdots\right);\\
\beta_6 &= -\frac{bC_1^6}{2304\Omega^2}\left(1-\frac{C_1^2}{28}+\cdots\right); \qquad \beta_8 = \frac{bC_1^8}{5160960\Omega^2}(1+\cdots).
\end{aligned} \tag{2.31}$$

Substituting (2.24) and (2.30) into (2.23), we obtain the expression

$$\begin{aligned} f(u_0) = {} & [\alpha_1 + (\beta_2 + \beta_4)C_2 + (\beta_4 + \beta_6)C_3 + (\beta_6 + \beta_8)C_4] \cos\tau \\ & + [\alpha_3 + (2\beta_0 + \beta_6)C_2 + (\beta_2 + \beta_8)C_3 + \beta_4 C_4] \cos 3\tau \\ & + [\alpha_5 + (\beta_2 + \beta_8)C_2 + 2\beta_0 C_3 + \beta_2 C_4] \cos 5\tau \\ & + (\alpha_7 + 2\beta_4 C_2 + 2\beta_2 C_3 + 2\beta_0 C_4) \cos 7\tau \\ & + (\alpha_9 + 2\beta_6 C_2 + 2\beta_4 C_3 + 2\beta_2 C_4) \cos 9\tau + \cdots . \end{aligned} \tag{2.32}$$

Equation (2.5)(i) becomes

$$\begin{aligned} u_1'(\tau) = {} & -[\alpha_1 + (\beta_2 + \beta_4)C_2 + (\beta_4 + \beta_6)C_3 + (\beta_6 + \beta_8)C_4] \sin\tau \\ & - \frac{1}{3}[\alpha_3 + (2\beta_0 + \beta_6)C_2 + (\beta_2 + \beta_8)C_3 + \beta_4 C_4] \sin 3\tau \\ & - \frac{1}{5}[\alpha_5 + (\beta_2 + \beta_8)C_2 + 2\beta_0 C_3 + \beta_2 C_4] \sin 5\tau \\ & - \frac{1}{7}(\alpha_7 + 2\beta_4 C_2 + 2\beta_2 C_3 + 2\beta_0 C_4) \sin 7\tau \\ & - \frac{1}{9}(\alpha_9 + 2\beta_6 C_2 + 2\beta_4 C_3 + 2\beta_2 C_4) \sin 9\tau + \cdots . \end{aligned} \tag{2.33}$$

Finally, (2.8) becomes

$$\begin{aligned} u_1(\tau) = {} & [\alpha_1 + (\beta_2 + \beta_4)C_2 + (\beta_4 + \beta_6)C_3 + (\beta_6 + \beta_8)C_4] \cos\tau \\ & + \frac{1}{9}[\alpha_3 + (2\beta_0 + \beta_6)C_2 + (\beta_2 + \beta_8)C_3 + \beta_4 C_4] \cos 3\tau \\ & + \frac{1}{25}[\alpha_5 + (\beta_2 + \beta_8)C_2 + 2\beta_0 C_3 + \beta_2 C_4] \cos 5\tau \\ & + \frac{1}{49}(\alpha_7 + 2\beta_4 C_2 + 2\beta_2 C_3 + 2\beta_0 C_4) \cos 7\tau \\ & + \frac{1}{81}(\alpha_9 + 2\beta_6 C_2 + 2\beta_4 C_4 + 2\beta_2 C_4) \cos 9\tau. \end{aligned} \tag{2.34}$$

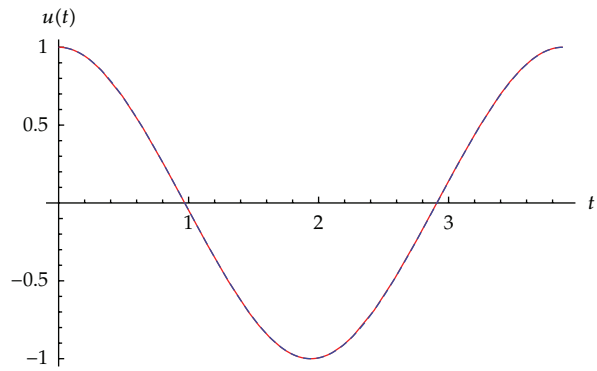

Figure 1: Comparison between the approximate solution (2.34) and numerical solution of (2.13) in Case a: $a = b = A = 1$: dashed red line: numerical solution, dashed blue line approximate solution.

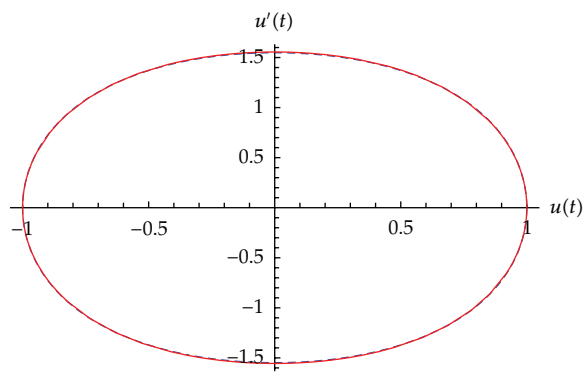

Figure 2: Comparison between the approximate solution (2.34) and numerical results of (2.13) in terms of phase plane in Case a: $a = b = A = 1$: dashed red line: numerical solution, dashed blue line approximate solution.

From (2.33) we obtain that (2.7)(i) becomes identity and (2.7)(ii) becomes

$$\alpha_1 + \frac{1}{9}\alpha_3 + \frac{1}{25}\alpha_5 + \frac{1}{49}\alpha_7 + \frac{1}{81}\alpha_9 + C_2\left(\frac{2}{9}\beta_0 + \frac{26}{25}\beta_2 + \frac{51}{49}\beta_4 + \frac{11}{81}\beta_6 + \frac{1}{25}\beta_8\right)$$
$$+ C_3\left(\frac{2}{25}\beta_0 + \frac{67}{441}\beta_2 + \frac{83}{81}\beta_4 + \beta_6 + \frac{1}{9}\beta_8\right) + C_4\left(\frac{2}{49}\beta_0 + \frac{131}{2025}\beta_2 + \frac{1}{9}\beta_4 + \beta_6 + \beta_8\right) - A = 0. \tag{2.35}$$

The frequency Ω and the constants C_1, C_2, C_3, and C_4 are determined by means of a collocation-type method.

3. Numerical Examples

We will illustrate the applicability, accuracy, and effectiveness of the proposed approach by comparing the analytical approximate periodic solution with numerical integration results obtained using a fourth-order Runge-Kutta method. The comparison is made in terms of displacements and phase plane. The error of the solution has been also computed. The results of these comparisons are presented in Figures 1–6 for several cases.

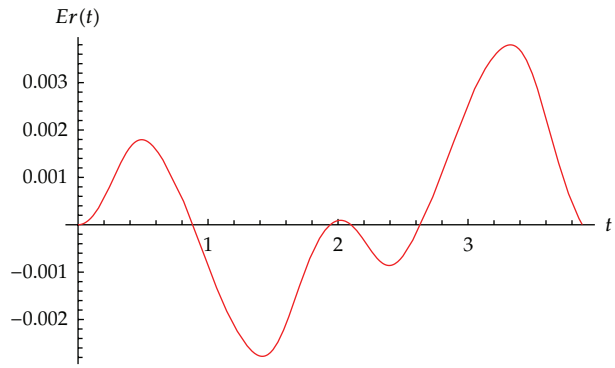

Figure 3: The error between the numerical and approximate solution (2.34) in Case a: $a = b = A = 1$.

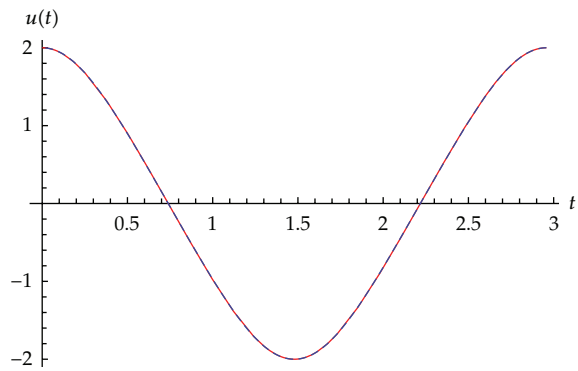

Figure 4: Comparison between the approximate solution (2.34) and numerical solution of (2.13) in Case b: $a = b = 1$, $A = 2$: dashed red line: numerical solution, dashed blue line approximate solution.

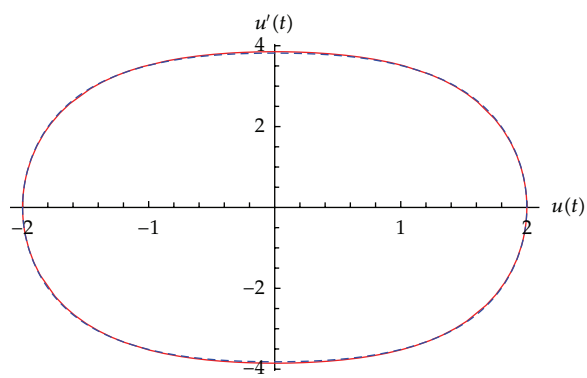

Figure 5: Comparison between the approximate solution (2.34) and numerical results of (2.13) in terms of phase plane in Case b: $a = b = 1$, $A = 2$: dashed red line: numerical solution, dashed blue line approximate solution.

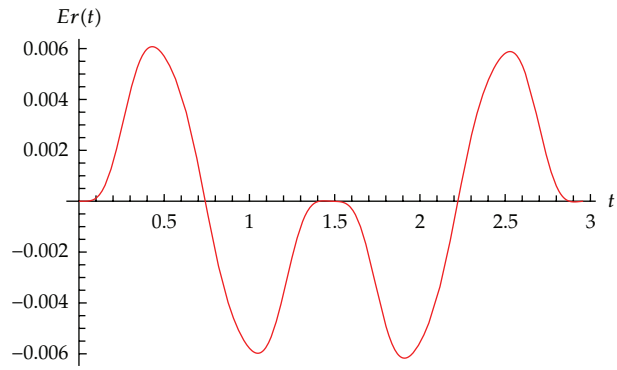

Figure 6: The error between the numerical and approximate analytical solution (2.34) in case b: $a = b = 1,\ A = 2$.

Case a. For $a = 1$, $b = 1$, $A = 1$, following the procedure described above we obtain the approximate periodic solution of (2.13) in the form

$$\begin{aligned} u_1(t) &= 0.988394597\cos\Omega t + 0.011310241\cos 3\Omega t + 0.000326978\cos 5\Omega t \\ &\quad + 0.000003994\cos 7\Omega t - 0.00003581\cos 9\Omega t, \end{aligned} \tag{3.1}$$

where $\Omega = 1.61923$. In Figure 1 is presented a comparison between the approximate solution (3.1) and the solution obtained through numerical simulations. Moreover, Figure 2 presents a comparison between the approximate solution (3.1) and the numerical results in terms of phase plane. In order to provide a comprehensive evidence of the accuracy of the results, the error of the solution has been computed:

$$Er(t) = u_N(t) - u_1(t), \tag{3.2}$$

where $u_N(t)$ is the numerical result and $u_1(t)$ is the approximate solution given by (2.34). A graphical representation of the error in the Case a is presented in Figure 3.

Case b. For $a = 1,\ b = 1,\ A = 2$, following the same procedure we obtain

$$\begin{aligned} u_1(t) &= 1.947052312\cos\Omega t + 0.052117923\cos 3\Omega t + 0.001198712\cos 5\Omega t \\ &\quad - 0.000241312\cos 7\Omega t - 0.000127635\cos 9\Omega t, \end{aligned} \tag{3.3}$$

where $\Omega = 2.12453$. Comparisons between the approximate and numerical results for Case b are presented in Figures 4–6.It can be seen from Figures 1–6 that the results obtained using OIPM are almost identical with those obtained through numerical simulations.

4. Conclusions

In this paper we have developed an analytical treatment of strongly nonlinear oscillators with cubic and harmonic restoring force using a new approximate analytical technique, namely,

the Optimal Iteration Perturbation Method (OIPM). This method accelerates the convergence of the solutions since after only one iteration we achieved very accurate results. The proposed approach is an iterative procedure, and iterations are preformed in a very simple manner by identifying optimally some coefficients and therefore very good approximations are obtained in few terms. Actually, the capital strength of OIPM is its fast convergence. An excellent agreement of the approximate periodic solutions and frequencies with the exact ones has been demonstrated. Two examples are given, and the results reveal that our procedure is very effective, simple, and accurate. This paper demonstrates the general validity and the great potential of the OIPM for solving strongly nonlinear problems.

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